ON A SPECIAL SCHEDULING PROBLEM OF POSETS

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الخلاصـــة :

تبحث هذه الورقة في مسألة عدد الإرتطامات . حيث تدرسها على النحو الآتي : إذا كان هناك مجموعة (س) فـأوْجِد امتداد خطي يحفظ الحـدّ الأدنى لعلاقات التغطية الموجودة في (س) . ونركِّز بالخصوص على المجموعاتِ المرتبةِ الشَّرِةِ ؛ وهي المجموعات التي يوجد لها حلٌّ أمثلٌ بواسطة (الخوارزمية الشرة) .

ABSTRACT

Given a finite ordered set P. Find a linear extension of P which preserves a minimum number of covering relations of P. This article mainly concentrates on this problem, called the bump number problem. In particular we present some new results concerning the "greedy" ordered sets; these are the ordered sets for which the "greedy algorithm" always produces an optimal solution.

ON A SPECIAL SCHEDULING PROBLEM OF POSETS

Perhaps one of the fundamental theorems in the theory of partially ordered sets, or posets, is the following [1]:

Every poset has a linear extension, that is, an extension of the ordering of the poset into a total ordering.

Since then, a wide variety of problems and results in the theory of linear extensions have emerged. One general question that has interested many authors is this:

Given a poset P and a property $\mathbb{S}$. Does there exist a linear extension of P that satisfies $\mathbb{S}$?

When P is finite, some instances of this problem appeared to be important. For instance, linear extensions of finite posets play a central role in some problems in the theory of scheduling, such as the m-machine problem and the jump number problem [2, 3]. Recently, a related problem has been introduced by Fishburn and Gehrlein [4]. Given a finite poset P, find a linear extension of P which preserves a minimum number of covering relations of P. This problem is appropriate for a very special case of scheduling problems. Suppose a set of tasks are to be performed, one at a time by a single machine. Precedence constraints imply that a task cannot be scheduled unless all of its predecessors have been scheduled already. From a desire for diversity, a penalty is incurred whenever a task is processed immediately after one of its predecessors. The problem is this:

Schedule the tasks to minimize the number of penalties.

In terms of ordered sets, this can be rendered as follows. Let P be a finite poset and let $L=\{x_1,\dots,x_n\}$ be a linear extension of P. An ordered pair (x_i, x_{i+1}) is a *bump* in L if $x_i<x_{i+1}$ in P. Let $b(P,L)$ or simply $b(L)$, the *number of bumps* in L. Put:

$$b(P) = \min\{b(P,L): L \text{ is a linear extension of } P\}.$$

The bump number problem is to construct a linear extension L of P such that $b(L) = b(P)$. We call this linear extension L *optimal.*

One can show easily that for an ordered set P of length one, either $b(P) = 1$ [If P is the linear sum of two disjoint antichains] or $b(P) = 0$. Also, it was not hard finding polynomial algorithms to solve the bump number problem for many classes of ordered sets, *e.g.* interval orders and partial semiorders [4], series-parallel and decomposable ordered sets [5], and ordered sets of width two [6]. Actually, for posets of width two the algorithm is based on the simple idea of having the first bump as late as possible in the linear extension. Lately, two papers appeared simultaneously which solve the bump number problem for arbitrary ordered sets. In the first, Habib, Môhring, and Steiner [7], design a clever algorithm that constructs an optimal linear extension which is greedy and which traverses the levels of the ordered set one after the other. If P is an arbitrary ordered set given by its diagram and the cardinality of P is n, then this algorithm computes $b(P)$ in less than $O(n^2)$ time. For a finite ordered set P the *levels* $L_1,\dots,L_m$ of P are defined inductively by: L_i as the set of minimal elements of $P\text{-}\bigcup_{k<i} L_k$.

In the second paper, Schaffer and Simons [8] have also obtained a polynomial algorithm for computing the bump number by using a technique from two-processor scheduling developed by Gabow [9].

One of the ways of constructing the linear extensions of P which comes to mind is to use the greedy algorithm, that is, to minimize the number of bumps in each step of the construction. Choose x_1 any minimal element in P. Suppose $x_1,\dots,x_i$ are already defined, choose x_{i+1} minimal in $P\text{-}\{x_1,\dots,x_i\}$ such that x_i is noncomparable with x_{i+1} in P, whenever possible. A linear extension constructed by such method is called *greedy*. We denote by $O(P)$ and by $G(P)$ the sets of all optimal and greedy linear extensions of P, respectively. In the study of many combinatorial optimization problems the greedy algorithm plays an important role. For instance most of the significant results for the jump number problem that is, the problem of maximizing bumps, are concerned with the greedy algorithm [10–12]. For instance, it is easy to check that all ordered sets illustrated in Figure 1 have the property that every greedy linear extension is optimal. This is not the case for the ordered sets illustrated in Figure 2. An ordered set P is called *greedy* if $O(P)\supseteq G(P)$. Call an ordered set *reversible* if for every L in $G(P)$, the dual of L is a greedy linear extension of the dual of P. [For an ordered set P, we denote P^d the dual of P, to be the ordered set with the same underlying set as P and with the ordering defined by $x\leq y$ in P if and only if

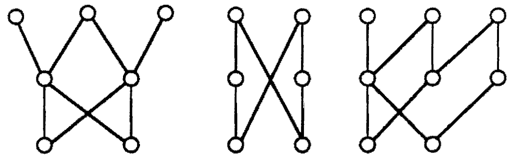


Figure 1.

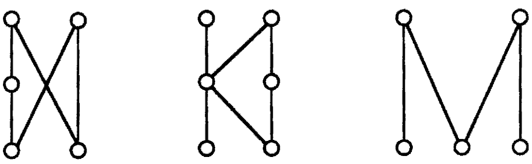


Figure 2.

$y \le x$ in P^d.] Say that x *covers* y in P, [or x is an *upper cover* of y] if $x > y$ and whenever $x > z \ge y$ then $y = z$.

A first step on the way to characterizing the greedy ordered sets is the reduction of the problem to those greedy ordered sets whose bump number equal zero. These are the ordered sets in which every greedy linear extension has no bumps at all.

Theorem 1 [13]. Let P be a greedy ordered set such that $b(P) = k$. Then P is a linear sum of $(k+1)$ greedy ordered sets, each of bump number zero.

The main idea used is to prove that P is greedy if and only if $O(P) = G(P)$.

Theorem 2. Let P be an ordered set. Then the following are equivalent.

(i) P is greedy.
(ii) $O(P) = G(P)$.
(iii) P is reversible.

Proof.

(i) $\Longleftrightarrow$ (ii), see Al-Thukair and Zaguia [13].

(ii) $\Longrightarrow$ (iii), suppose $O(P) = G(P)$ and let $L = \{x_1, \dots, x_n\}$ be in $G(P)$. If $L^d \notin G(P^d)$ then there is a bump (x_{i+1}, x_i) in L^d and $j < i$ such that x_j *is noncomparable to* x_k in P for every $j < k \le i+1$. But then

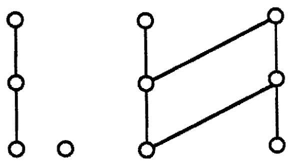
$$L_1 = \{x_1 < \dots < x_{j-1} < x_{j+1} < \dots < x_i < x_j < x_{i+1} < \dots\}$$

is an optimal linear extension of P. [We deleted the bump (x_i, x_{i+1}) and since L is optimal, surely we did create a new one (x_{j-1}, x_{j+1}) for otherwise L_1 would have less bumps than L.] Also it is obvious that L_1 is not greedy. This contradicts $O(P) = G(P)$. Therefore $L^d \in G(P^d)$ and P is reversible.

(iii) $\Longrightarrow$ (ii): by induction on the cardinality of P. Suppose P is reversible and let $L = \{x_1 < \dots < x_n\}$ be a greedy linear extension of P. Let (x_i, x_{i+1}) be the first bump in L. Thus $x_k > x_i$ in P, for every $k \ge i+1$. Since L^d is in $G(P^d)$, then $x_j < x_{i+1}$ for every $j \le i$. [Notice that (x_{i+1}, x_i) is a bump in L^d and L^d is greedy.] Consider u and v such that x_{i+1} covers u and v covers x_i in P. Let $L_1 = \{x_1 < \dots < x_i < v < \dots\}$ be any greedy linear extension of P which coincides with L on the first i elements. Applying the same argument for L_1 as we did for L shows that $x_j < v$ in P for every $j \le i$ and $y > x_i$ in P whenever $y > x_i$ in L_1. Thus $u < v$ in P. Thus $P = P_1 \oplus P_2$ [$\oplus$ denotes the linear sum of two ordered sets], where the maximal elements of P_1 are the lower covers of x_{i+1} in P and the minimal elements of P_2 are the upper covers of x_i in P. Therefore a linear extension of P is greedy if and only if it is the linear sum of a greedy linear extension of P_1 and a greedy linear extension of P_2. Thus P is reversible if and only if both P_1 and P_2 are reversible too. By the induction hypothesis $O(P_1) = G(P_1)$ and $O(P_2) = G(P_2)$. Thus $O(P) = G(P)$. Finally, notice that if no greedy linear extension of P has bumps then obviously $O(P) = G(P)$. This ends the proof of Theorem 2.

All greedy ordered sets illustrated in Figure 1 have the common property that every greedy linear extension must exhaust the level C_i before it begins on the consecutive level C_{i+1}. Moreover, it either always or never produces a bump in going from C_i to C_{i+1}. Contrary to the nongreedy ordered sets illustrated in Figure 2 for which always one, at least, of the two properties described above fails. Fishburn and Gehrlein [4] have conjectured that an ordered set is greedy if and only if the conjunction of these two properties holds. A proof of this conjecture has been given by Zaguia [14]. Actually the proof shows that we can decide in polynomial time whether or not an ordered set of bump number zero is greedy. Notice that the problem of deciding whether or not an ordered set is greedy for the jump number problem [maximizing-bumps problem] is NP-complete [16, 17].

In the last part of this paper we shall give some results concerning greedy linear extensions. There are ordered sets P such that for every L in $G(P)$ we have $L^d \notin G(P^d)$. [See Figure 3(a).] In fact, a natural question is the following:



(*a*) (*b*)

Characterize the ordered sets P for which there is L in $G(P)$ such that L^d is in $G(P^d)$.

Theorem 3. Let P be an ordered set. Then $G(P) \supseteq O(P)$ if and only if $G(P^d) \supseteq O(P^d)$.

Proof. Assume that $G(P) \supseteq O(P)$ and suppose that there exists $L = \{x_1 < \ldots < x_n\}$ a nongreedy optimal linear extension of P^d. Let (x_i, x_{i+1}) be the first bump in L which is not constructed in a greedy way. So, there is $j > i+1$ such that x_j is noncomparable with x_k in P for every $i \leq k < j$. Let

$$L_1 = \{x_1 < \ldots < x_i < x_j < x_{i+1} < \ldots < x_{j-1} < x_{j+1} < \ldots\}.$$

Since we deleted the bump (x_i, x_{i+1}) and L is optimal, then (x_{j-1}, x_{j+1}) is a bump in L_1. But now the dual of L_1 is an optimal linear extension of P, [notice that $b(P) = b(P^d)$] which is not greedy since the bump (x_{j+1}, x_{j-1}) is not constructed in a greedy way. This contradicts that $G(P) \supseteq O(P)$. The proof would be the same in the reverse direction.

An easy consequence of Theorem 3 is this. If $G(P) \supseteq O(P)$ then for every L in $O(P)$, we have L^d in $O(P^d)$ and thus L^d is in $G(P^d)$.

What are the ordered sets P for which $G(P) \supseteq O(P)$?

An ordered set is called a *partial semiorder* if it does not contain a subset isomorphic to the ordered set illustrated in Figure 3(*a*). Here is a partial answer to this question.

Theorem 4. If P is a partial semiorder then $G(P) \supseteq O(P)$.

Proof. Let P be a partial semiorder. Suppose that $G(P) \supseteq O(P)$ and let $L = \{x_1 < \ldots < x_n\}$ be in $O(P) - G(P)$. Let (x_i, x_{i+1}) be the first bump in L which is not constructed in a greedy way. Thus there is $j > i+1$ such that x_j is noncomparable to x_k in P for every $i \leq k < j$. Consider the smallest such index j. Now put x_j between x_i and x_{i+1} in L. Since L is optimal then we must create a new bump (x_{j-1}, x_{j+1}). But, according to the choice of j, $x_{j-1} > x_i$ in P. Thus the subset $\{x_i, x_{j-1}, x_{j+1}, x_j\}$ of P contradicts the fact that P is a partial semiorder.

Notice that the converse of Theorem 4 is false. For instance the ordered set illustrated in Figure 3(*b*) is not a partial semiorder and still $G(P) \supseteq O(P)$.

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