NUMERICAL SOLUTION FOR THE DIFFERENTIAL EQUATIONS GOVERNING THE FREE VIBRATIONS OF SPACE HELICOIDAL BARS

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الخلاصة :

يقدم هذا البحث طريقة عددية بديلة لدراسة الاهتزازات الذاتية لقضبان ثلاثية الأبعاد . فبعد تحديد الشكل الهندسي، نُقدم مُعادلات التوازن والتشوه في أطار المرونة الخطية ومن ثم نستعرض طريقة تغيير الوسائط وطريقة مصفوفة المرونة الديناميكية كطريقتين عدديتين بديلتين لحل المعادلات الناجمة. بعد ذلك تُقدم طريقة مصفوفة المرونة الديناميكية كطريقة عددية عملية لحل مسالة القيمة الحدية. وقد تم تطوير هذه الطريقة بَدْءًا من مصفوفة النقل الديناميكية واستخدمت لتطوير عنصر حلزوني محدد وحقيقي. ولتأكيد فاعلية هذه الطريقة العددية البديلة فقد استخدمت لدراسة الاهتزازات الذاتية في القضبان الحلزونية الدائرية وقورنت النتائج مع نتائج الطريقة العامة للأجزاء المحددة.

ABSTRACT

This paper presents an alternative numerical procedure for the analysis of free vibrations of three dimensional bars. First, for a specified geometry the twelve governing differential equations of equilibrium and deformations are presented within the framework of linear elasticity. Then the method of variation of parameters and the method of dynamic stiffness matrix are outlined as two alternative methods for solving the resulting equations. Finally, the dynamic stiffness matrix procedure is introduced as a more practical numerical procedure to solve the boundary value problem. The procedure is developed starting from the dynamic transport matrix and is used to develop a true helicoidal finite element. This procedure is outlined by performing the analysis of free vibrations of helicoidal circular bars and the results are compared to those of a general finite element code.

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INTRODUCTION

Uses of space bars are common in many structural applications for a wide variety of reasons ranging from necessity to practicality and even for aesthetic reasons. Examples of such applications include, but are not limited to, rectilinear staircases, helical staircases, and helical springs. Also, from a structural point of view, the efficient design of such structures is possible only with the availability of robust method(s) of analytical and/or numerical analysis. The availability of adequate methods of analysis will render the assessment of structural response under any conceivable set of forcing functions quite manageable at a reasonable cost and time.

The literature on the general subject of analysis and design of space bars includes several studies which have been devised to address fundamental issues of static and dynamic aspects of these structures. Some of these studies [1–7] have, however, been limited to either one of few particular cases including: (*a*) simplified structural geometry; (*b*) simple loading conditions; (*c*) particular boundary conditions; (*d*) static cases; or (*e*) free vibrations of simplified cases of geometry. The objective of this paper is twofold. First, the general case of a space bar (see Figure 1) is presented in a unified procedure (starting from a differential element level) that would make the process of analysis easily tractable considering the three dimensional nature of the problem. Second, the procedure is then specialized by developing a numerical solution to assess the characteristics of free vibrations of circular helicoidal bars.

BASIC RELATIONSHIPS OF GEOMETRY AND STRUCTURAL ANALYSIS

1. Geometry and Transformation

A general form of space bar is depicted in Figure 1 (*a* and *b*). In this form, it is seen that the space location of any material point is readily defined by a space position vector $\mathbf{R}(s)$. This position vector may be expressed in rectangular, polar or spherical coordinates, depending on bar geometry, but its key importance and relevance to the process of structural analysis remains the same [8]. Specifically, as a typical procedure of structural analysis undertakes the task of determining the state vectors (*i.e.* deformations and stress resultants under specified set of external effects), it is found that a concise description of local and global coordinates systems is essential. Once this description is clearly defined, an analyst can readily assess the dual relationship between state vectors in local and global coordinate systems by the construction and repetitive use of angular transformations matrix $\boldsymbol{\pi}^{lo}$. This concept is easily illustrated with reference to the case of a circular helicoidal bar shown in Figure 1(*c*). For this particular bar geometry, it is readily seen [9] that for a specified opening angle a, and helix angle $\overline{\alpha}$, the description of $\mathbf{R}(s)$ in rectangular coordinates and the use of Frenet's formulas [10], lead to the following matrix relationships:

$$\mathbf{U}^l = \mathbf{e}^l \cdot \mathbf{U}^o \,, \tag{1}$$

where $\mathbf{U}^l$ is a state vector in local system and $\mathbf{U}^o$ is a state vector in global system while the transformation vector $\mathbf{e}^l$ is obtained through the application of Frenet's equations. Based on this, it is readily found that Equation (1) takes the following form:

$$\mathbf{U}^l = \boldsymbol{\pi}^{lo} \cdot \mathbf{U}^o \,, \tag{2}$$

where the angular transformation matrix is given as:

$$\boldsymbol{\pi}^{lo} = \begin{bmatrix} -\bar{c}S_a & \bar{c}C_a & \bar{s} \\ -C_a & -S_a & 0 \\ \bar{s}S_a & -\bar{s}C_a & \bar{c} \end{bmatrix}, \tag{3}$$

in which: $\bar{c} = \cos\overline{\alpha}$; $\bar{s} = \cos\overline{\alpha}$; $C_a = \cos a$; and $S_a = \sin a$.

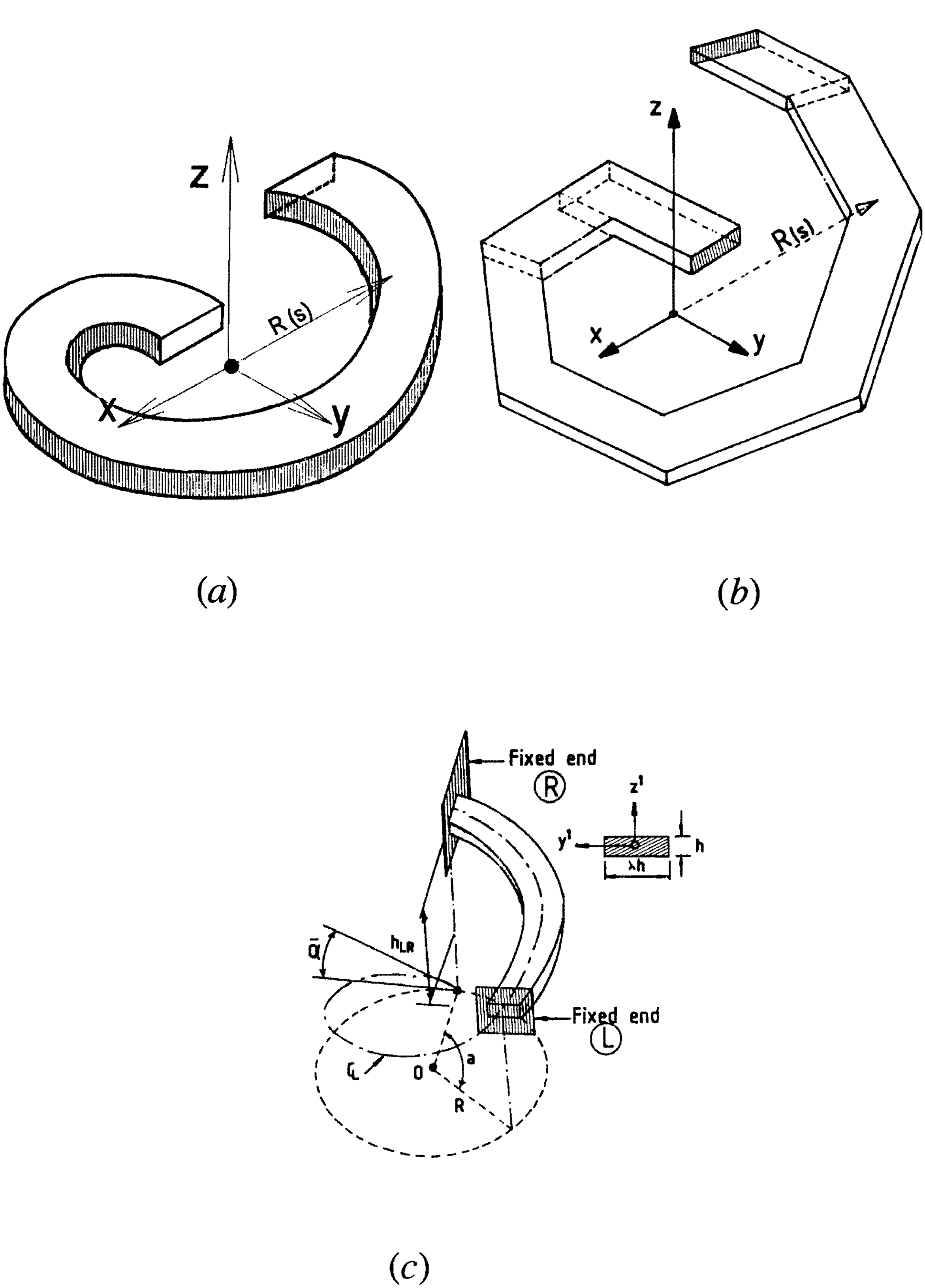

Figure 1. Geometry of Typical Space Beams. (a) *Continuous helicoidal element;* (b) *Discretized helicoidal element;* (c) *Typical helicoidal finite element.*

2. Comparison of Methods of Analysis

The methods of structural analysis are generally classified as analytical or numerical depending on how the solution for a specified problem is obtained. And with the multitude of possible structural geometries, material properties, boundary conditions and/or load cases, very often analytical solutions cannot be practically developed even for the modest combination of structural configurations and/or external effects [8, 11]. Within the scope of this paper, the inherent limitations of analytical procedures are outlined by deriving the mathematical relationships governing the characteristics of free vibrations of a circular helicoidal bar. These limitations necessarily lead to the development of practical numerical procedures which make the process of structural analysis relatively simple. For this purpose and as a prelude to later developments and/or extensions, the dual relationships between the transport matrix method and the stiffness method are reviewed with reference to the respective sign conventions of deformations and stress resultants described in Figure 1 for a generic bar segment *LR*. For this bar segment and within the framework of linear elasticity, the conditions of equilibrium and deformations are used to relate state vectors at ends *L* and *R* in their respective local coordinate systems. This results in the development of the transport matrix $\overline{T}_{RL}^{RL}$ [8] such that:

$$\overline{\mathbf{H}}_R^R = \overline{\mathbf{T}}_{RL}^{RL}\,\overline{\mathbf{H}}_L, \tag{4}$$

includes stress resultants and deformations in vector $\overline{\mathbf{H}}$ at either end, while matrix $\overline{\mathbf{T}}_{RL}^{RL}$ is function of bar geometry and is independent of boundary conditions.

It is implicitly assumed in Equation (4) that scaling factors can be introduced to overcome the numerical difficulties which may arise due to large disparities in magnitudes of elements of the absolute matrix T_{RL}^{RL}. The scaled matrix $\overline{T}_{RL}^{RL}$ may be further written in the following partitioned form:

$$\begin{bmatrix} 1 \\ \overline{S}_R \\ \overline{\Delta}_R \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \overline{G}_s & \overline{T}_{ss} & \overline{T}_{s\Delta} \\ \overline{G}_\Delta & \overline{T}_{\Delta s} & \overline{T}_{\Delta\Delta} \end{bmatrix}_{RL} \begin{bmatrix} 1 \\ \overline{S}_L \\ \overline{\Delta}_L \end{bmatrix}, \tag{5}$$

in which: external-effects-dependent vectors corresponding to stress resultants and deformations are designated, respectively, as $\overline{\mathbf{G}}_s$ and $\overline{\mathbf{G}}_\Delta$; submatrices $\overline{T}_{ss}$, $\overline{T}_{s\Delta}$, $\overline{T}_{\Delta s}$, $\overline{T}_{\Delta\Delta}$ are all of order (6×6) and form a concise representation of the conditions of equilibrium and compatibility.

The development of Equation (4) for a general space bar segment, in the form of Equation (5) is of key importance to ensuing analysis. Its importance is easily seen by noting that, as compared to the stiffness matrix method, the sign convention of deformations *remains* the same while the sign convention of stress resultants is related by the following matrix relationship:

$$\begin{bmatrix} \overline{S}_L \\ \overline{S}_R \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{S}_{LR} \\ \overline{S}_{RL} \end{bmatrix}, \tag{6}$$

in which: $\overline{S}_L$ and $\overline{S}_{LR}$ are scaled stress resultant vectors at end *L* according to the sign convention of the transport matrix method and the stiffness matrix method, respectively.

Using Equation (6) directly into Equation (4) and its inverse (assuming a case of free vibrations) leads to the following matrix equation:

$$\begin{bmatrix} \overline{S}_{LR} \\ \overline{S}_{RL} \end{bmatrix} = \begin{bmatrix} \overline{T})_{\Delta s}^{-1}\,\overline{T}_{\Delta\Delta} & -\overline{T})_{\Delta s}^{-1} \\ \overline{T})_{\Delta s}^{-1} & -\overline{T})_{\Delta s}^{-1}\,\overline{T}_{\Delta\Delta} \end{bmatrix} \begin{bmatrix} \overline{\Delta}_L \\ \overline{\Delta}_R \end{bmatrix}, \tag{7}$$

which is easily seen to be the required stiffness matrix equation. This matrix may be written in a short form as:

$$\overline{\mathbf{S}} = \overline{\mathbf{K}}\,\overline{\boldsymbol{\Delta}}\,, \tag{8}$$

and the comparison between Equations (7) and (8) immediately leads to the physical make-up of the respective submatrices of the stiffness matrix $\overline{\mathbf{K}}$ for a true helicoidal finite element bar.

EQUATIONS OF FREE VIBRATIONS AND SOLUTIONS

A differential segment of a circular helicoidal bar is a typical segment of a space bar for which the analytical description of geometry is possible as shown in Figure 1. With reference to a differential segment, the following geometric and statical parameters are defined:

$$l = R\sec\bar{\alpha} \quad ; \quad \alpha = \cos\bar{\alpha} \quad ; \quad \beta = \sin\bar{\alpha}$$

$$r_1 = I_z / l^2 A_x \quad ; \quad r_2 = \frac{E}{G}\frac{I_z}{I_x} \quad ; \quad r_3 = I_z / I_y \,. \tag{9}$$

Moreover, the following scaling parameters are written in terms of helix length aL and height ah as

$$\xi_1 = \frac{EI_z}{hL} \quad ; \quad \xi_2 = \frac{EI_z}{h^2 L} \,. \tag{10}$$

These scaling parameters are used to express forces, moments, deformations, and mass, respectively, as follows

$$\bar{F} = \xi_1 F \quad ; \quad \bar{M} = \xi_2 M$$

$$\bar{\Delta}_1 = \xi_1 \xi_2 \Delta_1 \quad ; \quad \bar{\Delta}_2 = \xi_1^2 \Delta_2$$

$$\bar{\mu} = \mu / \xi_2 \quad ; \quad \bar{\rho} = \xi_2 / \xi_1^2 \rho \,, \tag{11}$$

where: μ is mass per unit length and ρ is mass polar moment of inertia; Δ_1 and Δ_2 are generic names for either one of (u ; v ; w) displacements and either one of (φ; θ; ψ) slopes, respectively. Then based on the superposition of modal shapes, the normal modes of vibrations for displacements and slopes are expressed as:

$$\bar{\Delta}(a,t) = \sum_{m=1}^{\infty} \bar{\Delta}_m \exp(i\omega_m t) \,. \tag{12}$$

Equation (12) is incorporated in the conditions of equilibrium and deformations of a differential segment and the results are a set of twelve first order differential equations [9] which may be written in a matrix form as:

$$\frac{d}{da}\begin{bmatrix} \bar{S} \\ \cdots \\ \bar{D} \end{bmatrix} = \begin{bmatrix} \bar{M}_1 & \vdots & \bar{M}_{3,m} \\ \cdots & & \cdots \\ \bar{M}_2 & \vdots & -\bar{M}_1^T \end{bmatrix} \begin{bmatrix} \bar{S} \\ \cdots \\ \bar{D} \end{bmatrix} \,. \tag{13}$$

In Equation (13), submatrix $\bar{M}_1$ (6×6) is essentially a geometric matrix based on the equilibrium of a differential element, $\bar{M}_2$ (6×6) is used to express the force-deformation relations for the bar, and the submatrix $\bar{M}_{3,m}$ is a scaled mass matrix which describes the mass content of the bar and is written as:

$$\bar{M}_{3,m} = \frac{-\mu}{EI_z}(R\sec\bar{\alpha})^4 \omega_m^2 \begin{bmatrix} I & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} , \tag{14}$$

where: I is a (3×3) unit matrix and 0 is a (3×3) zero matrix; and ω_m designates a particular modal frequency.

As the full forms of submatrices $\bar{M}_1$ and $\bar{M}_2$ have been reported elsewhere [6], a close scrutiny of Equation (13) indicates that they are highly coupled. An attempt is made here to decouple these twelve equations [12]. The decoupling process is quite lengthy and laborious but the use of scaling parameters of Equations (11) has made the intermediate and final forms relatively easy to handle and to report. The resulting set of four differential equations represents the first stage of the decoupling process (in terms of scaled displacements $\bar{u}, \bar{v}, \bar{w}$, and scaled axial rotation $\bar{\varphi}$) as:

$$r_1 r_3 \bar{\mu} \ddot{\bar{u}} = r_3(1+r_1\alpha^2)\bar{u}^{(2)} - r_1\alpha^2\beta^2\bar{u} + r_1 r_3 \alpha \bar{v}^{(3)}$$

$$- \left[r_3 + r_1(2+r_3)\beta^2\right]\alpha\bar{v}^{(1)} - r_1(1+2r_3)\alpha\beta\bar{w}^{(2)}$$

$$+ r_1\alpha\beta^3\bar{w} + r_1\alpha^2\beta\bar{\varphi} \qquad (15\text{-}1)$$

$$r_1 r_2 r_3 \bar{\mu} \ddot{\bar{v}} = - r_1 r_2 r_3 \bar{v}^{(4)} + 2 r_1 r_2 (2+r_3)\beta^2 \bar{v}^{(2)}$$

$$- r_3\left[r_2\alpha^2 + r_1(\alpha^2 + r_2\beta^2)\beta^2\right]\bar{v}$$

$$- r_1 r_2 r_3 \alpha \bar{u}^{(3)} + r_2\left[r_3 + r_1(2+r_3)\beta^2\right]\alpha\bar{u}^{(1)}$$

$$+ 2 r_1 r_2 (1+r_3)\beta\bar{w}^{(3)} - r_1\left[2 r_2(1+r_3)\beta^2 + r_3\alpha^2\right]\beta\bar{w}^{(1)}$$

$$- r_1(2r_2 + r_3)\alpha\beta\bar{\varphi}^{(1)} \qquad (15\text{-}2)$$

$$r_2 r_3 \bar{\mu} \ddot{\bar{w}} = - r_2 \bar{w}^{(4)} + \left[r_3\alpha^2 + 2r_2(1+2r_3)\beta^2\right]\bar{w}^{(2)} - r_2\beta^4\bar{w}$$

$$- r_2(1+2r_3)\alpha\beta\bar{u}^{(2)} + r_2\alpha\beta^3\bar{u} - 2r_2(1+r_3)\beta\bar{v}^{(3)}$$

$$+ \left[r_3\alpha^2 + 2r_2(1+r_3)\beta^2\right]\beta\bar{v}^{(1)} + (r_2 + r_3)\alpha\bar{\varphi}^{(2)}$$

$$- r_2\alpha\beta^2\bar{\varphi} \qquad (15\text{-}3)$$

$$r_2 r_3 \bar{\rho} \ddot{\bar{\varphi}} = r_3\bar{\varphi}^{(2)} - r_2\alpha^2\bar{\varphi} + r_2\alpha^2\beta\bar{u} + (2r_2 + r_3)\alpha\beta\bar{v}^{(1)}$$

$$+ (r_2 + r_3)\alpha\bar{w}^{(2)} - r_2\alpha\beta^2\bar{w} \qquad (15\text{-}4)$$

in which the parameters r_1 to r_3, α, and β have all been defined in Equation (9).

It is here noted that Equations (15) will reduce to the equations of free vibrations for the case of a circular bar or straight bar when $\beta = 0$.

Further attempts to decouple Equations (15) have been undertaken in a previous research [13]. The result is a twelfth order differential equation in $\bar{u}$ and its even derivatives with respect to opening angle a with the coefficients uniquely defined for each modal shape based on the assumptions implied in Equation (12). The use of this equation to determine the corresponding eigenvalues and eigenvectors is, however, limited by practical and computational requirements. Moreover, analysis of Equations (15) indicates that *practical* solutions to extract natural frequencies are quite impossible and one has to resort to a numerical scheme. For this purpose, two alternative solutions are presented in the sequel as follows. The first method of solution starts by analyzing Equation (13) for the eigenvalues and eigenvectors of its coefficient matrix when the submatrix $\overline{M}_{3,m}$ is identically zero. This analysis then leads to developing a perturbation-method [10] based solution. Then the dynamic stiffness procedure is presented as a second more viable method of solution using the dynamic counterpart of Equations (4) and (5) [9] and the developments outlined in Equations (6–8).

Perturbation Method Solution

The solution of Equation (13) can be presented as a perturbation of the solution to the same equation with submatrix $\overline{M}_{3,m}$ being *identically* zero. For this purpose, the equation is rewritten in the following two forms:

$$\left(\frac{d}{da} - \overline{\mathbf{M}}\right) \overline{\mathbf{H}}(a) = \omega_m^2 \, \overline{\overline{\mathbf{P}}} \, \overline{\mathbf{H}}(a) \, , \tag{16-1}$$

$$\frac{d\overline{\mathbf{H}}}{da} = \overline{\mathbf{M}} \, \overline{\mathbf{H}} + \overline{\mathbf{P}} \, \overline{\mathbf{H}} \, , \tag{16-2}$$

in which the matrix $\overline{\overline{\mathbf{P}}}$ is a modified scaled mass matrix $\overline{\mathbf{P}}$ such that:

$$\overline{\mathbf{P}} = \omega_m^2 \, \overline{\overline{\mathbf{P}}} = \begin{bmatrix} 0 & \overline{\mathbf{M}}_{3,m} \\ 0 & 0 \end{bmatrix} , \tag{17}$$

is a (12×12) matrix. It is here noted that with a suitable discretization of domain of the bar (using, for example, the method of finite differences [10] or the method of finite elements [11]), Equation (16-1) leads to an algebraic eigenvalue problem which can be solved numerically for ω_m^2.

On the other hand, using the method of variation of parameters the solution based on Equation (16-2) is written in the form of an integral Equation such that:

$$\overline{\mathbf{H}}(a) = \exp\,(\overline{\mathbf{M}}a) \, \overline{\mathbf{H}}_o + \int_o^a \exp\,(\overline{\mathbf{M}}(a-s)) \, \overline{\mathbf{P}} \, \overline{\mathbf{H}}(s) \, ds \, , \tag{18}$$

in which: $\overline{\mathbf{H}}_o = \overline{\mathbf{H}}(a=0)$.

It is noted that Equation (18) is a *Volterra integral equation* and is written in a condensed form using an integral operator $T[\overline{\mathbf{H}}]$. The equation then becomes:

$$\overline{\mathbf{H}}(a) = \exp\,(\overline{\mathbf{M}}a) \, \overline{\mathbf{H}}_o + T[\overline{\mathbf{H}}]\,(a) \, , \tag{19-1}$$

in which:

$$T[\overline{\mathbf{H}}] = \int_o^a \exp(\overline{\mathbf{M}}(a-s)) \overline{\mathbf{P}} \, \overline{\mathbf{H}}(s) \, ds \, . \tag{19-2}$$

This integral operator has the following *iterative* property

$$T^n[\overline{\mathbf{N}}]\,(a) = T\left[T^{n-1}[\overline{\mathbf{N}}]\right](a) \tag{20}$$

for all $n \geq 1$. This property is then used to obtain the solution of Equation (16-2); and the solution is written as:

$$\overline{\mathbf{H}}(a) = \exp(\overline{\mathbf{M}}a) \, \overline{\mathbf{H}}_o + \sum_{n=1}^{\infty} T^n\left[\exp(\overline{\mathbf{M}}a) \, \overline{\mathbf{H}}_o\right] . \tag{21}$$

Further, in order to complete the analysis it is necessary to study the eigen-structure of matrix $\overline{\mathbf{M}}$. This matrix is found to have only *three distinct eigenvalues*; namely: 0 ; $+\sqrt{-1}$; $-\sqrt{-1}$, but *each one has a multiplicity of four*. The corresponding eigenvectors are obtained as follows:

$$(\overline{\mathbf{M}} - \lambda_i^* \, \mathbf{I}) \, \boldsymbol{\varphi}_{ij} = \boldsymbol{\varphi}_{ij-1} \tag{22}$$

where: $\boldsymbol{\varphi}_{io} = \overline{\boldsymbol{\varphi}}_i$; $1 \leq i \leq 4$; $1 \leq j \leq r_i - 1$ with $r_1 = r_2 = 2$ correspond to the zero eigenvalues, while $r_3 = 4$ and $r_4 = 4$ correspond to the complex eigenvalues $+\sqrt{-1}$ and $-\sqrt{-1}$, respectively; and $\overline{\boldsymbol{\varphi}}_i$ are the independent eigenvectors which are explicitly known in terms of the elements of matrix $\overline{\mathbf{M}}$.

It is then found that the solution of Equation (16) with $\overline{\mathbf{P}}$ being identically zero matrix can be written as:

$$\overline{\mathbf{H}}^*(a) = c_{10}\overline{\boldsymbol{\varphi}}_1 + c_{20}\overline{\boldsymbol{\varphi}}_2 + c_{30}\boldsymbol{\varphi}_3 e^{ia} + c_{40}\boldsymbol{\varphi}_4 e^{-ia}$$

$$+ (c_{11}\boldsymbol{\varphi}_{11} + c_{21}\boldsymbol{\varphi}_{21} + c_{31}\boldsymbol{\varphi}_{31} e^{ia} + c_{41}\boldsymbol{\varphi}_{41} e^{-ia})\, a$$

$$+ (c_{32}\boldsymbol{\varphi}_{32} e^{ia} + c_{42}\boldsymbol{\varphi}_{42} e^{-ia})\, a^2 + (c_{33}\boldsymbol{\varphi}_{33} e^{ia} + c_{43}\boldsymbol{\varphi}_{43} e^{-ia})\, a^3 \;, \tag{23}$$

or as:

$$\overline{\mathbf{H}}^*(a) = \sum_{i=1}^{4} \sum_{j=0}^{r_i - 1} c_{ij}\, \boldsymbol{\varphi}_{ij}\, a^j \exp(\lambda_i^* a) \;, \tag{24}$$

in which the twelve constants c_{ij} are to be determined depending on the boundary conditions and the superscript * corresponds to the static case (*i.e.* unperturbed case) and is used to distinguish this solution from the general solution given in Equation (19). This solution procedure — despite its mathematical elegance — is numerically quite involved for engineering applications when submatrix $\overline{\mathbf{M}}_{3,m}$ is not identically zero. For this case, the mathematical structure of eigenvectors $\boldsymbol{\varphi}_i$ is quite complicated and involved and more work is still being developed by the authors.

Dynamic Stiffness Matrix Solution

The dynamic stiffness matrix solution requires the construction of dynamic stiffness matrix $\overline{\mathbf{K}}$ [14]. This matrix is symbolically shown in Equation (7) and is noted to include the dynamic transport submatrices $\overline{T}_{\Delta s}$ and $\overline{T}_{\Delta\Delta}$ which are also constructed numerically [9].

Once the dynamic stiffness matrix $\overline{\mathbf{K}}$ is constructed for a typical space bar (*i.e.*: a true helicoidal bar), Equation (7) can be written in a partitioned form as:

$$\begin{bmatrix} \overline{S}_{LR} \\ \cdots \\ \overline{S}_{RL} \end{bmatrix} = \begin{bmatrix} \overline{k}_{LL} & \vdots & \overline{k}_{LR} \\ \cdots & & \cdots \\ \overline{k}_{RL} & \vdots & \overline{k}_{RR} \end{bmatrix} \begin{bmatrix} \overline{\Delta}_L \\ \cdots \\ \overline{\Delta}_R \end{bmatrix} , \tag{25}$$

in which each scaled submatrix is of order (3×3). This matrix equation is specific for a given geometry but its correspondence to a particular boundary condition is yet to be specified. A typical example, for this purpose, is the case of a cantilever bar with $\overline{\Delta}_R = 0$, and the resulting reduced matrix equation:

$$\overline{\mathbf{k}}_{LL}\overline{\boldsymbol{\Delta}}_L = \mathbf{0} \;, \tag{26}$$

can be solved to extract the corresponding natural frequencies ω_m using a modified bisection algorithm [14, 15].

NUMERICAL EXAMPLES AND DISCUSSIONS

As the numerical procedure outlined in Equations (4–8) results in the construction of a dynamic stiffness matrix using a dynamic transport matrix, it can be used to perform free vibration analysis of a three-dimensional space element. And since a true circular helicoidal bar is a typical structural space element where its geometry is defined completely by geometric parameters R, $\overline{a}$, $\overline{\alpha}$, and λ, the procedure can be *easily adapted* to the analysis with relative ease and less computational efforts as compared to a *typical* finite element method (FEM) code. This numerical procedure is applied to the analysis of free vibrations of typical circular helicoidal bars for selected boundary conditions and the results obtained are compared to those obtained using the finite element procedure [11, 16] capabilities of the FEM code GT STRUDL [17].

Two cases of a helicoidal beam are analyzed for the natural frequencies and the results are reported in Tables 1 and 2. These sample results indicate the practical efficiency of the numerical procedure developed and outlined herein as compared to the FEM with regard to the required number of finite elements, CPU time, and accuracy obtained for a particular discretization of the structure.

Table 1. Natural Frequencies of a Circular Helicoidal Beam*.

Case: Fixed-Free Boundaries

$\bar{a} = 360°$; $\bar{\alpha} = 10°$; $\bar{R} = 4$ m; $h = 0.5$ m; $\lambda = 1$

Dynamic Stiffness Method (*true helical elements*)				Finite Element Method [17] (*IPQS elements*)				
No. of Elements	n	ω_n(rad/sec)	CPU Time (sec)	No. of Elements	n	ω_n(rad/sec)	CPU Time (sec)	%-Diff.
	1	14.062			1	40.153		185.5
5	2	36.221	1.5	36	2	49.655	7.05	37.1
	3	91.607			3	83.119		–9.3
	1	13.866			1	28.181		103.2
15	2	30.382	3.0	72	2	34.287	15.04	12.9
	3	79.121			3	68.250		–13.7
	1	13.866			1	14.259		2.83
20	2	30.382	4.5	144	2	31.956	40.04	3.52
	3	79.121			3	73.056		–7.67

*Including shear deformations but neglecting rotary inertia; $E = 210$ GPa; $\nu = 0.3$; $\rho = 7992$ kg/m^3.

Table 2. Natural Frequencies of a Circular Helicoidal Beam*.

Case: Fixed-Free Boundaries

$\bar{a} = 360°$; $\bar{\alpha} = 10°$; $\bar{R} = 4$ m; $h = 1.0$ m; $\lambda = 0.25$

Dynamic Stiffness Method (*true helical elements*)				Finite Element Method [17] (*IPQS elements*)				
No. of Elements	n	ω_n(rad/sec)	CPU Time (sec)	No. of Elements	n	ω_n(rad/sec)	CPU Time (sec)	%-Diff.
	1	5.822			1	10.848		86.3
5	2	11.362	1.5	36	2	14.732	9.00	29.7
	3	34.700			3	40.037		15.4
	1	4.364			1	7.580		73.7
15	2	7.590	3.0	72	2	10.067	17.01	32.6
	3	17.376			3	26.264		51.2
	1	4.364			1	4.862		11.4
20	2	7.590	4.5	144	2	8.194	44.03	7.9
	3	17.376			3	17.927		3.2

*Including shear deformations but neglecting rotary inertia; $E = 210$ GPa; $\nu = 0.3$; $\rho = 7992$ kg/m^3.

Comparison of the results indicates that the dynamic stiffness matrix method (developed herein using the dynamic transport matrix) is quite efficient for two principal reasons. First, the number of structural elements required by the method is much less than that required by the FEM, and this means less computational costs and errors. The other key consideration which represents a prime advantage of this numerical procedure is the simple form of generating the structural mesh using only the geometric parameters R, $\bar{a}$, $\bar{\alpha}$, and λ. These two traits qualify the numerical code developed (based on the procedure) a potential candidate subroutine that can be adapted to automatically generate the dynamic stiffness matrix of a true helicoidal bar. This code can be easily interfaced with a general package of the FEM and thus allowing for the use of helicoidal beams (*e.g.*: staircases) *without* the need to *approximate* the true helicoidal bar by straight segments of a space beam in the form shown in Figure 1-*b*.

The numerical results obtained for the two cantilever bars were compared to the results obtained using a space bar idealization of the geometry. It was found that for the case with $\lambda = 0.25$, the values of the first three natural frequencies were, respectively, 4.364, 7.590, 17.376 rad/sec as compared to the values reported in Table 2. This favorable comparison with the results of this numerical procedure is, unfortunately, more of an exception than a rule. And in a more general case of cross section proportions (with higher values of λ), the type of a *general* FE should be selected very carefully and the analysis must be repeated enough number of times to ascertain that an *acceptable level of convergence* has been obtained. The dynamic stiffness matrix procedure (leading to a true helicoidal FE) greatly simplifies the analysis at minimal computational costs. The efficiency of the proposed procedure is *manifest* in the required number of finite elements, the required CPU time and the convergence rate in addition to the amount of pre-processing required to solve a problem. The comparison is made here with reference to a typical *compatible* finite element (IPQS [17, 18]) having a *quartic displacement field* within and along the edges of the element.

CONCLUDING REMARKS

Development of the dynamic stiffness matrix procedure is presented, based on the dynamic transport matrix $\overline{\mathbf{T}}_{RL}^{RL}$ method, as a viable practical numerical procedure to perform free vibration analysis of a general space bar. The procedure is general in nature once the $\overline{\mathbf{T}}_{RL}^{RL}$ is available in analytical or numerical form. It was shown, however, that *analytical* solutions for cases of space bars are beyond all practical means. This has been confirmed, herein, by considering the special case of a simple space geometry of a helicoidal beam. For this case, the resulting equations [Equations (15.1–15.4) and (19–24)] are highly coupled and *any further decoupling* does not lead to a practical method of solution for engineering applications.

The numerical procedure has been implemented in a computer code to generate the dynamic stiffness matrix $\overline{\mathbf{K}}$ for a *true helicoidal finite element* geometry. The code can be used to generate the stiffness matrix of a *helicoidal* finite element and the resulting structural contributions to the global stiffness matrix of a specified structure are easily identified through the assembly process of the finite element method. The results obtained for two typical helicoidal beam cases lead us to believe that the *effectiveness* of the procedure developed herein is quite *high* and will be further evaluated in a future work.

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