SINGLE FACILITY LOCATION WITH RECTILINEAR DISTANCE AND ROTATED AXES

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الخلاصـــة :

يناقش هذا البحث مشكلة إحلال مركز خدمة جديد بين المراكز الموجودة عندما تكون المسافات مستـقيمة ومحور النظام قابل للدوران . وبتـتبع النـتيجة التي حصل عليها وسولوسكي (Wesolowsky) أمكن إظهار أن عدداً محدوداً من زوايا الدوران مطلوب استخدامه لايجاد الموقع الأفضل لمركز الخدمة الجديد . ولقد قدمنا مثال لتوضيح الاجراءات اللازمة لاستغلال هذه الخاصية .

ABSTRACT

This paper examines the problem of locating a new facility amongst existing facilities when distances are rectilinear and the axes of the system may be rotated. Following a result by Wesolowsky, it is shown that only a finite number of angles of rotation needs to be considered to find the optimal location of the new facility. A worked example is solved to illustrate a procedure for exploiting this property.

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INTRODUCTION

Consider a set of m distinct points p_i, $i = 1,\ldots, m$ in the plane that represent the locations of existing facilities. It is required to determine the point q for the location of a new facility in order to minimize the total transportation cost of the system when the distances are assumed to be rectilinear. Let $q = (x, y)$, $p_i = (a_i, b_i)$, and w_i a positive real number denoting the weight associated to the existing facility i. The weight w_i represents the transportation cost between the new facility and the existing ith facility. Then the problem, to be denoted by P1, may be stated as the minimization of $f(x, y)$, such that

$$f(x,y) = \sum_{i=1}^{m} w_i(|x-a_i|+|y-b_i|), \qquad (1)$$

where the use of the rectilinear (or Manhattan) metric assumes that movement is always parallel to one or other of the coordinate axes.

Problems expressed as P1 arise in a variety of situations. Typical examples are in the layout of machines on a factory floor or in an urban situations when the movement of materials follows aisles of street laid out in a rectangular grid pattern. Problem P1 has been extensively examined in the literature and several algorithms have been proposed. See for example: Francis and White [1] or Love, Morris, and Wesolowsky [2]. In this paper we will examine an extension of problem P1 which arises when the axes of the system may be rotated through some angle θ.

In the next section we shall discuss problem P1 in the context of optimally rotating the coordinate axes. This combined problem (denoted P2) is shown to possess the same structure as the problem of locating a median line for a set of weighted points on a plane which was studied by Wesolowsky [3].

ROTATING THE AXES

It is simple to demonstrate that there could be cost advantages in terms of the optimal value of f in problem P1 through optimal rotation of the coordinate axes. Consider a point $p = (x, y)$ with the original alignment of the axes. The rectilinear distance between p and the origin is $x + y$. If, however, the axes were rotated through an angle θ such that the location of p under the new axes X' and Y' is $(x', 0)$, then the new rectilinear distance between p and the origin is x'. This is less than the original distance.

We now define problem P2. The locations of existing points under the new axes X' and Y' are denoted $p_i' = (a_i', b_i')$, $i = 1,\ldots, m$; where

$$a_i' = (a_i \cos\theta + b_i \sin\theta), \qquad b_i' = (-a_i \sin\theta + b_i \cos\theta).$$

Problem P2 is to minimize $f'(x', y', \theta)$, where

$$f'(x',y',\theta) = \sum_{i=1}^{m} w_i(|x'-a_i\cos\theta - b_i\sin\theta| + |y'+a_i\sin\theta - b_i\cos\theta|). \qquad (2)$$

Applications of P2 exist in cases where the realignment of the axes can feasibly take place. This can be difficult to apply when aisle are already defined for a fixed physical layout. However, at the design stage of the shop floor, the implementation of the solution to P2 would provide the optimum layout.

The earliest mention of problem P2 is believed to be by Hurriot and Perreur [4] who generalized the earlier problem P1 to include the rotation of the axes amongst other alternatives. Hurriot and Perreur show that the set of optimal locations is made up of arcs of circles. Since at least one optimum location with a given θ is on the abscissa x_i of an existing point p_i and the ordinate y_j of an existing point p_j (possibly the same point), the optimal location forms the third point of a right triangle with hypotenuse p_ip_j. For exposition we assume $i \neq j$. As long as the rotation leaves the point (x_i, y_j) as optimum, this point will take the form of an arc whose diameter is $p_i p_j$, as illustrated in Figure 1. Once the optimal point takes on the coordinates of points other than p_i and p_j, then the path of the optimal locations will take the form of an arc defined by the new points. Areas of indifference, which occur when the optimal location is not a unique point, will alter in a similar fashion. Hurriot and Perreur do not propose any procedure for determining the optimal angle of rotation.

Benkherouf and Watson Gandy [5] showed that problem P2 can be reduced to a concave quadratic programming problem which can be used as a heuristic since global optimality is not guaranteed. Next we shall propose a numerical method for solving problem P2.

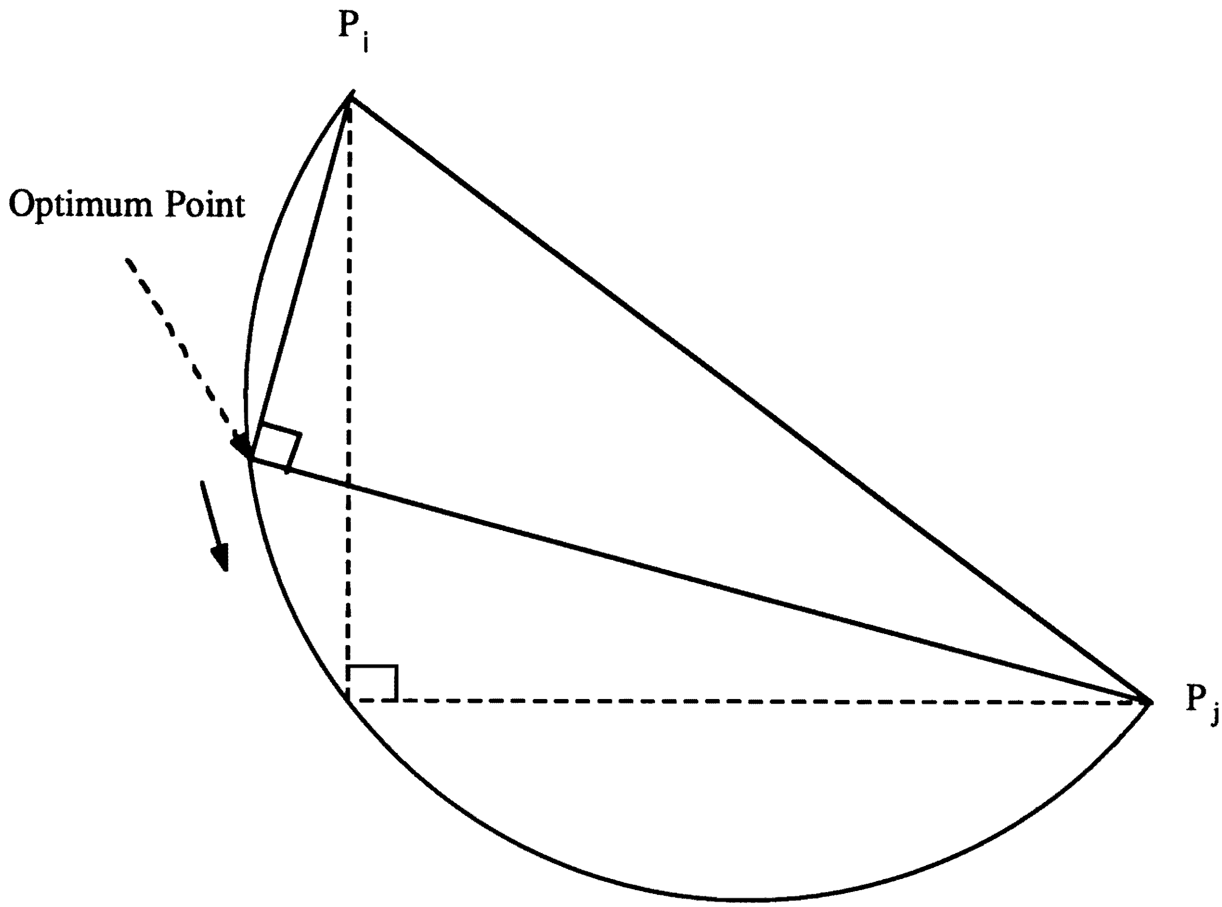

Figure 1. Locus of the Optimum Point as the Axes are Rotated.

Let $g(\theta)=f^*(x^*(\theta),y^*(\theta),\theta)$ denote the optical objective value in relation (2) for a given value of θ, where $x^*(\theta)$ and $y^*(\theta)$ are the optimal values of x and y, respectively, for that value of θ. It can be shown graphically that a piecewise concave pattern emerges from the curve. This suggests that the optimal location for a fixed value of θ is also optimal for a continuous range of θ. This range corresponds to the value of θ where the function $g(\theta)$ is concave. Moreover, this range corresponds to the arc mentioned by Hurriot and Perreur where the solution remains unchanged.

A related problem was investigated by Wesolowsky [3] and Morris and Norback [6], when they considered the problem of locating the median line for a set of weighted points on a plane. They employed a procedure involving the rotation of the axes. Wesolowsky suggested a computational procedure for finding the optimum median line which requires considering only a finite number of possible angle rotations. We shall show in the next section that the Wesolowsky approach can be generalized to cater for problem P2.

GENERAL RESULTS

In this section we shall follow Wesolowsky's approach to show that only a finite number of angles of rotation θ needs to be considered to solve problem P2. To do that we initially assume that θ is known and drop the primes (′) from the notation. It follows that problem P2 reduces to two independent problems, denoted problem P3:

$$\text{minimize } f_1(x,\theta) = \sum_{i=1}^{m} w_i(|x-a_i\cos\theta - b_i\sin\theta|), \tag{3}$$

and problem P4:

$$\text{minimize } f_2(y,\theta) = \sum_{i=1}^{m} w_i(|y+a_i\sin\theta - b_i\cos\theta|). \tag{4}$$

Expression (4) is equivalent to the problem investigated by Wesolowsky [3] and Morris and Norback [6].

It turns out that the optimal solutions of problems P3 and P4 are very much related as the following Lemma shows.

Lemma 1. If the solution to problem P3 is known for $\theta \in [0°, 180°]$, then the solution to problem P4 is easily deducible from the solution to problem P3.

Proof. If $\theta \in [0°, 90°]$ ($\theta \in [90°, 180°]$), then problem P3 is equivalent to problem P4 for $\theta + 90°$ ($\theta - 90°$). This can be easily seen by writing

$$-a_i \sin\theta + b_i \cos\theta = a_i \cos(\theta + 90°) + b_i \sin(\theta + 90°),$$

and

$$a_i \sin\theta - b_i \cos\theta = a_i \cos(\theta - 90°) + b_i \sin(\theta - 90°).$$

Suppose that $f_1(x^*(\theta+90°), \theta+90°)$ is the optimal solution to problem P3 for $\theta \in [0°, 90°]$, then it clear that $y^*(\theta) = x^*(\theta + 90°)$ and $f_2(y^*(\theta), \theta) = f_1(x^*(\theta+90°), \theta+90°)$ is the solution to P4 for $\theta + 90°$. Similarly, $f_2(y^*(\theta), \theta) = f_1(-x^*(\theta - 90°), \theta - 90°)$ and $y^*(\theta) = -x^*(\theta - 90°)$ is the solution to P4 for θ larger than 90°. □

As a consequence of Lemma 1, we need only solve problem P3 to solve problem P2. Next we shall present a computational procedure which is basically that suggested by Wesolowsky.

Let $A_{i\theta} = a_i \cos\theta + b_i \sin\theta$. For a given θ, arrange the $A_{i\theta}$'s in an increasing order. Hereafter, we will use superscript j to refer to the data (abscissa a_j, ordinate b_j, and weight w_j) related to the point placed at the jth position in the ordering of the $A_{i\theta}$'s. After arranging the $A_{i\theta}$'s in an increasing order, the objective function of problem P3 becomes

$$f_1(x, \theta) = \sum_{j=1}^{m} w^j |x - A_\theta^j|.$$

If $A_\theta^k \le x \le A_\theta^{k+1}$, then

$$f_1(x, \theta) = \sum_{j=1}^{k} w^j (x - A_\theta^j) - \sum_{j=k+1}^{m} w^j (x - A_\theta^j).$$

For a given θ,

$$\frac{\partial f_1(x, \theta)}{\partial x} = \sum_{j=1}^{k} w^j - \sum_{j=k+1}^{m} w^j = 2\sum_{j=1}^{k} w^j - w,$$

where $w = \sum_{j=1}^{m} w^j$.

Therefore, the solution $x^*(\theta)$ to P3 (see also Wesolowsky [3]) can be found as follows

$x^*(\theta) = A_\theta^{k^*}$, where k^* is the smallest k such that condition

$$(\text{C1}) : 2\sum_{j=1}^{k^*} w^j - w > 0 \text{ holds,}$$

or, $A_\theta^{k^*} \le x^*(\theta) \le A_\theta^{k^*+1}$, if condition

$$(\text{C2}) : 2\sum_{j=1}^{k^*} w^j - w = 0 \text{ is true.}$$

Next, we will show that the point placed at the k^*th position after the rearrangement will define the solution to problem P3 for a continuous range of θ values. This is in fact what Wesolowsky showed (see also Love, Morris, and Wesolowsky [2]) and we repeat the development here (with changes in notation for the purpose of clarity) only for completeness.

Lemma 2. The point that leads to the solution of problem P3 under conditions (C1) or (C2) is unchanged for a continuous range of θ values.

Proof. Note that condition (C1), if valid for a given θ, will continue to hold for a range of values of θ where the ordering of the $A_{i\theta}$'s is unchanged; that is,

for all $j < k^*$, $A_\theta^j \le A_\theta^{k^*}$, or equivalently

$$(b^j - b^{k^*}) \sin\theta \le (a^{k^*} - a^j) \cos\theta, \tag{5}$$

and for all $j > k^*$, $A_\theta^j \ge A_\theta^{k^*}$, or equivalently

$$(b^{k^*} - b^j) \sin\theta \le (a^j - a^{k^*}) \cos\theta. \tag{6}$$

So, as long as θ satisfies the $(k^* - 1)$ inequalities of relation (5) and the $(m - k^*)$ inequalities of relation (6), the point at the k^*th position will still satisfy (C1).

On the other hand condition (C2) will remain valid for a range of values of θ if the ordering of the $A_{i\theta}$'s is unchanged. Therefore, for all $j < k^*$, $A_\theta^j \le A_\theta^{k^*}$, or

$$(b^j - b^{k^*}) \sin\theta \le (a^{k^*} - a^j) \cos\theta, \tag{7}$$

and for all $j > k^* + 1$, $A_\theta^j \ge A_\theta^{k^*+1}$, or

$$(b^{k^*+1} - b^j) \sin\theta \le (a^j - a^{k^*+1}) \cos\theta, \tag{8}$$

and finally, $A_\theta^{k^*} \le A_\theta^{k^*+1}$, or

$$(b^{k^*} - b^{k^*+1}) \sin\theta \le (a^{k^*+1} - a^{k^*}) \cos\theta. \tag{9}$$

Hence, for θ satisfying relations (7), (8), and (9), condition (C2) will continue to hold for the point placed at the k^*th position. □

Note that by Lemma 2 we can determine all of the ranges of θ over which the solution to problem P3 remains unchanged. The ranges over which the solution to problem P4 is not altered can also be deduced from those of problem P3 using Lemma 1. Indeed, if $\theta \in [0°, 90°]$ ($\theta \in [90°, 180°]$) is an endpoint of a given range for problem P3, then $\theta + 90°$ ($\theta - 90°$) is an endpoint of a range for problem P4.

It can easily be shown that $f(x, \theta)$ is concave in θ within the range of values that satisfy relations (5) and (6) under Condition (C1) or relations (7), (8), and (9) under Condition (C2). Consequently, a minimum of $f(x, \theta)$ must occur at one of the endpoints of the ranges obtained by using Lemma 2.

Based on the above, it is straightforward to show that the following algorithm solves problem P2.

Algorithm

0. Set $t = 1$ and $\theta_{xt} = 0$.
1. Order the $A_{i\theta_{xt}}$'s in an increasing order, such that $A^1_{\theta_{xt}} \leq A^2_{\theta_{xt}} \leq \ldots \leq A^m_{\theta_{xt}}$.
2. Find the smallest superscript k^* such that (C1) or (C2) is satisfied.
3. If (C1) holds for k^*, then find the range $[\theta_{xt}, \theta_{xt+1}]$ of values of θ that satisfy relations (5) and (6).

   If (C2) holds for k^*, then find the range $[\theta_{xt}, \theta_{xt+1}]$ of values of θ that satisfy relations (7), (8), and (9).
4. Compute $x^*(\theta_{xt})$ and $f_1(x^*(\theta_{xt}), \theta_{xt})$.
5. If $\theta_{xt+1} = 180°$, then go to step 6, and otherwise increment t and go to step 1.
6. Let $\theta_{x1}, \ldots, \theta_{xr}$ be the endpoints that are smaller or equal to 90°, and $\theta_{xr+1}, \ldots, \theta_{xt}$ be the endpoints that are larger then 90°.

   The endpoints of the ranges relative to P4 are:

   $\theta_{ys} = \theta_{xt+1-s} - 90°$ for $s = 1, \ldots, t - r$,

   $\theta_{ys} = \theta_{xt+1-s} + 90°$ for $s = t - r + 1, \ldots, t$.
7. Compute $y^*(\theta_{ys})$ and $f_2(y^*(\theta_{ys}), \theta_{ys})$ for $s = 1, \ldots, t$.
8. Let $\theta_1, \theta_2, \ldots, \theta_q$ the endpoints obtained after arranging the θ_{xt}'s and the θ_{ys}'s in an increasing order and such that $\theta_q \leq 90°$.
9. The optimal objective value of P2 is:

   $f^* = \text{Min}\{f_1(x^*(\theta_i), \theta_i) + f_2(y^*(\theta_i), \theta_i),$

   $i \;\; = 1, \ldots, q\}$,

   the optical angle rotation is $\theta^* = \theta_{i^*}$,

   the optimal x coordinate is $x^* = x^*(\theta_{i^*})$,

   and the optimal y coordinate is $y^* = y^*(\theta_{i^*})$,

where

$$i^* = \{i : f^* = f_1(x^*(\theta_{i^*}), \theta_{i^*}) + f_2(y^*(\theta_{i^*}), \theta_{i^*})\}.$$

In steps 0 to 5, the algorithm finds the ranges of the angles of rotation for problem P3. In Step 4, the optimal x-coordinate as well as the optimal objective value for problem P3 are computed for each endpint of the different ranges of the angles of rotation. Then, in Step 6 the implied angles of rotation for problem P4 are obtained using Lemma 1. In Step 6, the optimal y-coordinate and objective value for problem P4 are calculated for each endpoint. Next, in Step 8 the algorithm combines the two ranges obtained in the previous steps to find the ranges for problem P2. Note that we did not consider the θ's that are larger than 90° since the solution to problem P2 is symmetric about 90°. Indeed, $f(x(\theta + 90°), y(\theta + 90°), \theta + 90°) = f(y(\theta), -x(\theta), \theta)$. Finally, in the last step the algorithm finds the minimum objective value for problem P2 over all possible θ that are smaller or equal to 90°. The optimal x and y coordinate are the ones corresponding to the θ that led to the optimal objective value.

WORKED EXAMPLE

The following is a worked example of the above algorithm.

Consider the problem of locating one new machine with respect to five existing facilities that are located as follows: $p_1 = (3, 4)$, $p_2 = (2, 8)$, $p_3 = (4, 5)$, $p_4 = (9, 3)$, $p_5 = (10, 2)$. The travel between facilities is assumed to be along a rectilinear aisle structure. The number of trips per day between the new machine and each existing facility is given as $w_1 = 6$, $w_2 = 3$, $w_3 = 2$, $w_4 = 4$, $w_5 = 5$. The main concern is to find the location of the new machine that would minimize the daily distance traveled between the new machine and the existing facilities.

The optimal location using the median method (see Francis and White [1] is $x^* = 4$, $y^* = 4$, with a total distance traveled of 90.

Table 1. Results of Problem P3.

t	1	2	3	4	5	6	7	8
θ	0	33.69	53.13	63.44	74.05	80.54	90	135
$x^*(\theta)$	4	6.1	7.6	6.3	4.7	4.4	4	0.7
$f_1(x^*(\theta),\theta)$	62	37.3	18.8	15.4	15.7	17.8	28	62.1

Starting with $\theta_{x1} = 0°$, we get the following points ordering: $(p_2, p_1, p_3, p_4, p_5)$. For this ordering, condition (C1) is satisfied for $k^* = 3$. Hence, $x^*(0°) = a_3 = 4$ and $f_1(x^*(0°), 0°) = 62$. Then, the use of relations (5) and (6) will yield the following four inequalities:

$j = 1$: $(8-5) \sin\theta \le (4-2) \cos\theta$, or $\tan\theta \le 2/3$,
$j = 2$: $(4-5) \sin\theta \le (4-3) \cos\theta$, or $\tan\theta \ge -1$,
$j = 4$: $(5-3) \sin\theta \le (9-4) \cos\theta$, or $\tan\theta \le 5/2$,
$j = 5$: $(5-2) \sin\theta \le (10-4) \cos\theta$, or $\tan\theta \le 2$.

The largest θ that still satisfy all of the above four inequalities is 33.69°. Therefore, the first range of θ values over which $x^*(0°) = 4$ remains unchanged is $[0°, 33.69°]$.

Next, for $\theta_{x2} = 33.69°$ the ordering of the points is changed to $(p_1, p_3, p_2, p_4, p_5)$. Again, condition (C1) is satisfied for $k^* = 3$ with $x^*(33.69°) = a_2 = 6.1$ and $f_1(x^*(33.69°), 33.69°) = 37.3$. The four inequalities obtained by using (5) and (6) are:

$j = 1$: $(4-8) \sin\theta \le (2-3) \cos\theta$, or $\tan\theta \ge 1/4$,
$j = 2$: $(5-8) \sin\theta \le (2-4) \cos\theta$, or $\tan\theta \ge 2/3$,
$j = 4$: $(8-3) \sin\theta \le (9-2) \cos\theta$, or $\tan\theta \le 7/5$,
$j = 5$: $(8-2) \sin\theta \le (10-2) \cos\theta$, or $\tan\theta \le 4/3$.

The largest θ for which all the above inequalities are satisfied is 53.13°. Hence, the second range is $[33.69°, 53.13°]$.

The preceding and remaining computations required to find all the ranges for problem P3 are presented in Table 1.

The ranges of the angles of rotation, the optimals y-coordinates and f-values for problem P4 are obtained by executing steps 6 and 7. Finally, the optimal solution to P2 is:

$\theta^* = 74.05°$
$x^* = 4.7$ (with respect to the new axes X' and Y')
$y^* = -2.5$ (with respect to the new axes X' and Y')
$f^* = 82.5$.

Finally, a simulation study was conducted to invesigate the relative effect of the rotation on the cost. The above algorithm was applied to a sample of 420 problems generated randomly. The number n of existing facilities was varied from 5 to 25. The x and the y coordinates were chosen from a uniform distribution [1, 100]. The weight w_i were generated from a uniform [1, 1000]. Further, for each fixed number of existing facilities we took 20 replications. Table 2 summarizes the computational results of the simulation. It shows the variation of the cost savings as a function of the number of facilities. It can be observed from the table that an average saving of 27.29% can be achieved by rotating the axes. This supports the fact that a substantial saving can be gained by applying our proposed algorithm.

Table 2. Variation of the Cost Saving as a Function of the Number of Facilities.

Number of Depots	Maximum Cost Saving	Average Cost Saving
5	34.69	21.91
6	35.84	22.61
7	34.33	22.72
8	32.55	23.31
9	33.11	26.57
10	32.69	25.12
11	33.05	25.49
12	34.98	26.67
13	33.50	28.32
14	34.16	27.65
15	35.66	27.56
16	33.32	29.53
17	35.37	28.99
18	35.00	27.32
19	35.76	30.27
20	34.51	29.57
21	35.16	29.98
22	34.21	28.98
23	34.62	30.07
24	36.06	29.85
25	35.61	30.63
Overall	36.06	27.29

Note that it has been observed that the cost saving increases as the range of the generated data increases. This means that this cost is obviously data dependent. Therefore, no general conclusion about the cost performance of the rotated axes algorithm, can be drawn from this simulation study. In other words, it is impossible to put an upper bound on this performance measure.

CONCLUSION

The problem of finding a single new facility location assuming rectilinear distances and optimal rotation of the coordinate axes was investigated. An optimizing method was suggested based on Wesolowsky's method for the location of the median line for a set of weighted points on a plane.

It would be of interest to see how the procedure suggested can be extended to a multi-facility location problem.

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Paper Received 26 November 1991; Revised 30 June 1992.