ON INVERSE *M*-MATRICES OF THE FORM $A = sI + B$

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الخلاصـــة :

تَـمَّ في هذا البحث دراسة المصفوفات التي لا صورة (أ = س 1 + ب) حيث (س) عدد حقيقي بينما (1) المصفوفة المحايدة للضرب ، أما (ب) فهي مصفوفة غير سالبة . ويبدأ البحث بدراسة الحالة الخاصة (س = ١) حيث تـمَّ وضع الشروط الكافية على المصفوفة (ب) لجعل (أ) مصفوفة ميمية عكسية . كما تَـمَّ إيجاد حدٍّ أدنى لَـ (س) لجعل (س 1 + ب) مصفوفة ميمية عكسية عندما تكون (ب) متساوية القوي .

ABSTRACT

In this paper matrices of the form $A = sI + B$, where s is a real number, I is the identity matrix, and B is a nonnegative matrix, are considered. The case when $s = 1$ is considered first and sufficient conditions are imposed on B to insure that A is an inverse M-matrix. A lower bound for s such that $sI + B$ is an inverse M-matrix when B is idempotent is given.

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1. PRELIMINARIES

An $n \times n$ real matrix $M = (m_{ij})$ is called a Z-matrix if and only if $m_{ij} \leq 0$ for all $i \neq j$. An important subclass of Z-matrices is the class of M-matrices. There is a number of equivalent definitions of these matrices [1]. For the purpose of this paper, a Z-matrix M is called an M-matrix if and only if it is of the form $M = \alpha I - P$ where P is entry-wise nonnegative and α exceeds the spectral radius of P. This is equivalent to saying that M is nonsingular and that its inverse is entry-wise nonnegative. An $n \times n$ nonnegative and nonsingular matrix A is called an inverse M-matrix provided that A^{-1} is an M-matrix. It is well-known that the zero−nonzero patterns of inverse M-matrices are power invariant and that any inverse M-matrix may be scaled by suitable diagonal matrices with positive diagonal entries to have prescribed diagonal entries [2]. Let B be an $n \times n$ nonnegative matrix such that $sI+B$ has a power invariant zero−nonzero pattern for $s > 0$. It has been shown that there is a real number s_0 such that $sI+B$ is an inverse M-matrix for $s > s_0$ and that $sI+B$ is not an inverse M-matrix for $s \leq s_0$. In this paper we give a characterization of this real number s_0 for certain special types of matrices B.

The following lemmas are easy to prove. Thus, their proofs have been omitted.

Lemma 1. Let B be a nonnegative $n \times n$ idempotent matrix. Then $0 \leq b_{ii} \leq 1$ for $i = 1,\ldots, n$.

Lemma 2. Let B be an $n \times n$ idempotent matrix such that $B \neq 0$ and $B \neq I$. Let s be a real number. Then $sI+B$ is invertible if and only if $s \neq 0$ and $s \neq -1$. In this case:

$$(sI+B)^{-1} = s^{-1}I - [s(s+1)]^{-1}B.$$

2. RESULTS

Theorem 1. Let A be a nonnegative nonsingular $n \times n$ matrix satisfying the polynomial

$$x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 - a_1x + a_0$$

in which a_0 is positive. Then A is an inverse M-matrix if and only if $A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2A$ is the negative of a Z-matrix.

Proof. Since A satisfies the given polynomial, it follows that

$$a_0A^{-1} = a_1I - [A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2A] .$$

From this it follows that A^{-1} is a Z-matrix and therefore A is an inverse M-matrix if and only if $A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_2A$ is the negative of a Z-matrix.

We may apply Theorem 1 to several situations in which it is easily seen that its requirements are met. Each is proven by translating the given information into coefficients and then checking them against Theorem 1.

Corollary 1. Let A be a nonnegative, nonsingular $n \times n$ matrix. If $A = I+B$, $B \geq 0$, and $B^2 = 0$, then A is an inverse M-matrix.

Proof. Let A be as given. Then $A^2 = I+2B = 2A-I$. Thus A satisfies the polynomial x^2-2x+1. Since $-A$ is a Z-matrix then Theorem 1 yields that A is an inverse M-matrix.

Corollary 2. Let A be a nonnegative, nonsingular $n \times n$ matrix. If $A = I+B$, where B is a nonnegative idempotent, then A is an inverse M-matrix.

Proof. It can be easily shown that A satisfies the polynomial x^2-3x+2. Moreover since $-A$ is a Z-matrix then Theorem 1 yields the desired result.

Corollary 3. Let A be a nonnegative, nonsingular $n \times n$ matrix. If $A = I+P$, where $P = uv^T$, and u and v are nonnegative vectors such that $v^Tu = 1$, then A is an inverse M-matrix.

Corollary 4. Let A be a nonnegative, nonsingular $n \times n$ matrix. If $A = sI+B$, where B is an idempotent matrix such that $B \neq I$ and $B \neq 0$, and s is a real number. Then A is an inverse M-matrix if and only if s is positive.

Proof. It is easy to show that A satisfies a minimal polynomial $x^2-a_1x+a_0$ with $a_1 = (2s+1)$, $a_0 = s(s+1)$. Moreover since $-A$ is a Z-matrix, then Theorem 1 proves the if part. The only if part follows from lemmas 2 and 1.

Theorem 2. Let A be an $n \times n$ matrix of the form $A = I+B$ where B is nonnegative and the spectral radius $\rho(B) < 1$. Then $B^2 \leq B$ implies that A is an inverse M-matrix.

Proof. A^{-1} has the convergent power series

$$\begin{aligned} A^{-1} &= I - B + B^2 - B^3 + \ldots \\ &= I - (B + B^3 - B^5 + \ldots) + (B^2 + B^4 + \ldots) \\ &= I - (B + B^2)(I + B^2 + B^4 + \ldots) \, . \end{aligned}$$

Since $(B - B^2)$ and $(I + B^2 + B^4 + \ldots)$ are nonnegative it follows that A^{-1} is a Z-matrix and since A is nonnegative then it is an inverse M-matrix.

REFERENCES

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