MAGNETOHYDRODYNAMIC FLOW OF SECOND ORDER FLUID THROUGH A POROUS MEDIUM ON AN INCLINED POROUS PLANE

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الخلاصة :

ندرس في هذا البحث تدفق مائع غير نيوتونى من نوع (رفلن - أركسن) من الدرجة الثانية غير قابل للانضغاط ، وهو موصل للكهربية خلال وسط مساميّ على مستوى مساميّ مائل في وجود مجال مغناطيسى منتظم .

ولقد تَمّ الحصول على الصورة التحليلية لكلّ من توزيع السرعة وقوى القص (الإجهاد) على المستوى . وقد نوقش تأثير كلّ من بارامتر (متغيّر وسيط) نفاذية الوسط ، وبارامتر المرونة ، وبارامتر المغناطيسية على كلّ من السرعة والأجهاد .

ABSTRACT

This paper considers the flow of an incompressible, conducting non-Newtonian Rivlin–Ericksen fluid of second order through a porous medium on an inclined permeable plane. In the presence of uniform magnetic field, an analytic solution to the volume-averaged momentum equation is obtained. The velocity profiles are illustrated for several combinations of the porous medium shape parameter, elastic parameter, and magnetic parameter. During the course of discussion, the drag force on the permeable wall is obtained and the effect of properties of the problem are studied and shown numerically and graphically.

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INTRODUCTION

There is increasing interest in magnetohydrodynamic flows within fluid-saturated porous media, because of numerous applications in geophysics and energy-related problems, such as thermal insulation of buildings, enhanced recovery of petroleum resources, geophysical flows, packed bed reactors, and sensible heat storage beds. Most of the previous studies of the flow through porous media [1–5] are based on the assumption that the fluid is Newtonian.

The understanding of non-Newtonian flows through porous media represents interesting challenges in geophysical systems, chemical reactor design, certain separation processes, polymer engineering, and in petroleum production.

In this technical brief, a theoretical study of the fully-developed magnetohydrodynamic non-Newtonian flow through a porous medium on an inclined permeable wall is presented (see Figure 1). The differential equation which describes the velocity distribution of the fluid have been solved using the method of series expansion in terms of a suitable parameter assumed to be small.

The main idea of our work is to show the relation between the different parameters of the motion and the external forces, in order to investigate how to control the velocity of the fluid by changing these parameters and external forces.

PRELIMINARIES

The Cauchy stress τ in an incompressible Rivlin–Ericksen fluid of second order is related to the fluid motion in the following manner [6, 7]:

$$\tau = -P\mathbf{I} + \mu\mathbf{A}_1 + \mu^*\mathbf{A}_2 + \mu^{**}\mathbf{A}_1^2, \qquad (1)$$

where μ is the coefficient of viscosity, μ^* and μ^{**} are the normal stress moduli, $-P\mathbf{I}$ denotes the indeterminate pressure and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical Rivlin–Ericksen tensors defined through:

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^T$$

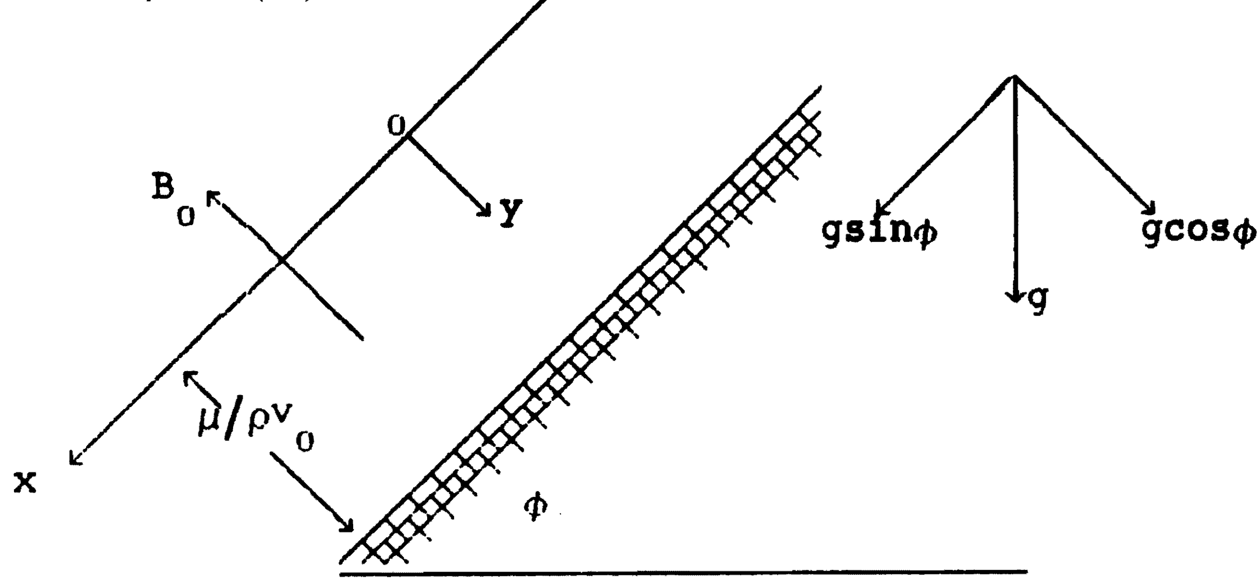

Figure 1. Physical Model and Its Coordinates.

and

$$\mathbf{A}_2 = \mathring{\mathbf{A}}_1 + \mathbf{A}_1(\nabla\mathbf{V}) + (\nabla\mathbf{V})^T\mathbf{A}_1$$

where the dot denotes material time differentiation and **V** denotes the velocity field.

We now develop the main field equations for the velocity **V** when the constitutive expression (1) is substituted into the volume-averaged momentum equation for an incompressible conducting fluid flowing through a porous medium with uniform porosity on an inclined permeable wall. The system is stressed by the normal magnetic field of uniform strength $\mathbf{B}_0$. The following conventional assumptions are considered: (*i*) the physical properties ρ, μ, μ^* and μ^{**} are considered to be constant; (*ii*) Darcy's law and Boussinesq approximation are employed; (*iii*) the fluid is injected through the inclined solid porous plate with constant velocity $\mathbf{V}_0$; and (*iv*) the induced magnetic field is neglected, which is valid for small magnetic Reynolds number; the external electric field is zero and the electric field due to polarization of charges is negligible [8]. Also, the fields generated by the fluid motions are negligible with respect to external fields.

We select a rectangular cartesian system with the axis of x in the direction of motion and axis of y perpendicular to it.

The governing equations are:

$$\left.\begin{aligned}
&\text{The continuity equation:}\\
&\quad \nabla\cdot\mathbf{V} = 0,\\
&\text{The momentum equation:}\\
&\quad \rho\left(\frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = \nabla\cdot\tau - \mu\frac{\mathbf{V}}{k} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g},\\
&\text{Maxwell's equations:}\\
&\quad \nabla\cdot\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\cdot\mathbf{B} = 0,\\
&\text{Ohm's law:}\\
&\quad \mathbf{J} = \sigma[\mathbf{E} + \mathbf{V}\times\mathbf{B}],
\end{aligned}\right\} \tag{2}$$

where ρ is the density, k the permeability of the medium, **J** the current density, **g** the acceleration gravity, **E** is the electric field, σ is the electrical conductivity, and **B** is the magnetic induction.

For two-dimensional flow, the velocity vector **V** and magnetic flux vector **B** have the components $\mathbf{V} = (U, V, 0)$ and $\mathbf{B} = (B_x, B_y, 0)$. Since the plate is infinite, all physical quantities are independent of x. With this assumption the equation of continuity reduces to $\dfrac{\partial V}{\partial y} = 0$ and we get $V = V_0(t)$. We will, however, assume the simple case when suction velocity is uniform, *i.e.*, $V = V_0 =$ constant. Maxwell's equations give $\dfrac{\partial B_y}{\partial t} = 0$ and $\dfrac{\partial B_y}{\partial y} = 0$, hence $B_y =$ constant $= B_0 =$ strength of the imposed magnetic field which is constant in space and time. If the magnetic Reynolds number is small, as we mentioned before, the component B_x of the induced magnetic field may be neglected. These assumptions result in great simplification. Also, we assume the situation that satisfies the condition $\mathbf{E} = 0$ and uncoupling of the hydrodynamic and hydromagnetic equations. Taking account of these assumptions the system of Equations (2) for steady state are reduced to:

$$\rho V_0 \frac{dU}{dy} = -\frac{\mu}{k}U + \mu\frac{d^2U}{dy^2} + V_0\mu^*\frac{d^3U}{dy^3} - \sigma B_0^2 U + \rho g\sin\phi \tag{3}$$

and

$$\frac{dP}{dy} = (2\mu^* + \mu^{**})\frac{d}{dy}\left(\frac{dU}{dy}\right)^2 - \left(\frac{\mu V_0}{k} + \rho g \cos\phi\right) \tag{4}$$

where ϕ the angle of inclination. Equation (3) determines the velocity distribution of the fluid in terms of the problem properties and from the velocity expression we can obtain the modified pressure formula by using Equation (4).

THE BOUNDARY CONDITIONS

At the inclined wall, the fluid velocity is zero. At the free surface, the shear stress is zero.

ANALYSIS

Using the following non-dimensional variables

$$\left.\begin{array}{lll} U' = \dfrac{U}{V_0}, & y' = y\dfrac{\rho V_0}{\mu}, & K = \dfrac{k\rho^2 V_0^2}{\mu^2} \\ \alpha = \dfrac{\rho V_0^2 \mu^*}{\mu^2}, & \beta = \dfrac{\sigma B_0^2 \mu}{\rho^2 V_0^2}, & \gamma = \dfrac{\mu g \sin\phi}{\rho V_0^3} \\ P' = \dfrac{P}{\rho V_0^2}, & S = \dfrac{\rho V_0^2 \mu^{**}}{\mu^2}, & \gamma^* = \dfrac{\mu g \cos\phi}{\rho V_0^2} \end{array}\right\} \tag{5}$$

Here, K is the permeability parameter, α is the elasticity parameter, β is the magnetic parameter, and S is the cross-viscosity parameter. The Equations of motion (3, 4) and the boundary conditions are obtained in the dimensionless form as follows (after dropping the primes)

$$\left.\begin{array}{l} \alpha\dfrac{d^3U}{dy^3} + \dfrac{d^2U}{dy^2} - \dfrac{dU}{dy} - \left(\beta + \dfrac{1}{K}\right)U = -\gamma \\ \dfrac{dP}{dy} = (2\alpha + S)\dfrac{d}{dy}\left(\dfrac{dU}{dy}\right)^2 - \left(\dfrac{1}{K} + \gamma^*\right) \end{array}\right\} \tag{6}$$

with the boundary conditions

$$\left.\begin{array}{ll} U = 0 & \text{at } y = 1 \\ \dfrac{dU}{dy} = 0 & \text{at } y = 0 \end{array}\right\} \tag{7}$$

It may be pointed out here that $\alpha = 0$ leads to the flow of ordinary Newtonian viscous fluid. For a solid inclined wall ($V_0 = 0$), when the fluid is an ordinary Newtonian ($\alpha = 0$) and in the absence of an external magnetic field ($\beta = 0$) Equation (6) reduces to that equation which was studied in reference [9].

METHOD OF SOLUTION

We seek the solution of Equations (6) and (7) in terms of perturbation elasticity parameter α in the following series expansion form

$$U = U^{(0)} + \alpha U^{(1)} + \cdots \tag{8}$$

thus, using the perturbation scheme (8) in Equations (6) and (7) and collecting the coefficients of like powers of α we get the following sets of equations with boundary conditions up to first order

$$\frac{d^2U^{(0)}}{dy^2} - \frac{dU^{(0)}}{dy} - \left(\beta + \frac{1}{K}\right)U^{(0)} = -\gamma \tag{9}$$

$$\frac{d^2U^{(1)}}{dy^2} - \frac{dU^{(1)}}{dy} - \left(\beta + \frac{1}{K}\right)U^{(1)} = -\frac{d^3U^{(0)}}{dy^3} \tag{10}$$

$$U^{(0)} = 0, \quad U^{(1)} = 0 \text{ at } y = 1 \tag{11}$$

$$\frac{dU^{(0)}}{dy} = 0, \quad \frac{dU^{(1)}}{dy} = 0 \text{ at } y = 0. \tag{12}$$

Equation (9) along with the conditions (11) and (12) give velocity component $U^{(0)}$ as

$$U^{(0)} = \frac{\gamma}{(\beta + 1/K)}\left[\frac{\lambda_1 e^{\lambda_2 y} - \lambda_2 e^{\lambda_1 y}}{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}} + 1\right] \tag{13}$$

where

$$\lambda_1 = \left\{1 + \sqrt{1 + 4(\beta + 1/K)}\right\} / 2$$

and

$$\lambda_2 = \left\{1 - \sqrt{1 + 4(\beta + 1/K)}\right\} / 2$$

Equation (10) when solved using the expression for $U^{(0)}$ with the conditions (11) and (12), gives the first order component $U^{(1)}$ as:

$$U^{(1)} = -\frac{\gamma}{(\beta + 1/K)}\left[\frac{-\lambda_1^3\lambda_2 m_1}{(\lambda_1 - \lambda_2)m_0^2} + \frac{\lambda_2^3\lambda_1(\lambda_2 - 1)e^{\lambda_2}}{(\lambda_2 - \lambda_1)m_0^2} + \frac{\lambda_1^3\lambda_2 Y}{(\lambda_1 - \lambda_2)m_0}\right]e^{\lambda_1 Y}$$
$$+\left[\frac{-\lambda_1^3(1 - \lambda_1)\lambda_2 e^{\lambda_1}}{(\lambda_1 - \lambda_2)m_0^2} + \frac{\lambda_2^3 m_2\lambda_1}{(\lambda_2 - \lambda_1)m_0^2} - \frac{\lambda_2^3\lambda_1 Y}{(\lambda_1 - \lambda_2)m_0}\right]e^{\lambda_2 Y} \tag{14}$$

where

$$m_0 = \lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2},$$

$$m_1 = \lambda_2 e^{\lambda_1} - e^{\lambda_2},$$

and

$$m_2 = e^{\lambda_1} - \lambda_1 e^{\lambda_2}.$$

Hence, the velocity distribution of our problem and the pressure may be written in the form:

$$U = \frac{\gamma}{(\beta + 1/K)}\left\{1 + e^{\lambda_1 Y}\left[-\frac{\lambda_2}{m_0} - \frac{\lambda_1^3\lambda_2 m_1\alpha}{m_0^2(\lambda_1 - \lambda_2)} + \frac{\lambda_2^3\lambda_1(\lambda_2 - 1)\alpha e^{\lambda_2}}{m_0^2(\lambda_2 - \lambda_1)} + \frac{\lambda_1^3\lambda_2\alpha\ Y}{m_0(\lambda_1 - \lambda_2)}\right]\right.$$
$$\left. + e^{\lambda_2 Y}\left[\frac{\lambda_1}{m_0} - \frac{\lambda_1^3\lambda_2(1 - \lambda_1)\alpha e^{\lambda_1}}{m_0^2(\lambda_1 - \lambda_2)} + \frac{\lambda_2^3\lambda_1 m_2\alpha}{m_0^2(\lambda_2 - \lambda_1)} - \frac{\lambda_2^3\lambda_1\alpha\ Y}{m_0(\lambda_2 - \lambda_1)}\right]\right\} \tag{15}$$

$$P = P_0 + (2\alpha + S)\left(\frac{\gamma}{(\beta + 1/K)}\right)^2\left\{e^{\lambda_1 Y}\left[\frac{-\lambda_1\lambda_2}{m_0} - \frac{\lambda_1^4\lambda_2 m_1\alpha}{m_0^2(\lambda_1 - \lambda_2)} + \frac{\lambda_1^2\lambda_2^3(\lambda_2 - 1)\alpha e^{\lambda_2}}{m_0^2(\lambda_2 - \lambda_1)}\right.\right.$$
$$\left. + \frac{\lambda_1^3\lambda_2\alpha}{m_0(\lambda_1 - \lambda_2)}(\lambda_1 Y + 1)\right] + e^{\lambda_2 Y}\left[\frac{\lambda_1\lambda_2}{m_0} - \frac{\lambda_1^3\lambda_2^2(1 - \lambda_1)\alpha e^{\lambda_1}}{m_0^2(\lambda_1 - \lambda_2)} + \frac{\lambda_2^4\lambda_1 m_2\alpha}{m_0^2(\lambda_2 - \lambda_1)} -\right.$$
$$\left.\left. - \frac{\lambda_2^3\lambda_1\alpha}{m_0(\lambda_2 - \lambda_1)}(\lambda_2 Y + 1)\right]\right\}^2 - (1/K + \gamma *)\ Y. \tag{16}$$

WALL SHEAR STRESS

Non-dimensional wall shear stress τ_w is given as:

$$\tau_w = \left(\frac{dU}{dy} + \alpha\frac{d^2U}{dy^2}\right)_{y=1}$$
$$= \frac{\gamma}{(\beta + 1/K)}\left\{e^{\lambda_1}\left[-\frac{\lambda_1\lambda_2(1+\lambda_1)}{m_0} - \frac{\lambda_1^4\lambda_2 m_1\alpha(1+\lambda_1)}{m_0^2(\lambda_1 - \lambda_2)} + \frac{\lambda_2^3\lambda_1^2(\lambda_2 - 1)(1+\lambda_1)\alpha e^{\lambda_2}}{m_0^2(\lambda_2 - \lambda_1)}\right.\right.$$
$$\left. + \frac{\lambda_1^4\lambda_2\alpha(2+\lambda_1)}{m_0(\lambda_1 - \lambda_2)} + \frac{\lambda_1^3\lambda_2\alpha(1+\lambda_1)e^{\lambda_1}}{m_0(\lambda_1 - \lambda_2)}\right] + e^{\lambda_2}\left[\frac{\lambda_1\lambda_2(1+\lambda_2)}{m_0} - \frac{\lambda_1^3\lambda_2^2(1-\lambda_1)(1+\lambda_2)\alpha e^{\lambda_1}}{m_0^2(\lambda_1 - \lambda_2)}\right.$$
$$\left.\left. + \frac{\lambda_2^4\lambda_1 m_2\alpha(1+\lambda_2)}{m_0^2(\lambda_2 - \lambda_1)} - \frac{\lambda_2^4\lambda_1\alpha(2+\lambda_2)}{m_0(\lambda_2 - \lambda_1)} - \frac{\lambda_2^3\lambda_1\alpha(1+\lambda_2)}{m_0(\lambda_2 - \lambda_1)}\right]\right\}. \tag{17}$$

RESULTS AND DISCUSSION

To study the effects of elastic parameter α, the magnetic parameter β, and the permeability parameter K on the velocity distribution and wall shear stress, the expressions (15) and (17) are evaluated by taking α = 0.01, 0.03, 0.05, β = 2, 4, 8, and K = 0.5, 0.7, and 0.85.

The values of U are plotted *versus* y in Figures (2–4). It is observed from these figures that the effect of non-Newtonian parameter α is to decrease the velocity profile when both of the magnetic parameter and permeability parameter are constants (Figure 2). Also in the case of constant permeability and elasticity parameters the effect of magnetic parameter β is to decrease the velocity distribution (Figure 3). Figure 4 indicates that the velocity profile increases with the increase of the permeability parameter K, this occurs when both of the magnetic and elasticity parameters are constants.

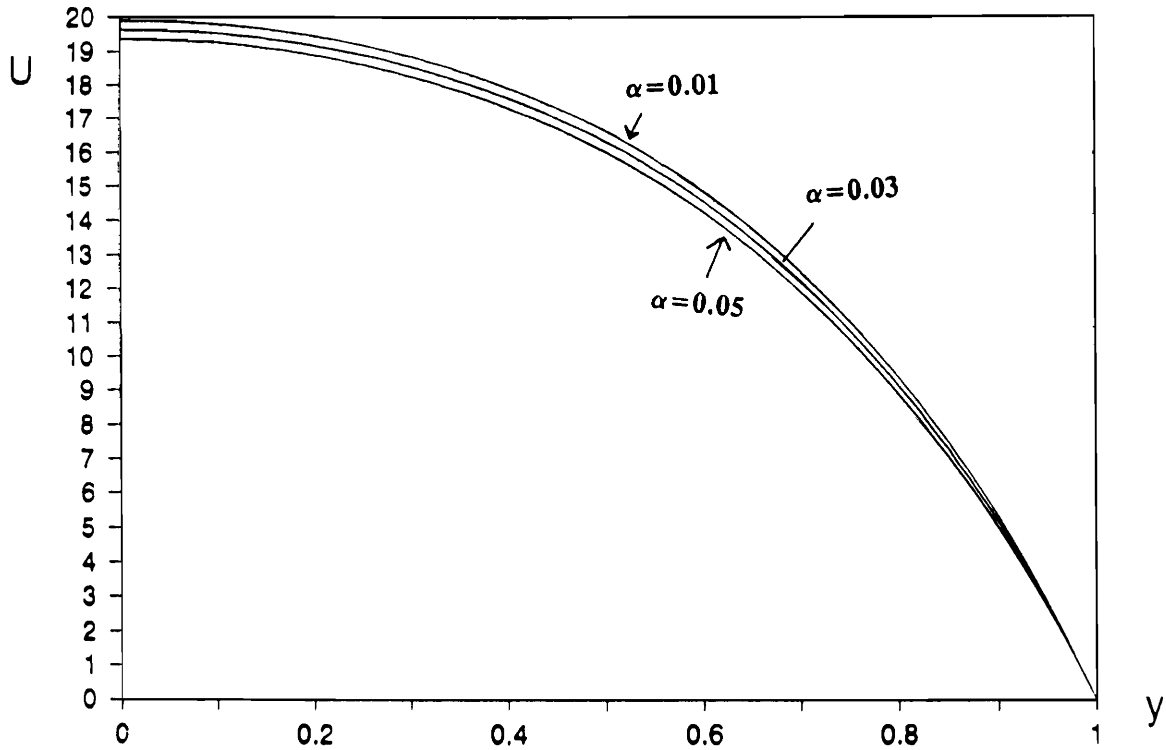

Figure 2. Velocity Profile Plotted Against y *for* β=2 *and* K=0.5.

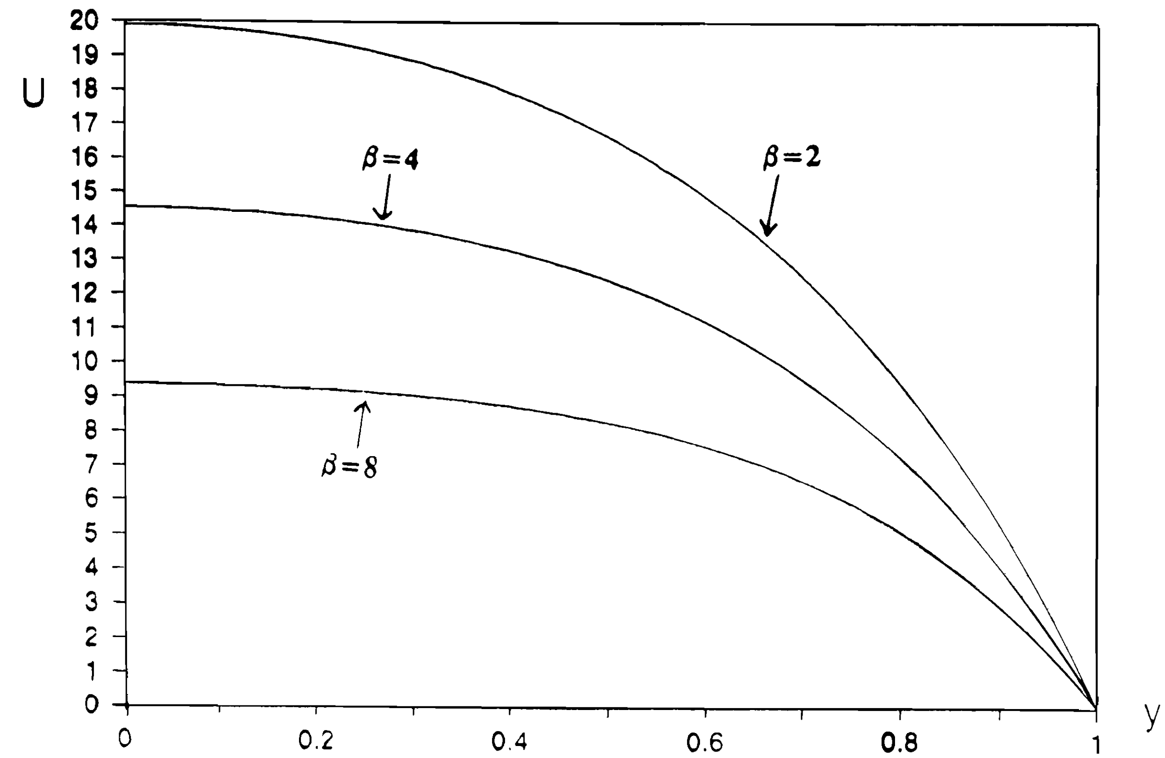

Figure 3. Velocity Profile Plotted Against y *for* α=0.01 *and* K=0.5.

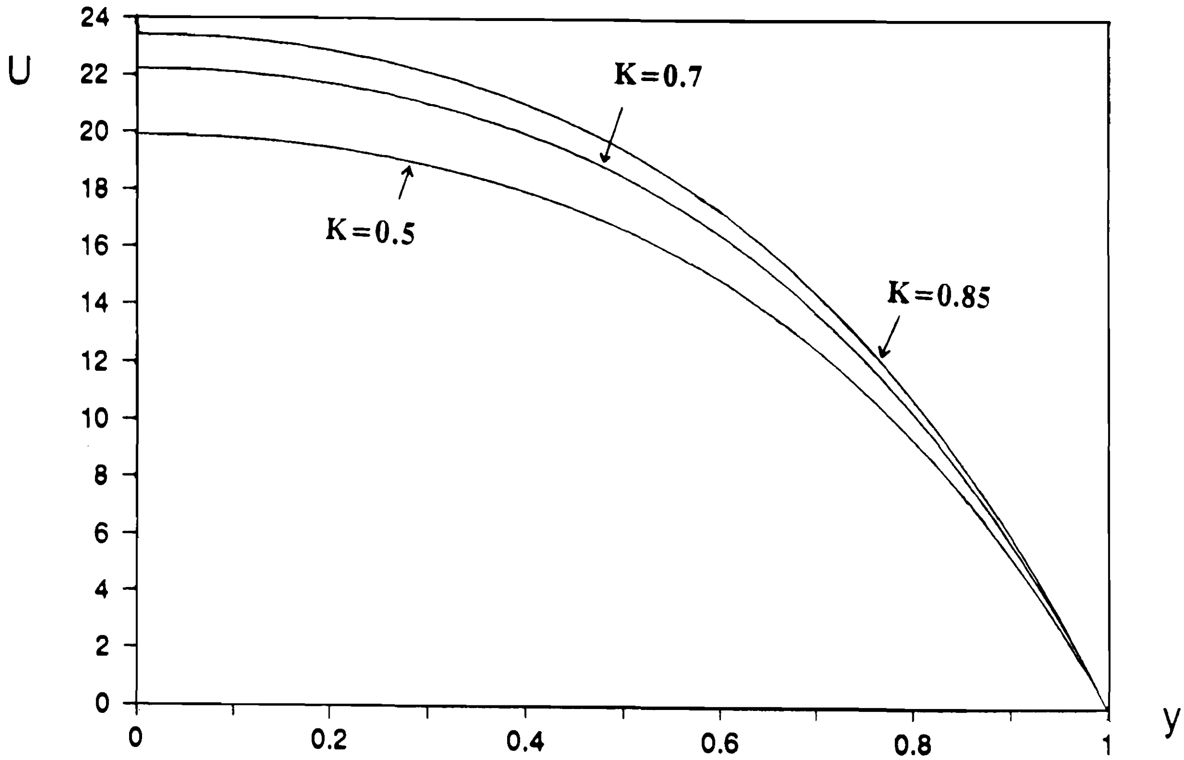

Figure 4. Velocity Profile Plotted Against y *for* α*=0.01 and* β*=2.*

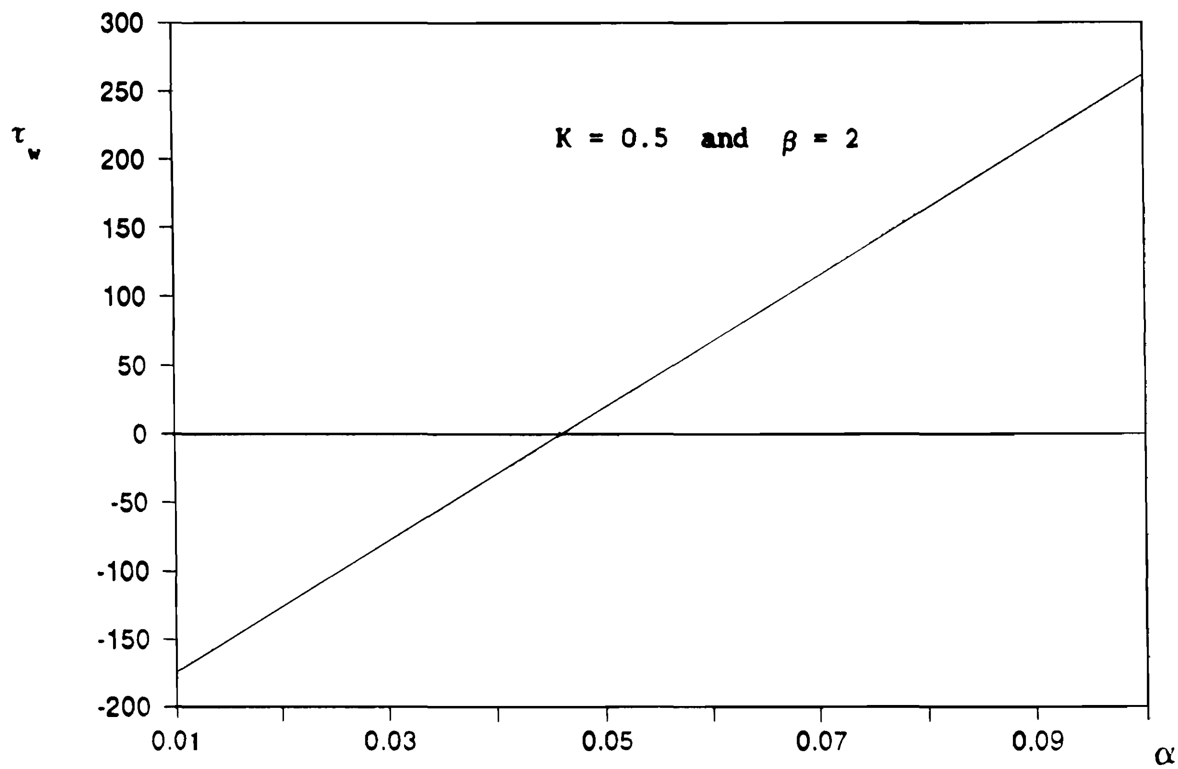

Figure 5. The Drag Force Plotted Against α*.*

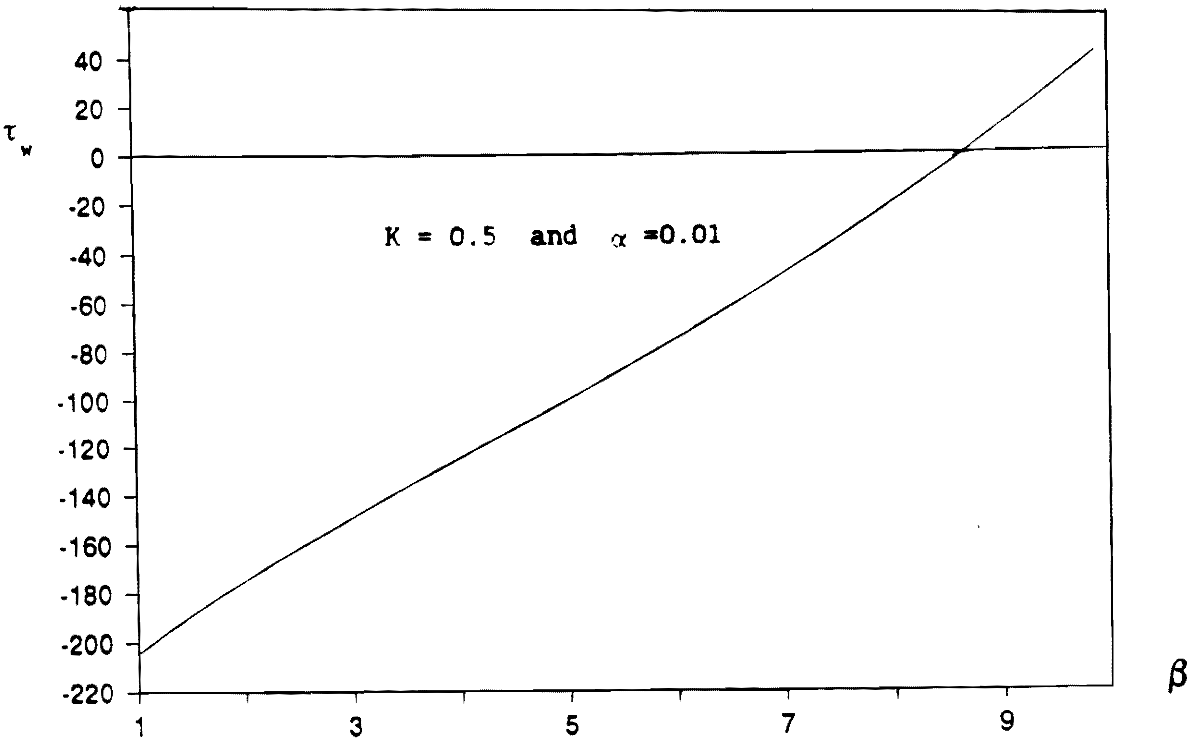

Figure 6. The Drag Force Plotted Against β.

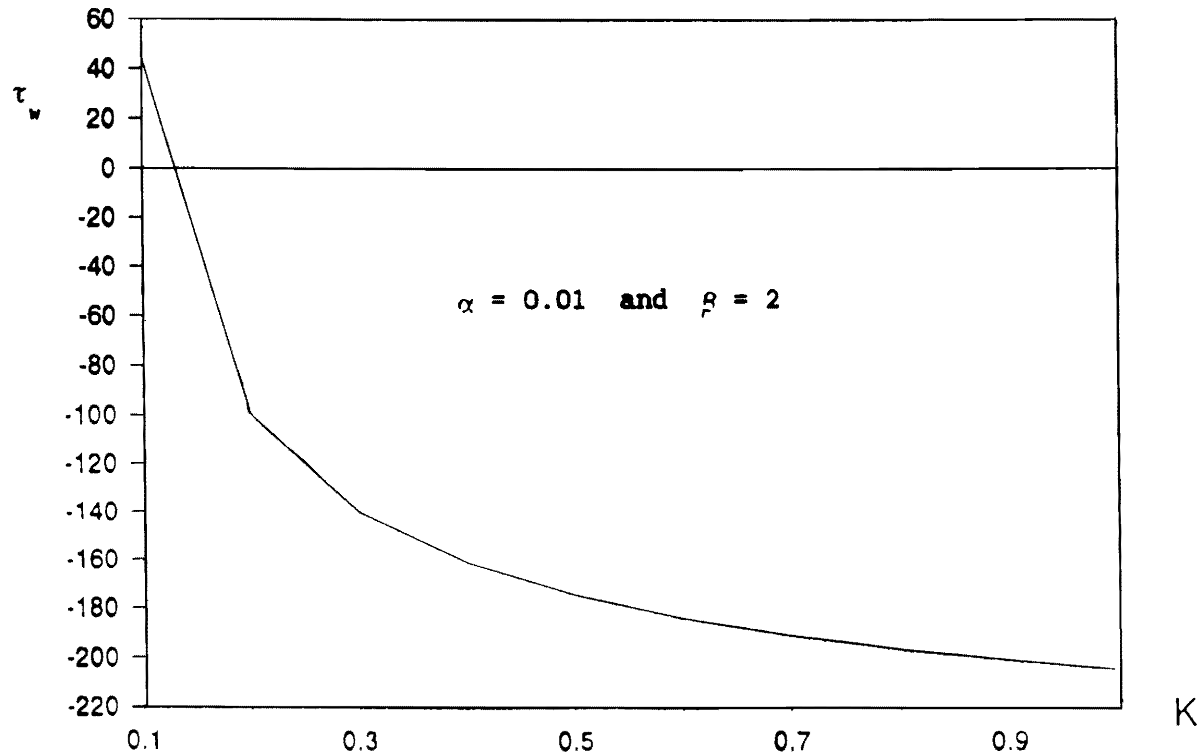

Figure 7. The Drag Force Plotted Against K.

The values of the drag force $\tau(\alpha, \beta, K)$ on the permeable wall are plotted in Figures 5–7. It is found that the value of the drag force increases with the elasticity parameter, when both of permeability and magnetic parameter are constants (Figure 5). Also, the value of the drag force at the inclined wall increases as the magnetic parameter increases in the case of constant elasticity and permeability parameters (Figure 6). Figure 7 shows that the drag force decreases as the permeability parameter increases when both elasticity and magnetic parameters are constants.

From the above analytical results and from figures we can conclude that the elasticity and magnetic terms α and β make an retardation of the flow, while the flow will accelerates for the medium of large porosity K. On other hand both of elasticity and magnetic terms will accelerate the drag force τ_w while the porosity K retards it.

Finally, it is found that the problem properties α, β, K played an important role in controlling the motion of the fluid under consideration.

CONCLUSIONS

The study of the physics of non-Newtonian fluid flow through porous media has become the basis for many scientific and engineering applications. This type of flow is of great importance to the petroleum engineer concerned with movement of oil, gas, and water through the reservoir of an oil or gas field, to the hydrologist in his study of the migration of underground water, and to the chemical engineer in connection with filtration processes. Beyond this, the study is widely applicable in soil mechanics, water purification, ceramic engineering, powder metallurgy, and mathematical medium.

The results of the problem are also of great interest in geophysics in the study of the interaction of the geomagnetic field with the fluid in the geothermal region. Water in the geothermal region is an electrically conducting liquid because of high temperature. With the fuel crisis deepening all over the developed world, attention is turning to the utilization of the enormous power beneath the earth's crust in the geothermal region.

Another potential geophysical application of the present results is in the exploration of geopressured reservoirs. In these reservoirs, water at elevated temperature exists at enormously high pressure because of the weight of the overlying rock and the geomagnetic field. The upflowing water from geopressured wells can run hydraulic turbines to produce electricity, while the heat in the water can simultaneously be extracted to run steam turbines, again producing electricity.

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