HYDRODYNAMIC-ANALYTICAL MODELING FOR IRRIGATION ANALYSIS

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الخلاصـــة :

يهدف النموذج الهيدروديناميكي التحليلي للجريان تحت السطحي إلى سد الفجوة بين نمذجة النفاذية وتطبيقات النماذج التجريبية للنفاذية غير الكافية. ويعتمد هذا النموذج أساسا على معادلتي ريتشارد ولابلاس المطورتين والمدمجتين من أجل الحصول على حل تحليلي لهذه المعادلات. لقد تَـمَّ تصميم النموذج الجديد باعتبار الخواص الهيدروليكية الشاملة للتربة خصيصا لعمليات توقع وحساب الترشيح لحساب النفاذية وتوزيع رطوبة التربة نتيجة لمعاملات الري المختلفة ويتم تحقيق ذلك ضمن عوامل أخرى باعتبار الخصائص الهيدروليكية للتربة. إن مجال تطبيقات هذا النموذج تشمل النفاذية وترشيح المياه خلال التربة أثناء عملية الترطيب باعتبار سطح التربة المشبعة وغير المشبعة كحدود عُليا. لقد أظهرت النتائج أن حسابات النماذج المطورة للنقاذية المتغيرة عالية الدقة وسهلة التطبيق نسبيا.

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ABSTRACT

The presented hydrodynamic–analytical subsurface flow model is intended to fill the gap between cumbersome numerical modeling of the infiltration process and the application of inadequate empirical infiltration models. It is based on both the Richards equation together with the Laplace equation, which are mathematically developed and integrated in order to obtain an analytical solution. The new model is especially designed for predicting infiltration and soil moisture distribution due to varying combinations of irrigation parameters. This is achieved, amongst other things, by a comprehensive consideration of the hydraulic soil characteristics. The field of model application encompasses the simulation of the infiltration process and soil water flow during the wetting phase, providing the option of both saturated or unsaturated soil surfaces as upper boundary conditions. The prognostic computation of the transient infiltration process by the proposed model is shown to be highly accurate and is, last but not least, relatively easy to apply.

HYDRODYNAMIC–ANALYTICAL MODELING FOR IRRIGATION ANALYSIS

1. INTRODUCTION

The purpose of irrigation is to cater for a particular moisture profile in the vicinity of the root zone of plants. The individualized form of this profile is characterized by the plant requirements and represents a plant-specific parameter which varies for different crops. On the one hand, such a transient soil moisture distribution is controlled by the fixed and given natural properties like the soil characteristics and the climatic conditions; on the other hand, it is dependent upon irrigation parameters such as the duration, frequency, and intensity of water application, which not only affect this process but can also be positively influenced in order to achieve the potentially best type of on-field irrigation efficiency. In many cases, the desired solution may additionally include the choice of cropping pattern as an optimization variable.

A realistic prediction of the infiltration process, taking into account its transient character during water application under changing conditions, is a first prerequisite of such investigations. This cannot be provided by the widely used empirical infiltration formulae as, *e.g.* the Kostiakov formula, the parameters of which have even been shown to be time-dependent [1]. On the contrary, the prognostic computation of the infiltration process requires full use of the information about the soil, which can be exclusively exploited by physically based models, such as by one of the various types of numerical solutions of the Richards equation [2, 3]. However, numerical modeling of irrigation phenomena on the field scale encounters serious problems. As with surface irrigation, for example, the flooding of dry areas always creates discontinuous infiltration processes with sharp wetting fronts, which are difficult to handle numerically due to the extremely high risk of instabilities and substantial truncation errors. The high computational effort for a repeatedly required solution, *e.g.* when attempting to optimize sprinkler irrigation parameters or computing infiltration during border or furrow irrigation [4], also represents a severe burden on this type of approach.

Being derived from physical considerations and avoiding numerical methods for solving partial differential equations, the well known, quasi-analytical series solution of Philip [5] seems on a first glance to satisfy the requirements for an application to irrigation hydraulics. Nevertheless, from a critical viewpoint it has to be admitted that the determination of the coefficients to be obtained from a (numerical) solution of a system of ordinary differential equations is not as simple as it seems. For the envisaged practical application, especially in the context of a prognostic computation of the infiltration process, this approach unfortunately involves some other restrictive disadvantages:

(*i*) At the soil surface, only a flux boundary condition can be used.

(*ii*) Saturation, often occurring due to water application in the upper soil layer, cannot be taken into account.

(*iii*) The series solution diverges after some time and, thus, is only applicable within an often regrettably short time interval (as short as 20 min for sand [6]).

(*iv*) The application of the series solution requires sound knowledge in mathematical and numerical calculus. The aforementioned disadvantages are most likely the reason why these approaches are rarely used in the current practice of determining irrigation strategies and/or irrigation parameters.

This contribution develops a physically based analytical infiltration model designed for predicting infiltration, as well as the transient distribution of soil moisture due to varying combinations of irrigation parameters, or, using a flux condition, to calculate the time required to saturate the soil surface and the depth of saturation. The physically based development, using the Richards equation in the domain of the unsaturated subsurface flow and the 1D-Laplace equation for saturated flow, leads to a reliable and accurate description of the flow phenomena by an unconditionally stable algorithm which is, in addition, relatively easy to run. It applies to both saturated or unsaturated soil surfaces but is restricted to simulating the infiltration process only. For long-term simulations of surface irrigation cycles, a coupled analytical–numerical approach could be favorable. The discontinuous infiltration processes with sharp wetting fronts and repeatedly-required solutions during border or furrow irrigation can be taken on board by the analytical model part, while the rather smooth redistribution of the soil moisture after irrigation can easily be taken over by a common numerical model.

In the context of a relatively easy and straightforward application, it has to be emphasized that the theoretical effort to be invested in analytical model development significantly differs from that required to design numerical infiltration models. For the latter, the mathematical calculus can be considered as almost marginal compared to the subsequently presented, sometimes cumbersome, mathematical development which is necessary to find an adequate analytical solution to the specified problem. However, this effort has to be expended only for the model development. Once the analytical solution is obtained, the effort pays off in terms of model application. This not only involves dealing with an unconditionally stable algorithm together with a dramatic reduction of computational requirements, but also results in a straightforward and easy-to-use model, which can be run without the need for comprehensive experience in numerical methods (which is certainly not the case with numerical models, especially if used to find optimal solutions).

2. BASIC EQUATIONS

Using the generalized form of Darcy's law together with the law of mass conservation, soil water movement in unsaturated soil can be described by:

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\Theta)\frac{\partial \Theta}{\partial z} - K(\Theta)\right] , \qquad (1)$$

with t = time, z = spatial coordinate oriented positively downward, Θ = volumetric soil moisture content, $K(\Theta)$ = hydraulic conductivity, and $D(\Theta)$ = diffusivity [7].

Water transfer in the saturated region is governed by the Laplace equation which can be expressed in the one-dimensional form as:

$$\frac{\partial^2 \psi}{\partial z^2} = 0 , \qquad (2)$$

with ψ = pressure head.

In the course of water application, a saturated area may develop, extending downwards from the soil surface (Figure 1). Denoting this time-dependent depth of the saturated region by $z_s = z_s(t)$ defines the domains of validity for the above equations, namely for $z > z_s$ Equation (1), and for $0 < z < z_s$ Equation (2), is relevant.

The initial condition of the natural system refers to an unsaturated soil with a uniform soil moisture distribution, *i.e.* $z_s(0) = 0$ and $\Theta(z, 0) = \Theta_i$ (Θ_i = initial moisture content), while the boundary condition at the soil surface may be considered, for the first case as $\psi(0, t) = h_0$ (constant water depth at the surface), and for the second case as $q(t) = q_0$ (constant infiltration rate). A semi-infinite soil column is characterized at the lower boundary by the condition $\Theta(+\infty, t) = \Theta_i$. At the advancing front between the saturated and unsaturated regions (if $z_s > 0$) the transition conditions are $\psi(z_s(t), t) = \psi_s$ and $\Theta(z_s(t), t) = \Theta_s$ [8], with ψ_s = air entrance pressure and Θ_s = soil moisture content at saturation.

The transformation of Equation (1), exchanging dependent and independent variables, namely expressing depth as a function of soil moisture and time, leads in its integral form [7] to:

$$z(\Theta,t) = z_s(t) + \int_{\Theta}^{\Theta_0(t)} \frac{D(\bar{\Theta})d\bar{\Theta}}{-\left[K(\bar{\Theta}) - K_i\right] + F(\bar{\Theta},t)[q(t) - K_i]} , \tag{3}$$

which still corresponds exactly to the Richards equation (1). The upper boundary of the integral $\Theta_0(t)$, is the volumetric moisture content at the soil surface. K_i denotes the initial hydraulic conductivity corresponding to the initial soil moisture and $F(\Theta,t)$ represents the flux–concentration relationship, first defined by Philip [9]:

$$F(\Theta,t) = \frac{q(\Theta,t) - K_i}{q(t) - K_i} , \tag{4}$$

with $q(\Theta,t)$ representing the flux as a function of moisture content and time.

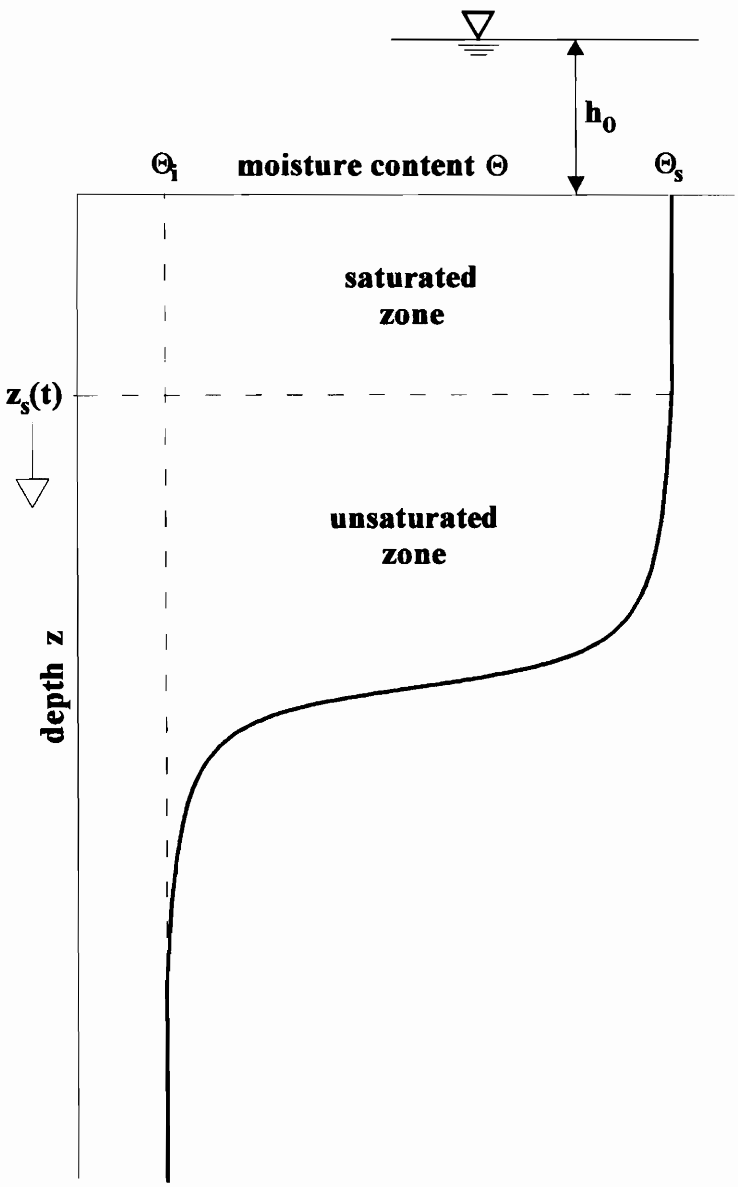

Figure 1. Infiltration Process and Qualitative Soil Moisture Profile During Irrigation.

3. INFILTRATION DURING SURFACE OR SPRINKLER IRRIGATION

In this Section, analytical solutions for the cumulative infiltration, the infiltration rate, and the extension of the saturated zone near the soil surface are presented. They are based on the mass balance by expressing the cumulative infiltration $I = I(t)$ as:

$$I(t) = K_i t + \int_{\Theta_i}^{\Theta_0(t)} z(\Theta, t) d\Theta \ . \tag{5}$$

For predicting the transient distribution of soil moisture for various applications in irrigated agriculture, a careful consideration of the hydraulic soil characteristics is a first prerequisite. This requires an adequate soil model which defines the functional relationships $K(\Theta)$ as well as $D(\Theta)$ in such a way that these functions reflect the hydraulic soil properties and, at the same time, allow for an explicit integration of Equation (5). The corresponding mathematical development is summarized in Appendix 1.

3.1. Solution for a Ponded Soil Surface

By employing the results derived in Appendix 1, a relationship between the cumulative infiltration $I(t)$ and the infiltration rate $q(t)$ can be obtained for a ponded soil surface, *i.e.* $\Theta_0(t) = \Theta_0$. Considerable mathematical calculus leads to:

$$\begin{aligned} I(t) &= K_i t + z_s(t)(\Theta_s - \Theta_i) + \frac{\sigma_s(K_s - K_i)(\Theta_s - \Theta_i)}{q(t) - K_s} \\ &+ \frac{S^2 - 2\sigma_s(K_s - K_i)(\Theta_s - \Theta_i)}{2\beta(K_s - K_i)} \ln\left[1 + \beta\frac{K_s - K_i}{q(t) - K_s}\right] , \end{aligned} \tag{6}$$

where S denotes sorptivity, K_s is the hydraulic conductivity at saturation, and β, σ_s are parameters of the soil characteristics with $0 < \beta < 1$ and $\sigma_s > 0$ (see Appendix 1).

Equation (6) still contains the unknown depth of saturation $z_s(t)$. The determination of $z_s(t)$ is performed by using the 1D-Laplace equation, which describes the flow in the saturated region $0 < z < z_s(t)$. Based on the boundary conditions defined in Section 2, one obtains:

$$\psi(z,t) = h_0 + (\psi_s - h_0)\frac{z}{z_s(t)} \tag{7}$$

for the pressure head $\psi(z,t)$ in the saturated area. Applying Darcy's law for saturated conditions $q(t) = -K_s \frac{\partial \psi}{\partial z} + K_s$, and using $\frac{\partial \psi}{\partial z} = \frac{\psi_s - h_0}{z_s(t)}$ for the gradient, yields for the transient depth of saturation:

$$z_s(t) = K_s \frac{h_0 - \psi_s}{q(t) - K_s} \ . \tag{8}$$

With this relationship and Equation (A-14) it is now possible to determine the infiltration rate from Equation (6) in terms of the parameters of the hydrodynamic soil characteristics and, among them, the global sorptivity S which as well is relatively easy to derive from on-site measurements or standard laboratory experiments:

$$q = \frac{dI}{dt} = K_i - \frac{[K_s(h_0-\psi_s)+\sigma_s(K_s-K_i)](\Theta_s-\Theta_i)}{(q-K_s)^2}\cdot\frac{dq}{dt} - \frac{S^2-2\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}{2[q-K_s+\beta(K_s-K_i)](q-K_s)}\cdot\frac{dq}{dt}\ . \tag{9}$$

Finally, Equation (9) can be integrated by separating the variables leading to the result of this first case, namely the analytical infiltration formula for a ponded soil surface:

$$t = a\frac{K_s-K_i}{q-K_s} + \frac{b}{\beta}\ln\left(1+\beta\frac{K_s-K_i}{q-K_s}\right) - (a+b)\ln\left(1+\frac{K_s-K_i}{q-K_s}\right)\ . \tag{10}$$

The constants a and b are determined by the hydrodynamic soil characteristics and the water depth at the soil surface:

$$a = \frac{[K_s(h_0-\psi_s)+\sigma_s(K_s-K_i)](\Theta_s-\Theta_i)}{(K_s-K_i)^2} \tag{11a}$$

$$b = \frac{S^2-2\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}{2(1-\beta)(K_s-K_i)^2} \tag{11b}$$

For a given ponding time t Equation (10) can now be solved iteratively for the infiltration rate $q = q(t)$. Finally, the cumulative infiltration $I(t)$ is easily obtained from Equation (6).

3.2. Solution for a Flux Condition at the Soil Surface

Let us now consider the case of a given flux condition $q(t) = q_0$ at the upper boundary, characterized by the cumulative infiltration:

$$I(t) = q_0 t\ . \tag{12}$$

In this case two of the questions to be answered refer to the existence of a saturated zone, *i.e.*, whether and at what time saturation at the soil surface will occur and, secondly, how the transient location of the zone of saturation advances with time [10].

If the water available at the soil surface is less than the value of the saturated hydraulic conductivity obviously no saturated zone will form, thus, its extension $z_s = 0$ and the moisture at the soil surface will always remain below its saturated value, *i.e.* $\Theta_0(t) < \Theta_s$.

If the flux available at the soil surface is greater than the value of the saturated hydraulic conductivity, *i.e.* $q_0 > K_s$, then the time to saturation t_s of the soil surface has to be evaluated. Taking $t = t_s$, $z_s(t_s) = 0$, and $\Theta_0(t_s) = \Theta_s$ in the mass balance (5), the expression for the cumulative infiltration (12) yields:

$$t_s = \frac{S^2-2\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}{2\beta(K_s-K_i)(q_0-K_i)}\cdot\ln\left(1+\beta\frac{K_s-K_i}{q_0-K_s}\right) + \frac{\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}{(q_0-K_i)(q_0-K_s)}\ . \tag{13}$$

For $t > t_s$ and $\Theta_0(t) = \Theta_s$, the mass balance (5) and the cumulative infiltration (12) can likewise be used to determine, together with Equation (13), the transient extension of the zone of saturation:

$$z_s(t) = \frac{q_0 - K_i}{\Theta_s - \Theta_i}(t - t_s, \; . \qquad (14)$$

4. PREDICTION OF THE TRANSIENT SOIL MOISTURE DISTRIBUTION DURING IRRIGATION

4.1. Soil Characteristics

For the prediction of the transient soil moisture distribution the soil characteristics given in Appendix 1 have to be specified. The following relationships quantifying the hydraulic conductivity K and the diffusivity D will be used throughout the remaining part of this paper:

$$K(\Theta) = K_i + (K_s - K_i) \cdot \frac{(1-\beta)\cdot\Theta^*}{1 - \beta \cdot \Theta^*} \qquad (15)$$

$$D(\Theta) = \frac{\sigma_c(1-\beta)(K_s - K_i)}{(\Theta_s - \Theta_i)(1 - \beta\Theta^*)^2} + \frac{\sigma_s(1-\omega)(K_s - K_i)}{\Theta_s - \Theta_i} \cdot \prod_{k=1}^{N+1}\left(1 + \frac{1-\omega}{k}\right) \cdot \frac{(\Theta^*)^N}{(1-\Theta^*)^\omega} \qquad (16) \; ,$$

using the expression:

$$\Theta^* = \frac{\Theta - \Theta_i}{\Theta_s - \Theta_i} \qquad (17)$$

for denoting the relative soil moisture content Θ^*. The mathematical development of Equations (15) and (16) is given in Appendix 2.

The functional relationship $K(\Theta)$ defined by Equation (15) is well suited for representing this kind of relationship for a broad variety of soils and corresponds to current soil models (Figure 2). The diffusivity function used in this approach is confirmed by a broad variety of observations and corresponds to other state-of-the-art types of diffusivity functions (Figure 3). The soil model, formed by Equations (15) and (16) additionally provides the rather unique possibility of a direct comparison between an analytical solution and numerical models.

4.2. Analytical Solution for the Soil Moisture Profile

The application of the aforementioned defined soil model to the basic equation for the analytical solution Equation (3) enables an analytic evaluation of all its integrals. This procedure involves considerable mathematical transformation work, leading to the analytical formula for the transient soil moisture profile:

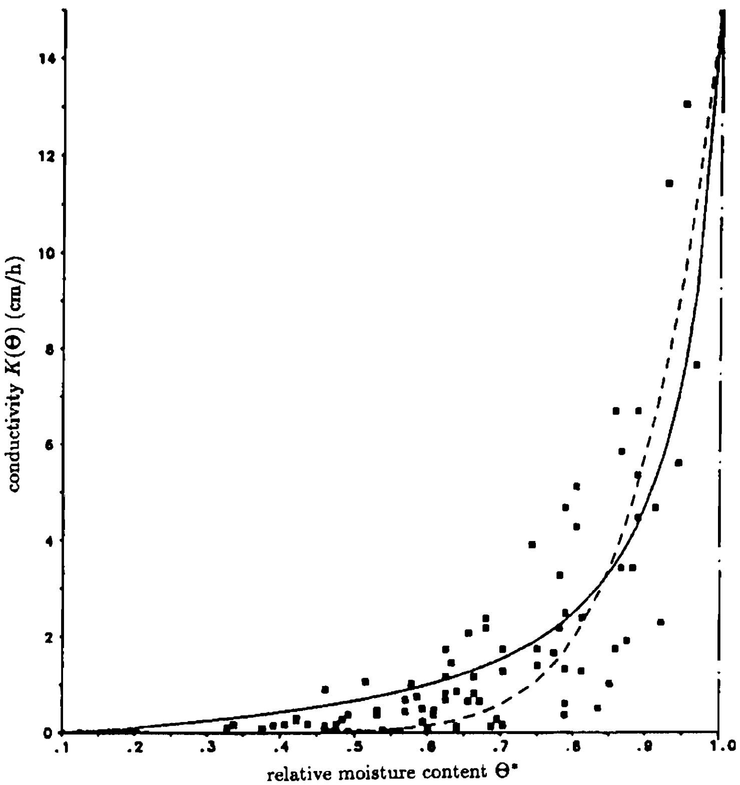

Figure 2. Conductivity Values for Varying Soil Moisture Obtained from a Field Experiment [11].
(——— Fitted curve according to Equation (15); – – – – fitted curve according to Brooks and Corey [12]).

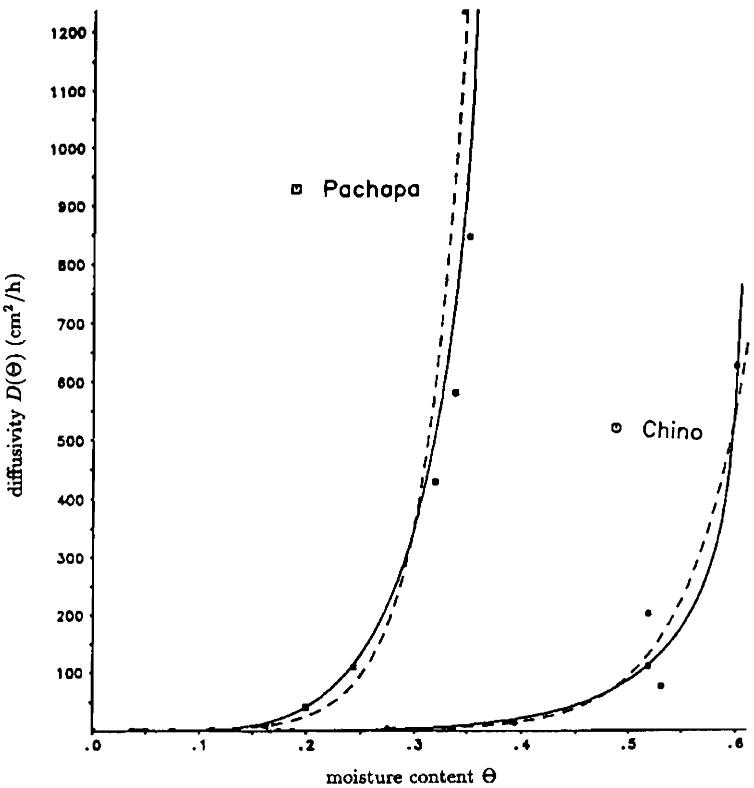

Figure 3. Diffusivity as a Function of Water Content for Pachapa and Chino Soil [13].
(——— Fitted curve according to Equation (16); – – – – fitted curve according to Gardner and Mayhugh [14]).

$$z(\Theta,t) = z_s(t) + \sigma_c \ln\frac{\Theta^*(1-\beta\Theta_0^*)}{\Theta_0^*(1-\beta\Theta^*)} + \frac{\sigma_c(q-K_i)}{q-K_i-(1-\beta)(K_s-K_i)\Theta_0^*}$$
$$\cdot \ln\left[\frac{(q-K_i)(1-\beta\Theta^*)-(1-\beta)(K_s-K_i)\Theta_0^*}{(q-K_i)(1-\beta\Theta_0^*)-(1-\beta)(K_s-K_i)\Theta_0^*}\cdot\frac{\Theta_0^*}{\Theta^*}\right]$$
$$+ \frac{\sigma_s(1-\omega)(K_s-K_i)}{q-K_i-(K_s-K_i)(1-\beta)\Theta_0^*/(1-\beta\Theta_0^*)}\cdot\prod_{k=1}^{N+1}\left(1+\frac{1-\omega}{k}\right) \tag{18}$$
$$\cdot\sum_{k=0}^{N}\binom{N}{k}(-1)^k\,\frac{(1-\Theta^*)^{k+1-\omega}-(1-\Theta_0^*)^{k+1-\omega}}{k+1-\omega}\,.$$

The analytical solution (18) employs the time-dependent relative soil moisture at the soil surface expressed as:

$$\Theta_0^* = \Theta_0^*(t) = \frac{\Theta_0(t)-\Theta_i}{\Theta_s-\Theta_i} \tag{19}$$

and still contains the depth of saturation z_s, which both have to be determined from the boundary conditions.

4.3. Soil Moisture Profile for a Ponded Soil Surface

In the case of a ponded soil surface with a prescribed ponding depth, the relative soil moisture at the surface, defined by Equation (19), equals unity and, thus, in the analytical solution (18), $\Theta_0^* = 1$. The transient infiltration rate $q = q(t)$ is determined by the analytical infiltration formula for a ponded soil surface (10), so that the depth of saturation can now be directly calculated on the basis of the 1D-Laplace equation together with Darcy's law for saturated conditions, which results in Equation (8). Thus, all the parameters in the analytical formula for the transient soil moisture profile are determined and the moisture distribution for a ponded soil surface can be calculated iteratively from Equation (18).

4.4. Soil Moisture Profile for an Unsaturated Soil Surface

In the case of a given flux at the upper boundary, *i.e.* $q(t) = q_0$, we first consider the case that $q_0 \leq K_s$ which indicates that the soil surface always remains unsaturated and $z_s = 0$. The relative soil moisture at the surface Θ_0^* is obtained by inserting Equation (18) into the mass balance equation (5) and integrating the resulting relationship. This leads to:

$$(q_0-K_i)t = \frac{\sigma_c}{\beta}(\Theta_s-\Theta_i)\ln\frac{q_0-K_i-(1-\beta)(K_s-K_i)\Theta_0^*}{q_0-K_i-(1-\beta)(K_s-K_i)\Theta_0^*/(1-\beta\Theta_0^*)}$$
$$+ \frac{\sigma_s(1-\omega)(K_s-K_i)(\Theta_s-\Theta_i)}{q_0-K_i-(K_s-K_i)(1-\beta)\Theta_0^*/(1-\beta\Theta_0^*)}\cdot\prod_{k=1}^{N+1}\left(1+\frac{1-\omega}{k}\right) \tag{20}$$
$$\cdot\sum_{k=0}^{N+1}\binom{N+1}{k}(-1)^k\,\frac{1-(1-\Theta_0^*)^{k+1-\omega}}{k+1-\omega}\,,$$

which provides for any given time t the corresponding value of Θ_0^*. Now, all the parameters in the analytical formula Equation (18) for the transient soil moisture profile are evaluated and the moisture distribution for the unsaturated soil surface can be calculated. For the upper limit of very long infiltration times, this yields the relationship

$$\Theta_0^*(t = +\infty) = \frac{q_0 - K_i}{\beta(q_0 - K_i) + (1-\beta)(K_s - K_i)} \tag{21}$$

which can be used directly for a relatively easy determination of β from data obtained by a standard experiment.

4.5. Soil Moisture Profile for an Unsaturated/Saturated Soil Surface

In the case that the given flux at the soil surface $q_0 > K_s$ this excess of the flux boundary value over the saturated hydraulic conductivity will result, after some time t_s, in the saturation of the soil surface [10]. During the time interval $0 < t < t_s$ the value of z_s equals 0. The relative soil moisture at the surface Θ_0^* can be calculated from Equation (20). After some time $t > t_s$ of applying the prescribed flux, saturation occurs and thus $\Theta_0^* = 1$. The transient depth $z_s(t)$ of saturation is then given by Equation (14), while the time t_s required to saturate the soil surface is given by Equation (13).

Thus, the parameters $\Theta_0^*(t)$ and $z_s(t)$ in the formula for the transient soil moisture profile are determined for both saturated and unsaturated boundary conditions and can be used in Equation (18) in order to evaluate for any depth z the corresponding relative moisture $\Theta^* = \Theta^*(z,t)$. For the soil moisture Equation (17) directly provides

$$\Theta(z,t) = \Theta_i + (\Theta_s - \Theta_i) \cdot \Theta^*(z,t) \; . \tag{22}$$

5. MODEL APPLICATIONS

In order to demonstrate the applicability of the analytical approach developed in this paper several examples will be presented. Cumulative infiltration and moisture profiles for three different soils are compared to results obtained by a numerical solution of the Richards equation employing the finite–difference scheme LOC1.B3 [15]. Additionally, soil moisture measurements of Touma and Vauclin [16] have been used to validate the new model. The comparisons have been performed for ponding conditions as well as for given infiltration rates (flux conditions).

The soil parameters and the initial and boundary values are listed in Table 1. The parameters for the sandy clay ("Yolo Light Clay") have been determined from measurements published by Haverkamp [6]. The sandy loam located near Texcoco (Mexico) has been characterized by Rendon *et al.* [17]. Data for a sandy soil near Grenoble (France) have been obtained from Touma and Vauclin [16].

In Figures 4–9 ponding conditions are considered, *i.e.* the soil surface is initially saturated. In the first example, infiltration in a sandy loam soil near Texcoco (Mexico) is modeled for a ponding depth of $h_0 = 3$ cm. The results presented in Figures 4 and 5 indicate that the differences between the analytical and the numerical solution are definitely negligible within a ponding period of 90 min.

A similar result is obtained for "Yolo Light Clay". Figures 6 and 7 show an excellent coincidence of the numerical and the analytical model during 10 h of ponding ($h_0 = 3$ cm).

Finally, the new approach has been validated against measurements of cumulative infiltration and soil moisture for a sandy soil from Grenoble (France). In the ponding experiment [16] there was a constant water depth $h_0 = 2.3$ cm for 24 min. Figures 8 and 9 indicate that the analytical model is able to reproduce the measured values with high accuracy. For cumulative infiltration, there are even slightly larger deviations between the measurements and the results of the finite difference approach which seems to have some difficulties in approximating the sharp moisture front.

Table 1. Input Data for the Computation of Infiltration and Soil Moisture Distributions in Sand, Loam, and Clay.

			sand	loam	clay
Ponding depth (Figures 4–9)	h_0	(cm)	2.3	3.0	3.0
Flux (Figures 10, 11)	q_0	(cm/h)	8.3	3.0	—
Flux (Figures 12–14)	q_0	(cm/h)	20.0	10.0	0.7
Moisture content at saturation	Θ_s	(-)	0.312	0.4	0.495
Initial moisture content	Θ_i	(-)	0.08	0.1	0.2376
Conductivity at saturation	K_s	(cm/h)	15.4	4.0	0.04330
Initial conductivity	K_i	(cm/h)	0.0006	0.0625	0.00003
Air entrance pressure	ψ_s	(cm)	−0.1	−24.25	−10.0
Sorptivity	S	(cm/$\sqrt{h}$)	8.0678	6.2099	0.5175
Parameters of the soil model	β	(-)	0.8802	0.6584	0.8497
	σ_s	(cm)	9.0004	5.7030	6.6446
	ω	(-)	0.8	0.5878	0.4004
	N	(-)	3	1	1

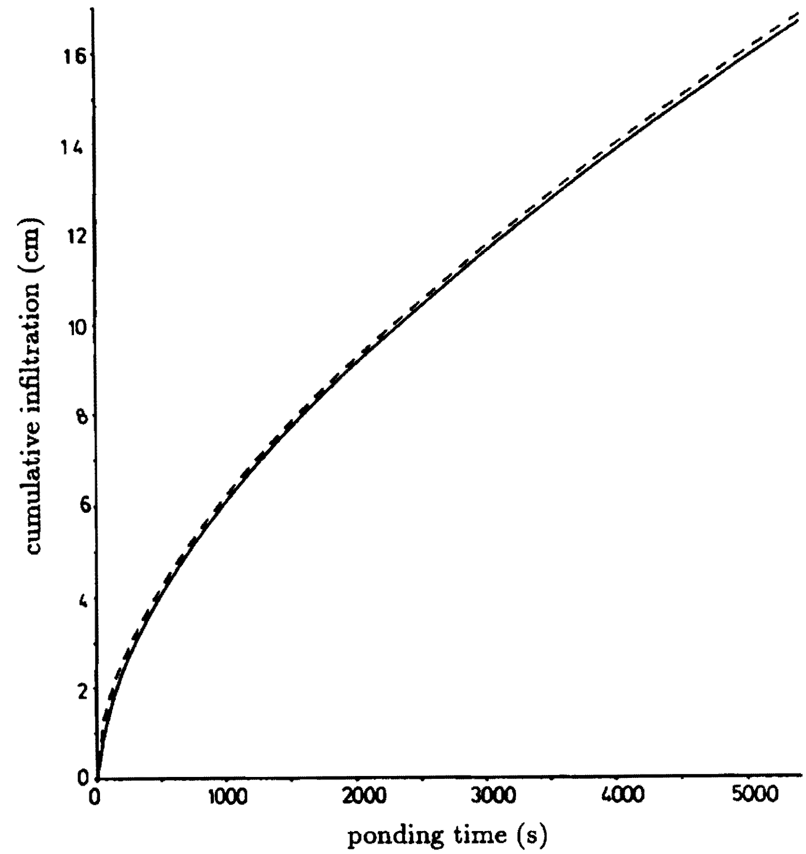

Figure 4. Cumulative Infiltration in Sandy Loam.
(——— Numerical model; — — — — analytical model).

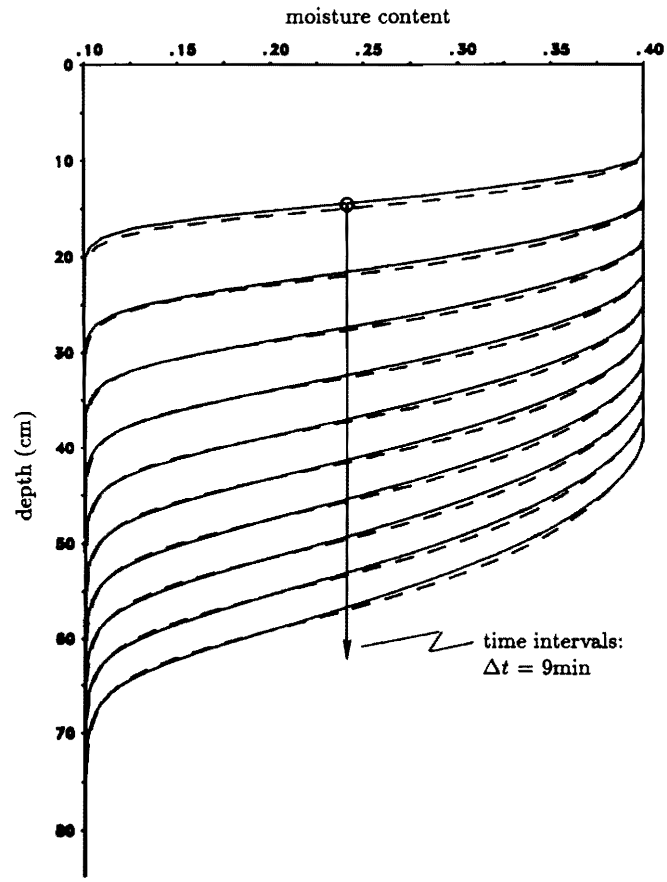

Figure 5. Soil Moisture Profiles in Sandy Loam for Ponding Conditions.
(——— Numerical model; — — — — analytical model).

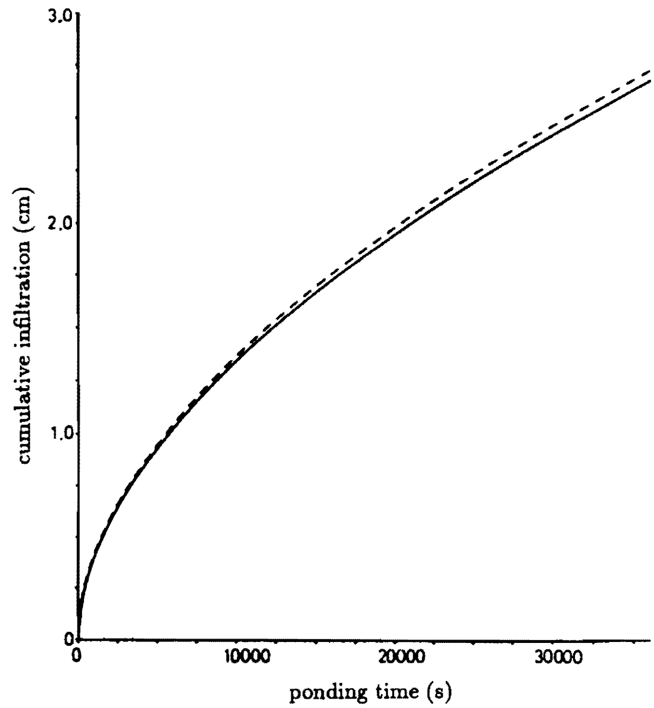

Figure 6. Cumulative Infiltration in Sandy Clay.
(——— Numerical model; — — — — analytical model).

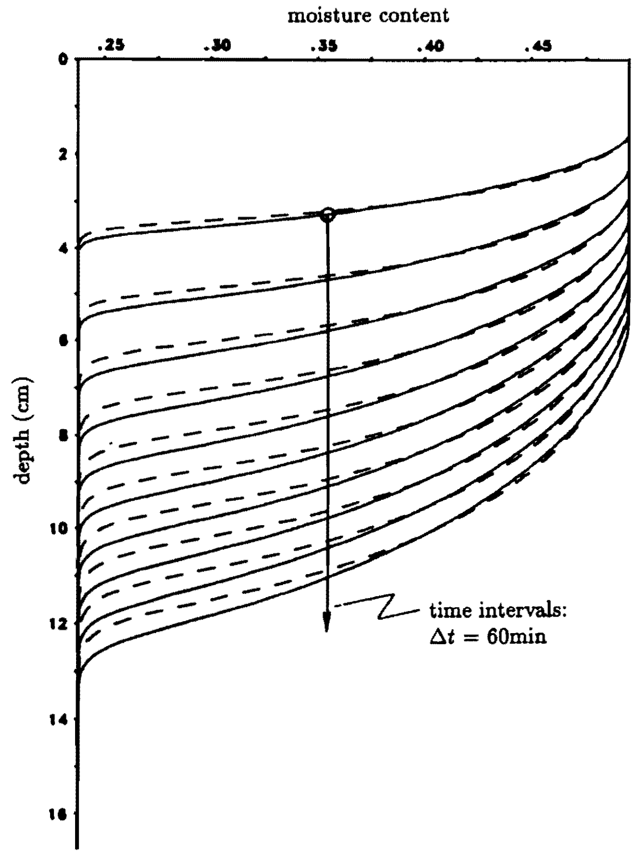

Figure 7. Soil Moisture Profiles in Sandy Clay for Ponding Conditions.
(——— Numerical model; — — — — analytical model).

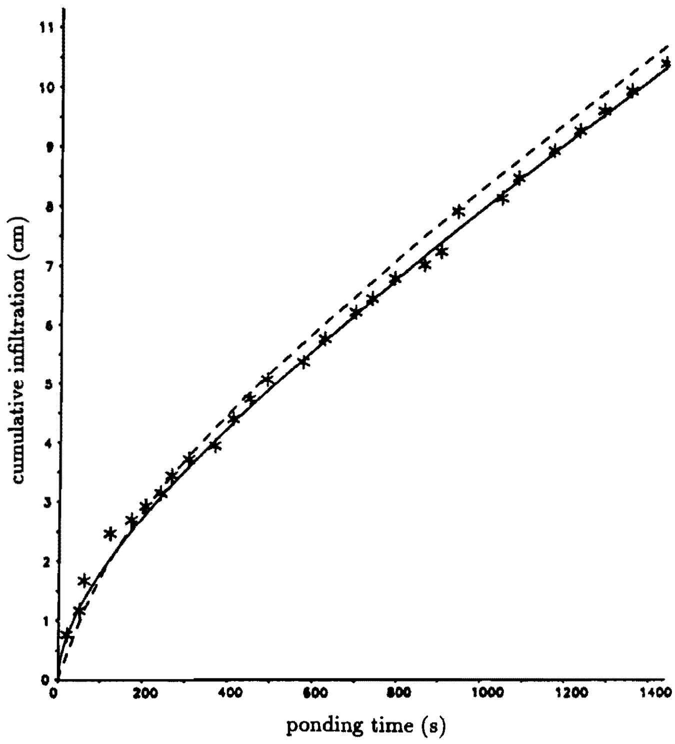

Figure 8. Cumulative Infiltration in Sand.
(——— Analytical model; — — — — numerical model; ∗ data from [16]).

It should be noted that, in the examples presented, the analytical model produces soil moisture fronts which are steeper than the fronts obtained by the numerical model. This might be attributed to the property of all finite–difference or finite–element methods to flatten high gradients ("numerical dispersion").

Next, model results for given infiltration rates are presented. Moisture profiles for an unsaturated soil surface ($q_0 \leq K_s$) have been modeled for the sandy loam (Figure 10) and for the Grenoble sand (Figure 11). Both examples indicate that there is a good coincidence of the analytical and the numerical approach. Additionally, for Grenoble sand the measured values of the soil moisture can be reproduced quite accurately. It should be added that, according to Equations (21) and (22), the analytical model predicts a maximum soil moisture content which is approximated for very long times. Here these limits are $\Theta(t = +\infty) = 0.369$ for the loam and $\Theta(t = +\infty) = 0.290$ for the sand. Both long-term limits have not been reached during the simulation periods of 2h (loam) and 30 min (sand).

The results of the analytical model and the finite–difference method, namely the soil moisture profiles for an unsaturated/saturated soil surface, show excellent agreement (Figures 12–14). In these cases, infiltration rates $q_0 > K_s$ are applied (Table 1) causing saturation of the soil surface after $t_s \approx 17$ min for the loam (Figure 12) and for the clay (Figure 13) and after $t_s \approx 21$ min for the sand (Figure 14), according to Equation (13). For the Grenoble sand, the analytical model is once more validated by using experimental data of Touma and Vauclin [16] for an observation period of 0.5 h and $q_0 = 20$ cm/h. It is found that the analytical model can predict the measured values even better than for the non-saturated case (Figure 11).

These very encouraging results for a broad variety of soils lead to the conclusion that the infiltration model presented in this paper is a highly reliable tool for predicting cumulative infiltration and transient soil moisture distributions. It should be added that it is also superior to numerical models with respect to computational effort. In the examples considered, computer time could be reduced by a factor of about 100, and the iterative procedures used to solve Equation (10) for the flux or Equations (18) and (20) for the moisture content converge quickly and are absolutely stable, so that any difficulties associated with numerical methods are avoided.

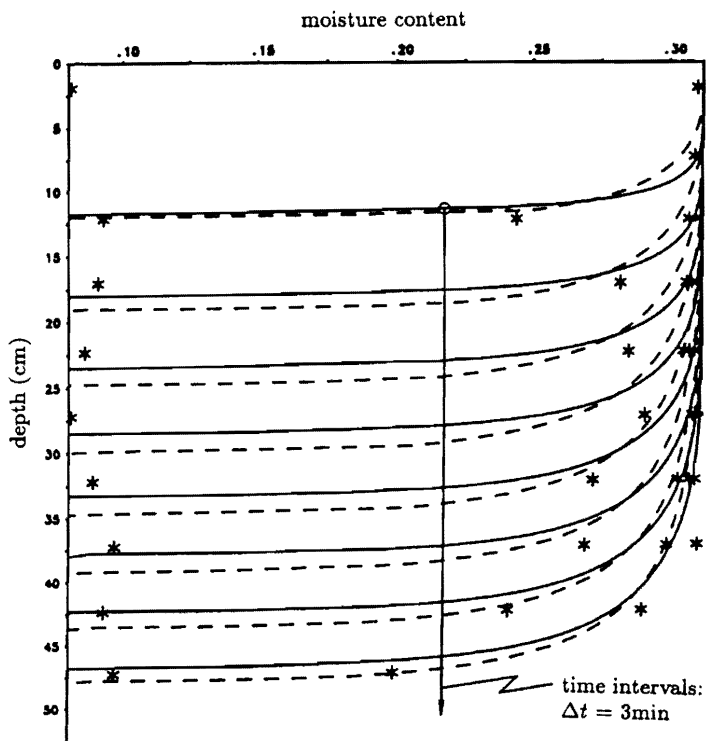

Figure 9. Soil Moisture Profiles in Grenoble Sand for Ponding Conditions.
(——— Analytical model; – – – – numerical model; ∗ data from [16]).

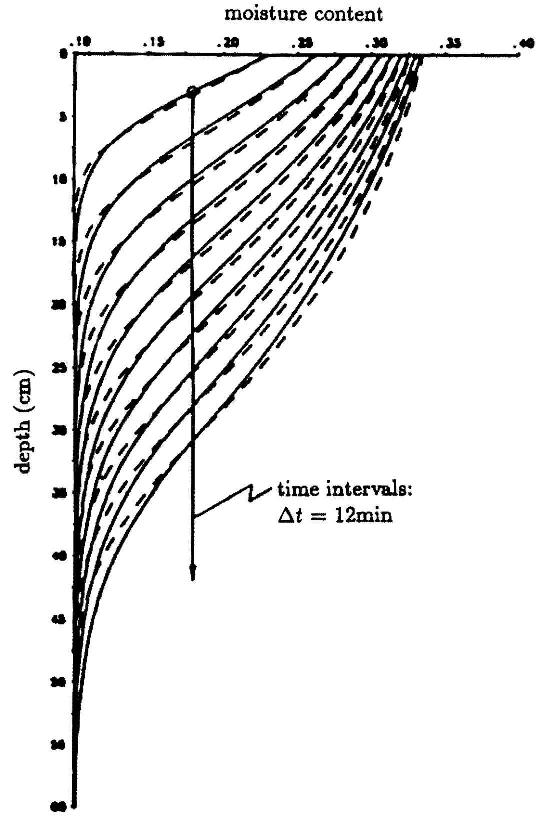

Figure 10. Soil Moisture Profiles in Sandy Loam for an Infiltration Rate of q_0 *= 3 cm/h.*
(——— Analytical model; — — — — numerical model).

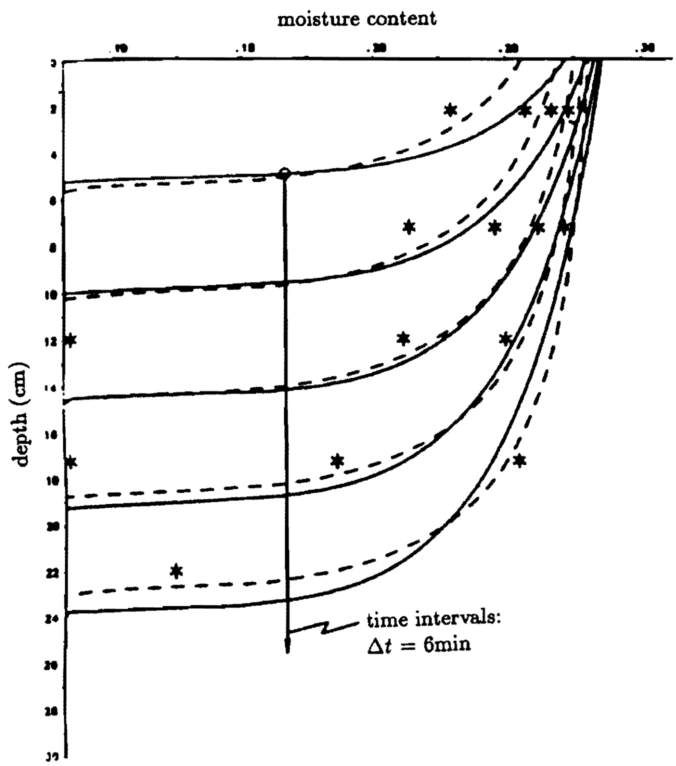

Figure 11. Soil Moisture Profiles in Grenoble Sand for an Infiltration Rate of q_0 *= 8.3 cm/h.*
*(——— Analytical model; — — — — numerical model; * data from [16]).*

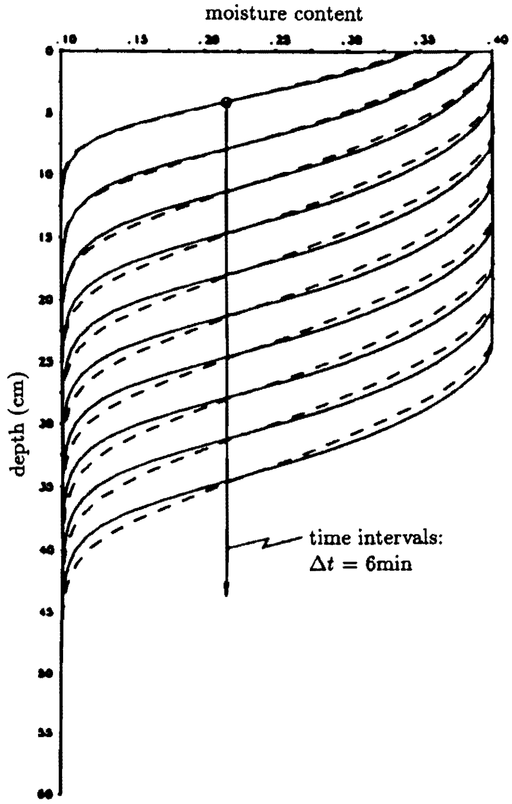

Figure 12. Soil Moisture Profiles in Sandy Loam for an Infiltration Rate of $q_0 = 10$ *cm/h.*
(——— Analytical model; — — — — numerical model).

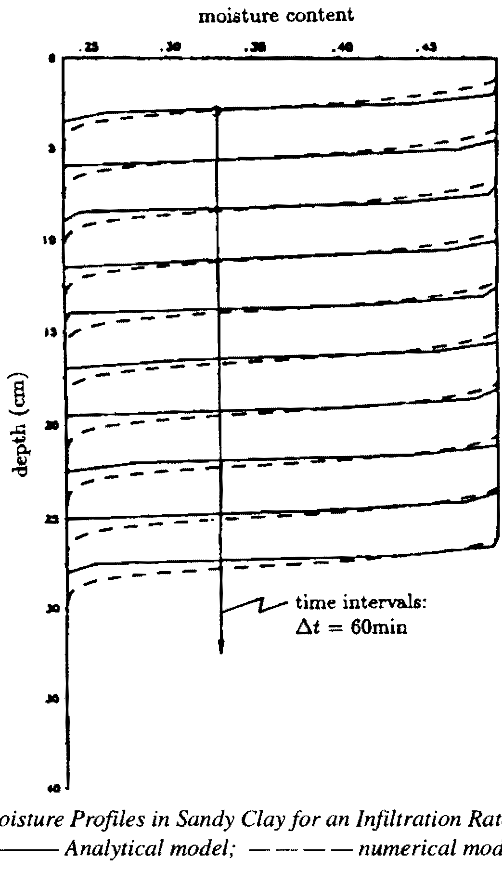

Figure 13. Soil Moisture Profiles in Sandy Clay for an Infiltration Rate of $q_0 = 0.7$ *cm/h.*
(——— Analytical model; — — — — numerical model).

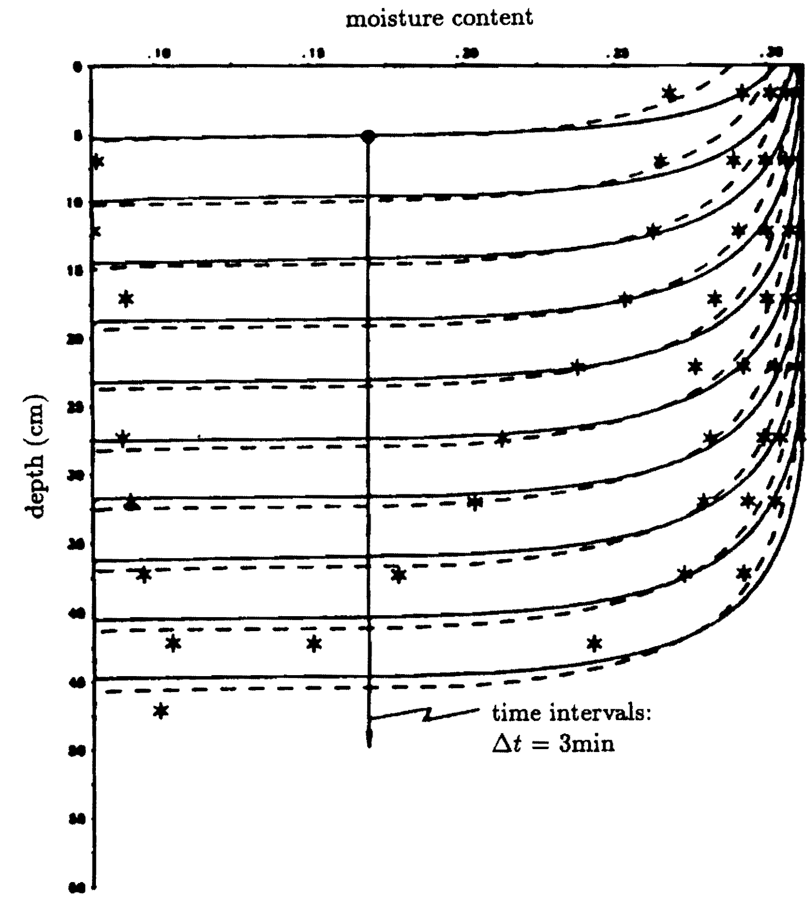

Figure 14. Soil Moisture Profiles in Grenoble Sand for an Infiltration Rate of q_0 *= 20 cm/h.*
*(——— Analytical model; — — — — numerical model; * data from [16]).*

6. SUMMARY AND CONCLUSIONS

A realistic prediction of the infiltration process during water application under a multitude of varying conditions is strictly reserved to physically-based models which alone are able to exploit measurable information about the specific hydrodynamic characteristics of the soil. The presented subsurface flow model is therefore derived from both the Richards equation and the Laplace equation, which are mathematically developed and integrated in order to obtain an analytical solution. The new model allows one to take into account a prescribed ponding depth or infiltration rate at the saturated/unsaturated soil surface, and provides information about the transient infiltration and soil moisture distribution, including the time to saturation of the soil surface and, in case of saturation, the increasing depth of the saturated domain.

The difficulties associated with the application of numerical methods (*e.g.* FEM, FDM) for describing irrigation processes on the field scale mainly relate to numerical instabilities and substantial truncation errors, which arise from discontinuous wetting processes together with the propagation of sharp wetting fronts. Thus, this task should be performed by a hydrodynamic–analytical model. However, taking into consideration the development of the soil moisture after irrigation, *i.e.* during the rather smooth redistribution process, the application of numerical methods (*e.g.* FEM, FDM) usually does not encounter any problems. Therefore, for long-term simulations of surface irrigation cycles in the course of optimizing on-field irrigation over longer periods, a coupled analytical–numerical approach could be favorable. The discontinuous infiltration processes with sharp wetting fronts and repeatedly required solutions during border or furrow irrigation could be taken on board by the analytical model part, while the rather smooth redistribution of the soil moisture after irrigation can easily be taken over by a common numerical model.

Therefore, the hydrodynamic–analytical subsurface flow model presented in this paper has been developed for predicting the infiltration and the transient distribution of soil moisture with respect to various applications in irrigated agriculture. Irrigation generally aims to establish a special form of moisture profile in the vicinity of the root zone of the plants. This root zone represents a plant-specific parameter which varies with the different crops in question. In the light of the fact that such a transient soil moisture distribution is controlled by the given natural soil characteristics, a careful consideration of these hydraulic soil properties by a realistic soil model was an important criterion for the development.

Its high computational efficiency suits the new approach for frequently required solutions, *e.g.* for the optimization of sprinkler irrigation parameters or computing infiltration during border or furrow irrigation. The prognostic computation of the transient infiltration process by the proposed model was shown to be an accurate method and it is, last but not least, relatively easy to apply.

APPENDIX 1

The diffusivity seen in Equation (3) has a major influence on unsaturated flow phenomena. Its functional relationship with the soil moisture Θ is often approximated by exponential functions. Due to the limited flexibility of these functions, this approach cannot provide satisfactory results for all soil types. A higher flexibility, allowing a more general and realistic consideration of the hydraulic soil characteristics, can be achieved by defining the diffusivity function $D(\Theta)$ by a superposition of two non-negative, steady-growing functions [18]:

$$D(\Theta) \;=\; D_c(\Theta) + D_s(\Theta)\;. \tag{A-1}$$

While the restricted component $D_c(\Theta)$ grows with increasing moisture from $D_c(\Theta_i)$ up to the finite value $D_c(\Theta_s)$, the unbounded function $D_s(\Theta)$ starts at $D_s(\Theta_i) = 0$ for growing towards infinity at $\Theta = \Theta_s$ (Figure A-1). Without affecting any physical considerations, a further degree of flexibility can be introduced by defining the free parameter $\sigma_s > 0$ such that:

$$\int_{\Theta_i}^{\Theta_s} (\Theta - \Theta_i)\, D_s(\Theta) d\Theta \;=\; \sigma_s (K_s - K_i)(\Theta_s - \Theta_i)\;, \tag{A-2}$$

with K_s = conductivity at saturation. Equation (3) yields with the diffusivity approach of Equation (A-1):

$$\begin{aligned} z(\Theta,t) \;=\; z_s(t) &+ \int_{\Theta}^{\Theta_0(t)} \frac{D_c(\bar\Theta)d\bar\Theta}{-\,[K(\bar\Theta) - K_i] + F(\bar\Theta,t)[q(t) - K_i]} \\ &+ \int_{\Theta}^{\Theta_0(t)} \frac{D_s(\bar\Theta)d\bar\Theta}{-\,[K(\bar\Theta) - K_i] + F(\bar\Theta,t)[q(t) - K_i]}\;. \end{aligned} \tag{A-3}$$

The following approximation refers to the flux–concentration Equation (4) at the saturated soil surface. Due to the fact that the infiltration process here is governed mainly by gravity [19], the investigations of Elrick and Robin [20] and Parlange *et al.* [21] recommend expressing the components of $F(\Theta,t)$ — referring to the bounded and unbounded diffusivities D_c and D_s — as:

$$F(\Theta,t) = F_c(\Theta,t) = \frac{\Theta - \Theta_i}{\Theta_0(t) - \Theta_i}\,, \tag{A-4}$$

and:

$$F(\Theta,t) = F_s(\Theta,t) = 1 - \frac{K_0(t) - K(\Theta)}{q(t) - K_i}\,, \tag{A-5}$$

where $K_0(t) = K(\Theta_0(t))$ denotes the conductivity at the soil surface. This yields the basic equation for the analytical solution, which reads:

$$z(\Theta,t) = z_s(t) + \int_{\Theta}^{\Theta_0(t)} \frac{D_c(\bar{\Theta})d\bar{\Theta}}{-[K(\bar{\Theta}) - K_i] + F_c(\bar{\Theta},t)[q(t) - K_i]} + \frac{1}{q(t) - K_0(t)} \int_{\Theta}^{\Theta_0(t)} D_s(\bar{\Theta})d\bar{\Theta}\,. \tag{A-6}$$

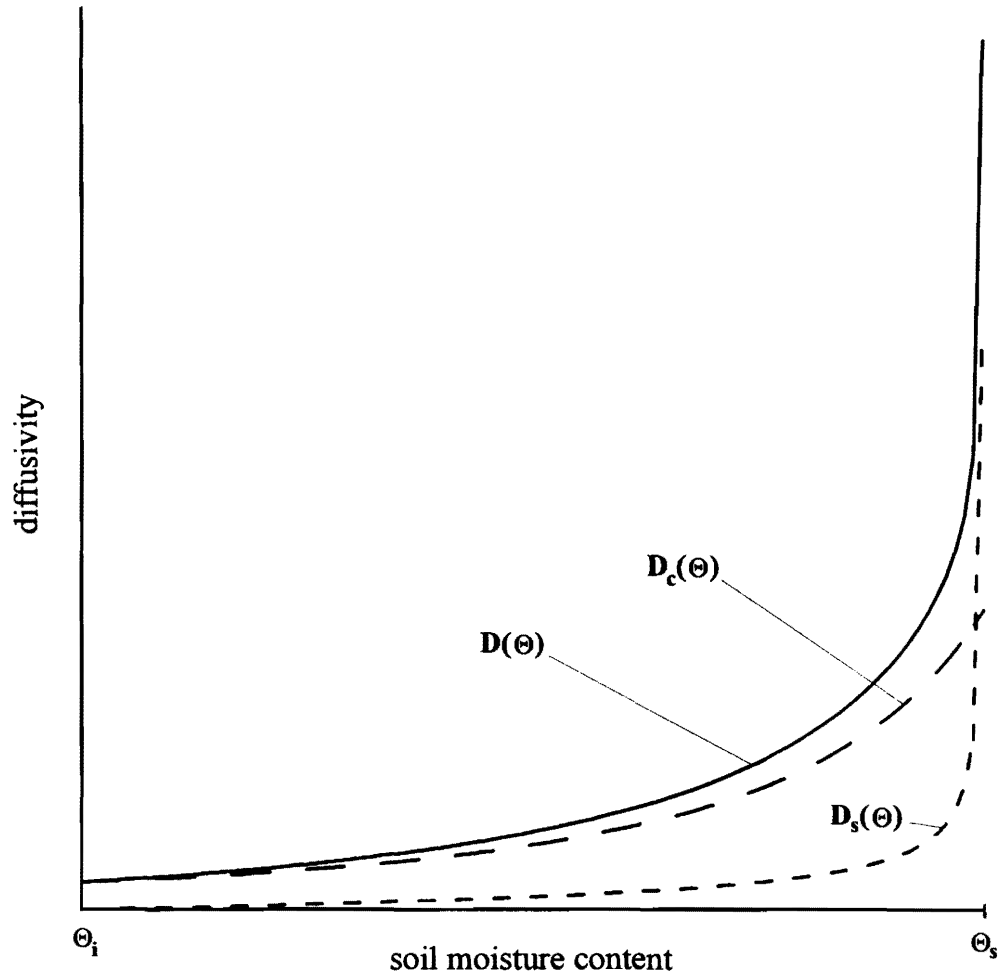

Figure A-1. Diffusivity $D(\Theta)$ *as Superposition of the Components* $D_c(\Theta)$ *and* $D_s(\Theta)$.

Considering the case of a ponded surface with a given water depth, the relevant expression for the flux–concentration relationship is:

$$F_c(\Theta,t) \;=\; F_c(\Theta) \;=\; \frac{\Theta-\Theta_i}{\Theta_s-\Theta_i}\;, \tag{A-7}$$

because $\Theta_0(t)=\Theta_s$ at the saturated soil surface. Inserting now the basic equation for the analytical solution, Equation (A-6), into the formula for the cumulative infiltration Equation (5) provides, together with the condition (A-2) referring to the free parameter σ_s:

$$\begin{aligned} I(t) \;&=\; K_i t + z_s(t)(\Theta_s-\Theta_i) \\ &+ \int\limits_{\Theta_i}^{\Theta_s} \frac{(\Theta-\Theta_i)D_c(\Theta)d\Theta}{-[K(\Theta)-K_i]+F_c(\Theta)[q(t)-K_i]} + \frac{\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}{q(t)-K_s}\;. \end{aligned} \tag{A-8}$$

The further development employs the sorptivity S, introduced by Philip [22]. To assure consistency with the diffusivity approach, it is simply decomposed analogously:

$$S^2 = S_c^2 + S_s^2\;, \tag{A-9}$$

without applying any restrictive conditions to the components of S. Defining β as a necessary free parameter in the $K(\Theta)$ function with $0<\beta<1$, the relationship between conductivity and diffusivity [6, 21]:

$$\frac{K(\Theta)-K_i}{K_s-K_i} \;=\; F_c(\Theta)\cdot\left\{1-\frac{2\beta}{S_c^2}\int\limits_{\Theta}^{\Theta_s}\left[\frac{\bar\Theta-\Theta_i}{F_c(\bar\Theta)}D_c(\bar\Theta)\right]d\bar\Theta\right\}\;, \tag{A-10}$$

employs the two components of the sorptivity as follows:

$$S_c \;=\; \sqrt{2\int\limits_{\Theta_i}^{\Theta_s}\left[\frac{\Theta-\Theta_i}{F_c(\Theta)}D_c(\Theta)\right]d\Theta} \tag{A-11}$$

$$S_s \;=\; \sqrt{2\int\limits_{\Theta_i}^{\Theta_s}\left[\frac{\Theta-\Theta_i}{F_s(\Theta,t)}D_s(\Theta)\right]d\Theta}\;. \tag{A-12}$$

In Equation (A-12) it is mainly the soil moisture values near saturation which contribute to the value of the integral (Figure A-1). This is because $D_s(\Theta)$ approaches infinity at saturation, resulting in a value of $F_s(\Theta,t)$ of approximately one. This yields together with Equation (A-2):

$$S_s \;=\; \sqrt{2\sigma_s(K_s-K_i)(\Theta_s-\Theta_i)}\;. \tag{A-13}$$

Using the decomposed sorptivity from Equation (A-9) the component S_c can be expressed as:

$$S_c^2 = S^2 - 2\sigma_s(K_s - K_i)(\Theta_s - \Theta_i) . \qquad \text{(A-14)}$$

The integral in Equation (A-8) can now be evaluated explicitly by simply employing the relationship (A-10) between conductivity and diffusivity yielding Equation (6).

The determination of the diffusivity parameter σ_s and the conductivity parameter β can be executed by measurement of the soil characteristics. Comprehensive investigations showed, that for a broad variety of soils, σ_s can be approximated with a good accuracy by the constant value $\sigma_s = 7.0$ cm. More details on the determination of these parameters are outlined in Appendix 2.

APPENDIX 2

Philip [22] found the following proportionality between the diffusivity and the conductivity:

$$D_c(\Theta) = \sigma_c \frac{dK}{d\Theta}(\Theta) . \qquad \text{(A-15)}$$

Thus, the relationship $K(\Theta)$ has to satisfy Equation (A-10) providing for the proportionality factor σ_c the condition:

$$\sigma_c = \frac{S^2}{2(K_s - K_i)(\Theta_s - \Theta_i)} - \sigma_s , \qquad \text{(A-16)}$$

by using Equation (A-14). Equation (A-16) yields together with Equations (A-10), (A-14), and (A-15):

$$\frac{K(\Theta) - K_i}{K_s - K_i} = \Theta^* \cdot \left(1 - \frac{\beta}{K_s - K_i} \int_{\Theta}^{\Theta_s} \frac{dK}{d\bar{\Theta}} d\bar{\Theta}\right) . \qquad \text{(A-17)}$$

This directly leads to the definition of the first part of the soil model, namely, to the required functional relationship $K(\Theta)$ as given in Equation (15).

The second part of the soil model focuses on the determination of the diffusivity function $D(\Theta)$. Its bounded component $D_c(\Theta)$, attaining a finite maximum value $D_c(\Theta_s)$ at Θ_s, can now be derived from Equations (A-14) and (A-15), yielding the first line on the right-hand side of Equation (A-16).

The definition of the soil model has to be rounded up by an adequate functional form of the unbounded component of the diffusivity $D_s(\Theta)$. This functional form has to satisfy the following criteria [23]:

(*i*) The sum $D(\Theta)$ of the components $D_c(\Theta)$ and $D_s(\Theta)$ has to correspond to the typical graph of diffusivity functions.

(*ii*) $D_s(\Theta)$ has to feature free parameters for higher flexibility in order to allow a more general and realistic consideration of the hydraulic soil characteristics for all soil types.

(*iii*) The function $D_s(\Theta)$ has to satisfy the condition (A-2).

From the above criteria, the following functional form of the unbounded component of the diffusivity $D_s(\Theta)$ is derived:

$$D_s(\Theta) = d^* \cdot \frac{(\Theta^*)^N}{(1-\Theta^*)^\omega}, \qquad \text{(A-18)}$$

employing the free parameters N and ω, with N representing a positive integer and $0 < \omega < 1$. The value of the coefficient d^* has to be chosen in order to satisfy the condition (A-2), from which the following relationship is obtained:

$$d^* \int_0^1 \frac{(\Theta^*)^{N+1}}{(1-\Theta^*)^\omega} d\Theta^* = \sigma_s \frac{K_s - K_i}{\Theta_s - \Theta_i}. \qquad \text{(A-19)}$$

After some mathematical calculus this leads to the expression for the coefficient d^*:

$$d^* = \frac{\sigma_s(1-\omega)(K_s - K_i)}{\Theta_s - \Theta_i} \cdot \prod_{k=1}^{N+1} \left(1 + \frac{1-\omega}{k}\right). \qquad \text{(A-20)}$$

The determination of the soil model parameters from experimentally established $K(\Theta)$ and $D(\Theta)$ relationships can be easily executed due to the low dimensionality of the curve fitting problem, while the sorptivity S can be evaluated directly from a rather simple standard experiment.

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