CONTACT PROBLEM FOR AN ELASTIC LAYER ON RIGID FLAT SUPPORTS

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الخلاصة :

يهتم هذا البحث بدراسة مسألة المرونة الساكنة المستوية لصفيحة مستقرة على حاملين صلبين لهما حواف حادة . تثبت الصفيحة على الحاملين بقوة ضاغطة موزعة بانتظام على جزء محدد من سطحها العلوي . ويفترض أن لا يكون هناك أي أحتكاك بين سطوح التماس للصفيحة والحاملين وأن الإجهادات الضاغطة العمودية فقط تستطيع أن تنقل عبر السطح البيني . قد يبقى التماس على طول السطح البيني متواصلاً وقد يبدأ الانفصال باتجاه الحافة الخارجية للحاملين معتمداً على الكمية والتوزيع النسبي للقوة الناتجة . شكلت المسألة على صيغة معادلة تكاملية مفردة لضغط التماس في حالة التماس المتواصل يوجد نقاط مفردة لضغط التماس على حواف الحاملين ، بينما يكون مقيدا (صفرا) في النقاط التي يبدأ فيها الأنفصال في حالة التماس غير المتواصل . بالنسبة للحالة الثانية يشكل قياس مسافة التماس مجهولاً إضافيا . النتائج العددية لضغط التماس والإجهاد المحوري والمسافة المحددة لمساحة الإنفصال معطاة في جداول ومنحنيات بيانية .

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ABSTRACT

This paper is concerned with the plane elastostatic problem of a layer resting on two rigid flat supports with sharp edges. The layer is pressed against the supports by a uniform clamping pressure applied over a finite portion of its top surface. It is assumed that the contact between the layer and the supports is frictionless and that only compressive normal tractions can be transmitted through the interface. The contact along the interface may remain continuous or separation may start towards the outer edge of the supports depending on the magnitude and the relative distribution of the resultant force. The problem is formulated in terms of a singular integral equation for the contact pressure. In case of continuous contact, the contact pressure has singularities at the edges of the supports whereas it is bounded (zero) at the points where separation starts in case of discontinuous contact. For this latter case, size of the contact area constitutes an additional unknown. Numerical results for contact pressure, axial stress, and distance determining the separation area are given in graphical and tabular forms.

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INTRODUCTION

The contact problem for an elastic layer has attracted considerable attention in the past. The layer usually rests on a continuous foundation, which may be either elastic (see, for example, [1–9]), or rigid (see, for example, [10–14]). In [15], the layer is supported by two elastic quarter planes. In most of the previous work the layer is pressed locally against the foundation and the effect of gravity is neglected. Consequently, in the absence of a clamping pressure, the contact area is finite and its size is independent of the magnitude of the load [16]. This property holds also for loading through a flat-ended rigid stamp with sharp edges. However, the size of the contact area is a function of the magnitude of the resultant compressive force for other stamp profiles. Some examples taking the effect of gravity into account may be found in [8–14].

In this paper, the plane elastostatic problem of an infinite layer resting on two rigid horizontal flat supports is considered. The supports have 90° sharp corners. It is assumed that the contact along the interface is frictionless and no tensile tractions can be transmitted across the interface (receding contact). The effect of gravity is neglected. The layer is subjected to a uniform clamping pressure over a finite portion of its top surface (see Figure 1). Clearly, the supports may be considered as flat-ended rigid stamps and the problem may be assumed to approximate a double contact problem where the contact pressure on the upper surface is approximated by a uniform distribution over a predetermined contact region. The size of the contact area between the layer and the supports depends on the size of the region over which the layer is subjected to pressure, which determines the magnitude and the distribution of the applied resultant force.

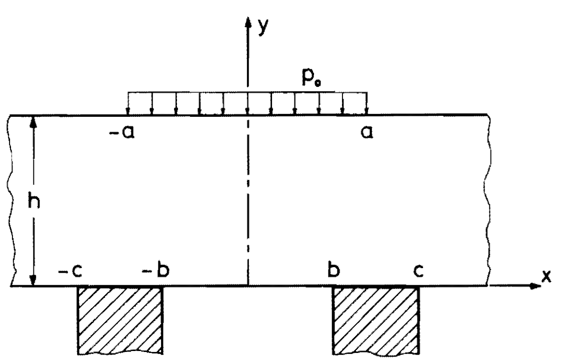

Figure 1. Elastic Layer Pressed Against Rigid Flat Supports

FORMULATION OF THE PROBLEM

Consider the isotropic, linearly elastic infinite layer of thickness h resting on two symmetrical rigid flat supports with 90° sharp corners shown in Figure 1. The contact between the layer and the supports is assumed to be frictionless. Only compressive tractions can be transmitted across the interface. The layer is subjected to a uniform clamping pressure of intensity p_0 over a portion of width $2a$ on its top surface.

In the absence of body forces, two-dimensional Navier equations may be written in the form

$$\frac{\kappa+1}{\kappa-1}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{2}{\kappa-1}\frac{\partial^2 v}{\partial x\partial y}=0,$$

$$\frac{2}{\kappa-1}\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}+\frac{\kappa+1}{\kappa-1}\frac{\partial^2 v}{\partial y^2}=0, \qquad \text{(1a, b)}$$

where u and v are the x and y-components of the displacement vector and $\kappa=3-4\nu$ for plane strain, $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν being the Poisson's ratio. The use of κ instead of ν reduces two separate formulations for plane strain and plane stress cases to a single formulation. The stress components may be expressed as

$$\sigma_x=\mu\left(\frac{\kappa+1}{\kappa-1}\frac{\partial u}{\partial x}+\frac{3-\kappa}{\kappa-1}\frac{\partial v}{\partial y}\right),$$

$$\sigma_y=\mu\left(\frac{3-\kappa}{\kappa-1}\frac{\partial u}{\partial x}+\frac{\kappa+1}{\kappa-1}\frac{\partial v}{\partial y}\right),$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \qquad \text{(2a–c)}$$

where μ is the shear modulus.

For the plane elastostatic problem under consideration Navier equations must be solved under the following boundary conditions:

$$\tau_{xy}(x,h)=0,\ (0\le|x|<\infty), \qquad (3)$$

$$\sigma_y(x,h)=\begin{cases}-p_0, & (0\le|x|<a),\\ 0, & (a<|x|<\infty),\end{cases} \qquad (4)$$

$$\tau_{xy}(x,0)=0,\ (0\le|x|<\infty), \qquad (5)$$

$$v(x,0)=0,\ (b<|x|<c), \tag{6}$$

$$\sigma_y(x,0)=0,\ (0\le|x|<b,\ c<|x|<\infty). \tag{7}$$

Observing that $x=0$ is a plane of symmetry it is sufficient to consider the problem in the region $0\le x<\infty$ only. In this case, in order to match the two halves at $x=0$ properly, the following conditions must also be satisfied:

$$u(0,y)=0,\quad \tau_{xy}(0,y)=0,\qquad (0\le y\le h). \tag{8a, b}$$

Taking the Fourier sine and cosine transforms [17] of (1a) and (1b), respectively, in the x direction, rearranging and solving the resulting ordinary differential equations and finally taking the inverse transforms one may obtain the solutions:

$$u(x,y)=\frac{2}{\pi}\int_0^\infty [(A+Bry)e^{-ry}+(C+Dry)e^{ry}]\sin(rx)dr,$$

$$v(x,y)=\frac{2}{\pi}\int_0^\infty [(A+Bry+\kappa B)e^{-ry}-(C+Dry-\kappa D)e^{ry}]\cos(rx)dr. \tag{9a, b}$$

These solutions satisfy the symmetry conditions (8) and when they are substituted in Equations (2), stress components may be obtained as

$$\sigma_x(x,y)=\frac{4\mu}{\pi}\int_0^\infty r[(A+Bry+\frac{\kappa-3}{2}B)e^{-ry}+(C+Dry-\frac{\kappa-3}{2}D)e^{ry}]\cos(rx)dr,$$

$$\sigma_y(x,y)=\frac{4\mu}{\pi}\int_0^\infty r[-(A+Bry+\frac{\kappa+1}{2}B)e^{-ry}-(C+Dry-\frac{\kappa+1}{2}D)e^{ry}]\cos(rx)dr,$$

$$\tau_{xy}(x,y)=\frac{4\mu}{\pi}\int_0^\infty r[-(A+Bry+\frac{\kappa+1}{2}B)e^{-ry}+(C+Dry-\frac{\kappa-1}{2}D)e^{ry}]\sin(rx)dr. \tag{10a–C}$$

Here A, B, C, D are yet unknown functions to be determined from the boundary conditions.

One may note that boundary conditions (6) and (7) are of mixed type. In order to have the same type of conditions we may replace Equation (6) by

$$\frac{\partial}{\partial x}v(x,0)=0,\qquad (b<x<c), \tag{11}$$

and (7) by

$$\sigma_y(x,0)=-p(x),\qquad (0\le x<\infty). \tag{12}$$

Here a new unknown function $p(x)$ is introduced for providing a more direct procedure in which the formulation is reduced to a singular integral equation instead of triple integral equations. Note that Equation (7) is satisfied if

$$p(x)=0,\qquad (0\le x<b,\ c<x<\infty). \tag{13}$$

Substituting Equation (10) in Equations (3–5) and (12) one may determine A, B, C, D within the unknown function $p(x)$ as follows:

$$\begin{aligned}
A&=[1-\kappa-2\kappa rh-(1-\kappa-2rh)e^{-2rh}]e^{-rh}\,p_0\sin(ar)/H\\
&\quad+\{\kappa-1+[(1-\kappa)(1-2rh)+4r^2h^2]e^{-2rh}\}rP(r)/H,\\
B&=2(1+2rh-e^{-2rh})e^{-rh}\,p_0\sin(ar)/H\\
&\quad-2[1-(1-2rh)e^{-2rh}]rP(r)/H,\\
C&=[\kappa-1-2rh+(1-\kappa+2\kappa rh)e^{-2rh}]e^{-rh}\,p_0\sin(ar)/H\\
&\quad+[(1-\kappa)(1+2rh-e^{-2rh})+4r^2h^2]e^{-2h}\,rP(r)/H,\\
D&=2[1-(1-2rh)e^{-2rh}]e^{-rh}\,p_0\sin(ar)/H\\
&\quad-2(1+2rh-e^{-2rh})e^{-2rh}\,rP(r)/H,
\end{aligned} \tag{14}$$

where

$$H=4\mu r^2[4r^2h^2e^{-2rh}-(1-e^{-2rh})^2], \tag{15}$$

$$P(r)=\int_b^c p(x)\cos(rx)dx, \tag{16}$$

so that Equation (13) is also satisfied. Hence the displacements and stresses are expressed in terms of the function $p(x)$.

THE INTEGRAL EQUATION

Now if Equations (14–16) are substituted in Equation (9b) and if the resulting expression is substituted in Equation (11), after some routine manipulations, one may find the following singular integral equation

$$\int_b^c\left[\frac{1}{t-x}-\frac{1}{t+x}+k(x,t)\right]p(t)dt=m(x),\qquad (b<x<c), \tag{17}$$

for $p(x)$ where

$$k(x,t)=\frac{2}{h}\int_0^\infty \frac{(1+2z+2z^2-e^{-2z})e^{-2z}}{1-2(1+2z^2)e^{-2z}+e^{-4z}}\left[\sin(t-x)\frac{z}{h}-\sin(t+x)\frac{z}{h}\right]dz,$$

$$m(x)=2p_0\int_0^\infty \frac{(1+z)e^{-z}-(1-z)e^{-3z}}{1-2(1+2z^2)e^{-2z}+e^{-4z}}\left[\cos(a+x)\frac{z}{h}-\cos(a-x)\frac{z}{h}\right]\frac{dz}{z}. \quad (18)$$

In Equation (17), the kernel $k(x,t)$ is bounded in the closed interval $b \leq (x,t) \leq c$, and the index of the integral equation is $+1$ [18]. Thus, the solution will contain an arbitrary constant which can be determined from the equilibrium condition

$$\int_b^c p(x)dx=ap_0. \quad (19)$$

Contact along the interface may be either continuous or discontinuous. If a/h is sufficiently small and c/h is sufficiently large separation starts on outer portion of the supports. These two cases must be analyzed separately.

(a) Continuous Contact

In order to simplify the numerical analysis introduce the following dimensionless quantities:

$$(x,t)=\frac{c-b}{2}(w,s)+\frac{c+b}{2},$$

$$M(w)=m\left(\frac{c-b}{2}w+\frac{c+b}{2}\right)\Big/p_0,$$

$$g(s)=p\left(\frac{c-b}{2}s+\frac{c+b}{2}\right)\Big/p_0. \quad (20)$$

Then, Equations (17) and (19) may be written as

$$\int_{-1}^{1}\left[\frac{1}{s-w}-\frac{1}{s+w}+K(w,s)\right]g(s)ds=M(w),$$

$$(-1<w<1), \quad (21)$$

$$\int_{-1}^{1} g(s)ds=\frac{2a}{c-b}, \quad (22)$$

where

$$K(w,s)=\frac{c-b}{2}k\left(\frac{c-b}{2}w+\frac{c+b}{2},\ \frac{c-b}{2}s+\frac{c+b}{2}\right). \quad (23)$$

Rigid flat supports have 90° sharp corners. Hence, the contact stress σ_y, and consequently the contact pressure p, will be singular at the corners and referring to references [6, 7, 14, 18] the solution of the integral equation will be in the form

$$g(s)=G(s)(1-s^2)^{-(1/2)}, \quad (24)$$

where $G(s)$ is bounded in $(-1\leq s\leq 1)$. Then, using the appropriate integration formula [19], Equations (21) and (22) are replaced by

$$\sum_{i=1}^{n} C_i\left[\frac{1}{s_i-w_j}-\frac{1}{s_i+w_j}+K(w_j,s_i)\right]G(s_i)=M(w_j),$$

$$(j=1,\ldots,n-1), \quad (25)$$

$$\sum_{i=1}^{n} C_iG(s_i)=\frac{2a}{c-b},$$

where

$$C_1=C_n=\frac{\pi}{2n-2},\quad C_i=\frac{\pi}{n-1},\quad (i=2,\ldots,n-1),$$

$$s_i=\cos\left(\frac{i-1}{n-1}\pi\right)\quad (i=1,\ldots,n),$$

$$w_j=\cos\left(\frac{2j-1}{2n-2}\pi\right),\quad (j=1,\ldots,n-1). \quad (26)$$

The system in Equations (25) contain n linear algebraic equations for n unknowns, $G(s_i)$, $(i=1,\ldots,n)$.

(b) Discontinuous Contact

If a/h is sufficiently small and c/h is sufficiently large, contact between the layer and the supports can be maintained on the inner portion of the supports along $b\leq|x|<d(d<c)$ only. Formulation of the problem up to Equation (23) is still valid for this case except for the fact that c must be replaced by d. At $|x|=d$ separation starts and the contact between the layer and the supports will be smooth near these points. Therefore, $g(1)$ vanishes and consequently the index of the integral Equation (21) will be zero [18]. Hence, the solution will be in the following form [15, 18]

$$g(s)=G(s)(1-s)^{1/2}(1+s)^{-1/2},\quad (-1<s<1), \quad (27)$$

where again $G(s)$ is bounded in $(-1\leq s\leq 1)$. In this case, the use of Gauss–Chebyshev integration formula [20] reduces Equations (21) and (22) to

$$\sum_{i=1}^{n}(1-s_i)\left[\frac{1}{s_i-w_j}-\frac{1}{s_i+w_j}+K(w_j,s_i)\right]G(s_i)$$

$$=\frac{2n+1}{2\pi}M(w_j),\quad (j=1,\ldots,n),$$

$$\sum_{i=1}^{n}(1-s_i)G(s_i)=\frac{(2n+1)a}{\pi(d-b)}, \tag{28}$$

where

$$s_i=\cos\left(\frac{2i\pi}{2n+1}\right), \qquad (i=1,\ldots,n),$$

$$w_j=\cos\left(\frac{2j-1}{2n+1}\pi\right), \qquad (j=1,\ldots,n). \tag{29}$$

Note that the system given by Equation (28) contains $n+1$ equations for $n+1$ unknowns, $G(s_i)$, $(i=1,\ldots,n)$, and d.

RESULTS

The Fourier integrals appearing in expressions for $k(x,t)$ and $m(x)$ can be evaluated numerically. Several quadrature formulas have been considered and it came out that Simpson's rule [21], which is considerably straightforward, yields sufficiently accurate results. However, the interval of integration $(0,\infty)$ has been divided into several subintervals. Around the lower limit $z=0$, the integrands show very rapid variations. Therefore, very closely spaced integration points are required (e.g., $\Delta z=0.0001$ which is not possible when more sophisticated integration formulas are employed). On the other hand, in order to get sufficiently close to the upper limit, distance between the integration points must be gradually increased in the consecutive subintervals.

Some of the calculated results are shown in Figures 2–8 and Table 1. Figures 2–4 show the normalized contact pressure $p(x)/p_0$ for the continuous contact case. The contact pressure becomes infinitely large at the corners of the rigid supports. The general trend for $p(x)$ which is an increase with increasing a/h ratio can be seen in Figure 2. One may note that a very small a/h (e.g., $a/h=0.01$) represents the case of a concentrated load $P=2ap_0$ at $x=0$. As a/h increases, this load is distributed over a larger portion of the layer and also the total load increases. For relatively small a/h values the location of minimum contact pressure is close to the outer edge and as a/h increases it moves toward the inner edge. This effect can be seen in Figures 3 and 4 easily.

Figure 5 shows the dimensionless stress $\sigma_x(0,y)/p_0$ along the symmetry plane $x=0$ for the continuous contact case. For relatively small a/h values σ_x is compressive in upper portion and tensile in lower portion whereas it is tensile in upper portion and

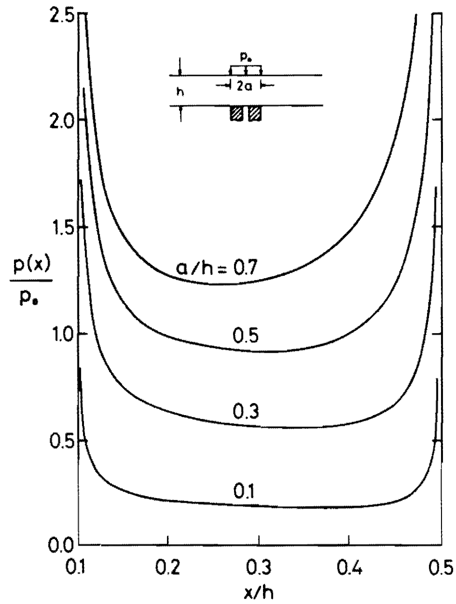

Figure 2. Contact Pressure Distribution for Continuous Contact Case ($b/h=0.1$, $c/h=0.5$)

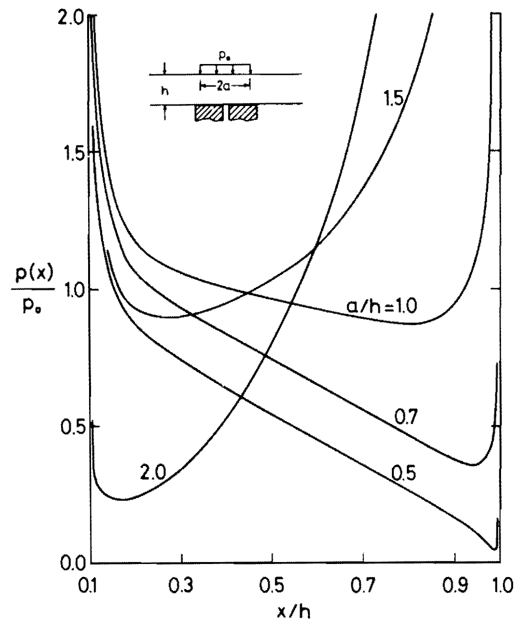

Figure 3. Contact Pressure Distribution for Continuous Contact Case ($b/h=0.1$, $c/h=1.0$)

compressive in lower portion for larger a/h values. If the layer is considered as a beam, σ_x represents the axial stress. For small a/h the layer behaves like a simply supported beam under a point load (positive bending moment and upward concaveness) whereas it behaves like an overhanging beam subjected to a distributed load (negative bending moment and

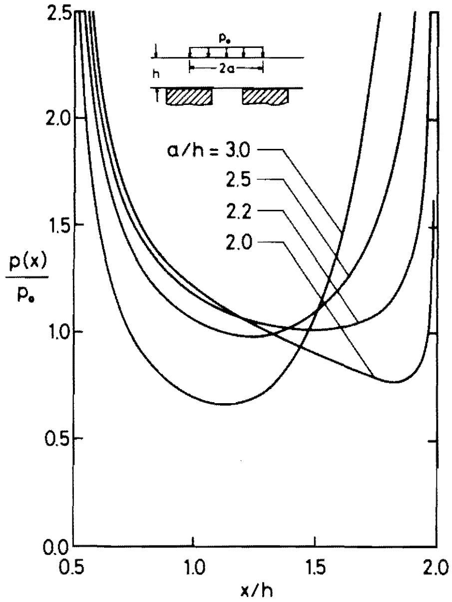

Figure 4. Contact Pressure Distribution for Continuous Contact Case ($b/h = 0.5$, $c/h = 2.0$)

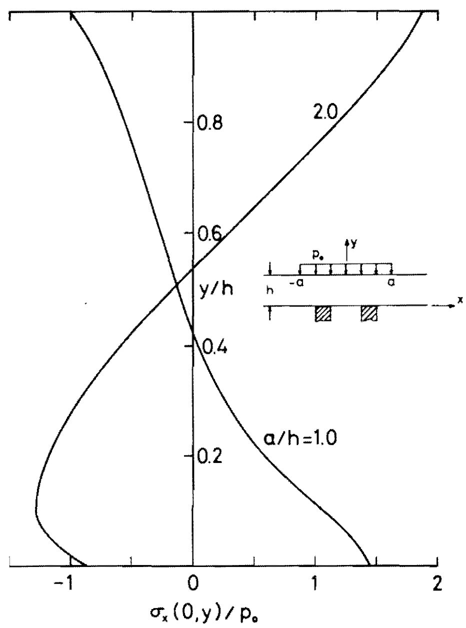

Figure 5. Axial Stress $\sigma_x(0,y)/p_0$ for Continuous Contact Case ($b/h = 0.5$, $c/h = 1.0$)

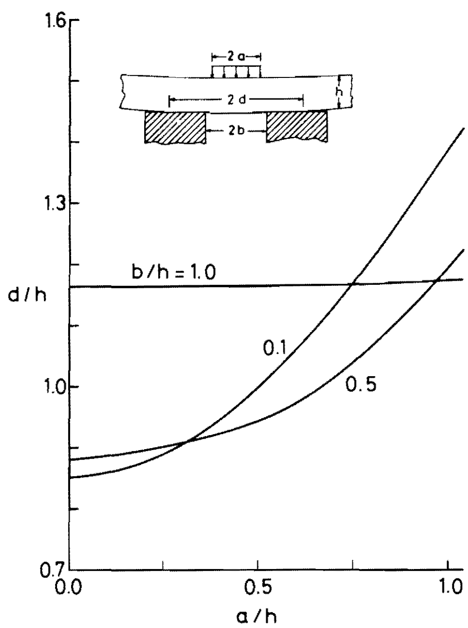

Figure 6. Location of Point Where Separation Starts

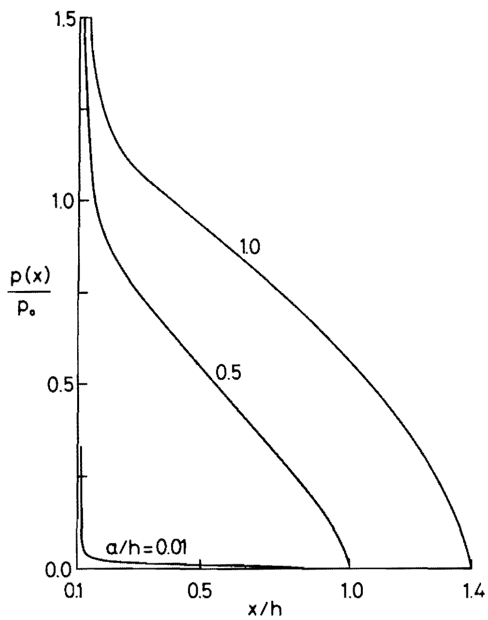

Figure 7. Contact Pressure Distribution for Discontinuous Contact Case ($b/h = 0.1$, $c > d$)

downward concaveness) for sufficiently large a/h. Therefore, the axial stress distribution may be completely different for various a/h values.

Many $a/h-b/h-c/h$ combinations have been considered and it has been shown that for certain combinations (for sufficiently small a/h and sufficiently large c/h) separation between the layer and the supports is possible only on the outer portion of the supports. Table 1 and Figure 6 give the location of point d where separation starts for various a/h–b/h combinations.

Figures 7 and 8 show the contact pressure $p(x)/p_0$ for discontinuous contact case. The contact pressure is infinitely large around the inner edge whereas it

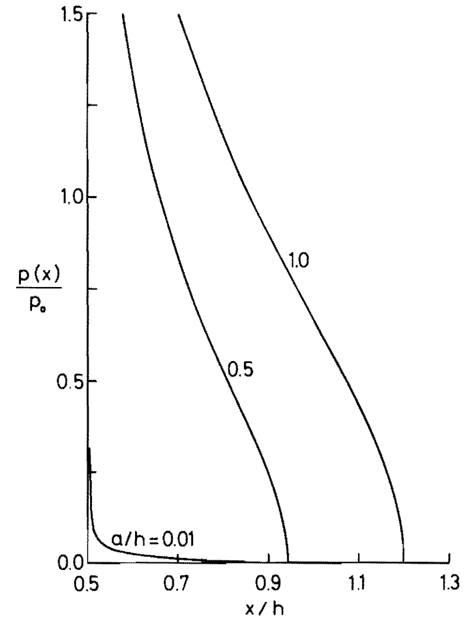

Figure 8. Contact Pressure Distribution for Discontinuous Contact Case ($b/h = 0.5$, $c > d$)

Table 1. Location of Point Where Separation Starts ($c > d$)

a/h	b/h	d/h
0.01	0.1	0.854100
0.50	0.1	1.000076
1.00	0.1	1.391872
0.01	0.5	0.880998
0.50	0.5	0.944992
1.00	0.5	1.200166
0.01	1.0	1.164082
0.10	1.0	1.164480
0.50	1.0	1.175165

decreases to zero at $x = d$. For very small a/h (e.g., for $a/h = 0.01$) contact pressure distribution is such that the reaction is accumulated around the inner edge as if the layer is subjected to three point loads.

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