A CLOSED FORM SOLUTION FOR THE DYNAMIC RESPONSE OF INDUCTION MACHINES

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الخلاصـــة :

يُقـدِّم هذا البحث حلاً جبريا لحساب الأداء الديناميكي لمحرك الحث ، وقد بُني هذا الحل على استخدام زاوية العزم في المحرك ، والذي أمكن من خلاله تمثيل الأداء الديناميكي لزاوية العزم بمعادلة تفاضلية خطية من الدرجة الثانية يمكن حلها جبريا .

ولبيان مدى دقة المعادلات المقترحة في حساب الأداء الديناميكي فقد تَـمَّ مقارنة نتائجها مع نتائج دقيقة تَـمَّ الحصول عليها باستخدام طريقة المحورين المتعامدين واللذين يدوران بالسرعة المتزامنة لمحركين أحدهما ذو مقنن عالي وآخر ذو مقنن منخفض . وفي جميع الحالات وُجِد أن المعادلات المقترحة - رغم بساطتها - تعطي نتائج متطابقة مع نظيراتها المحسوبة بدقة .

ABSTRACT

This paper presents a closed form solution for the dynamic response of an induction machine. Based on the load angle approach, the induction machine is represented by a simple linear second order differential equation whose solution can be obtained easily in a closed form. The accuracy of the closed form solution is demonstrated by comparing it with that predicted by a detailed *d/q* model in a synchronously rotating reference frame. The testing of the proposed model is carried out for very large as well as very small motors. In all cases the closed form solution gave excellent predictions despite its simplicity.

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NOMENCLATURE

T	Developed torque
T_L	Load torque
δ	Load angle
R_s	Stator resistance
R_r	Rotor referred resistance
L_s	Stator inductance
L_r	Rotor referred inductance
M	Magnetizing inductance
X_s	Stator leakage reactance
X_r	Rotor referred leakage reactance
X	$X_s + X_r$
P	Number of pair of poles
p	(d/dt) operator
J	Rotor inertia
V	Terminal voltage
ϕ	$\tan^{-1}(R_s/X)$
ω_s	Synchronous speed
ω_n	Natural frequency
ζ	Damping coefficient

1. INTRODUCTION

Simplified induction machine models play a major role in power system stability studies and in the prediction of induction machine dynamics. Various simplified models that predict the dynamic response of induction machines have been investigated [1–5]. Most of the reduced order models are based on small signal analysis and on neglecting the time rate of change of stator flux linkages ($p\lambda$-terms) [1–4]. The second approximation is usually referred to as the neglecting of the stator transients. But for induction machines, especially for those of the large horsepower rating, these models have limited accuracy.

A different approach for obtaining a reduced order model is described in reference [5]. Such a model represents the dynamic behavior of an induction machine by a non-linear second order differential equation similar to the swing equation of synchronous machines. The reduction in this model is based on the use of a simplified steady state equivalent circuit of the induction machine shown in Figure 1, and on the introduction of the load angle δ as defined in Figure 2. The dynamic response predicted by the load angle approach [5, 6] follows closely that obtained from the detailed d/q model [7].

In this paper, the non-linear second order differential equation of reference [5] has been simplified to a linear second order differential equation which has a simple closed form solution. The simplification is based on the introduction of a new angle θ as defined in Figure 2. The proposed approach predicts the dynamic behavior of induction motors fairly accurately at any time following a disturbance in the load torque without any numerical instability whatsoever. The proposed model is derived next.

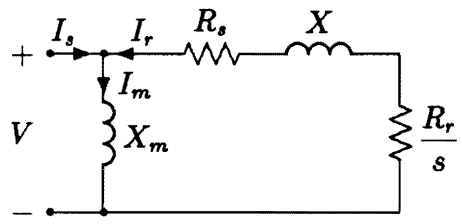

Figure 1. Simplified Equivalent Circuit of Induction Machine.

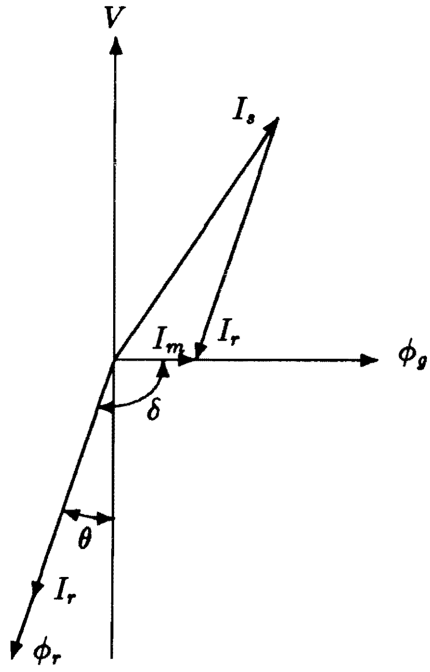

Figure 2. Phasor Diagram of the Simplified Circuit.

2. LOAD ANGLE MODEL

The load angle δ in an induction machine is defined as the angle between the synchronously rotating flux and the air-gap flux as shown in Figure 2. Based on this definition of δ, the developed electromagnetic torque as a function of the load angle can be expressed as [6]:

$$T(\delta) = -T_m(\sin(2\delta+\phi)+\sin\phi) \quad (1)$$

where

$$T_m = \frac{3PV^2}{2\omega_s X\cos\phi} \quad \text{and} \quad \tan\phi = \frac{R_s}{X}. \quad (2)$$

If the load torque applied to the motor is abruptly changed by a unit step, then the dynamic behavior of the machine load angle can be described by the following non-linear second order differential equation as given in [6]:

$$\frac{d^2\delta}{dt^2} + A(\delta)\frac{d\delta}{dt} + \frac{P}{J}(T(\delta)-T_L) = 0 \quad (3)$$

where

$$A(\delta) = \frac{\omega_s R_r \cos^2\phi}{X\sin^2(\delta+\phi)}, \quad (4)$$

and $T(\delta)$ is given by Equation (1).

3. PROPOSED MODEL

In this section, a closed form solution for the dynamic response of induction motors will be derived. The derivation is based on the linearization of $T(\delta)$ and $A(\delta)$ which are given by Equations (1) and (4) respectively. Consequently, Equation (3) will be become a linear second order differential equation.

3.1. Simplifying The Load Angle Equation

Equation (1) can be expanded as follows:

$$T(\delta) = -T_m(\sin 2\delta\cos\phi+\cos 2\delta\sin\phi+\sin\phi). \quad (5)$$

Substituting $\delta = \frac{\pi}{2}+\theta$, (see Figure 2), into the above equation gives:

$$T(\delta) = T_m(\sin 2\theta\cos\phi+\cos 2\theta\sin\phi-\sin\phi). \quad (6)$$

Figure 3 shows the variation in the angle θ for a 100% increase in the load torque for a large and a small motor. These responses are obtained from the detailed d/q model which will be outlined later in this paper. From this Figure, it is clear that θ is small enough which justifies the following approximation

$$\sin 2\theta \simeq 2\theta \quad \text{and} \quad \cos 2\theta \simeq 1.$$

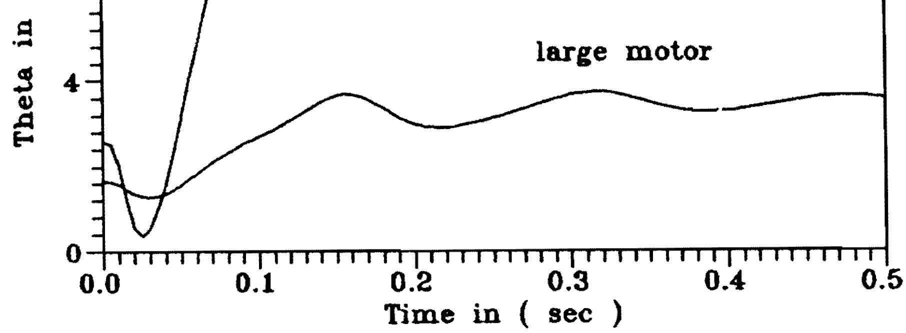

Figure 3. Variation of θ for a 100% Change in the Load Torque.

Substituting for $\sin 2\theta$ and $\cos 2\theta$ into Equation (6), the developed torque equation can be rewritten as follows:

$$T(\theta) = T_c\theta \tag{7}$$

where

$$T_c = 2T_m \cos\phi = \frac{3PV^2}{\omega_s X}\ . \tag{8}$$

From Equation (7) it is clear that the electromagnetic torque varies linearly with the new angle θ.

To complete the simplification process, $A(\delta)$ of Equation (3) must also be expressed in terms of the new angle θ. This can be accomplished by rewriting equation (4) as follows:

$$A(\theta) = \frac{\omega_s R_r \cos^2\phi}{X\sin^2(\theta + \frac{\pi}{2} + \phi)} = \frac{\omega_s R_r \cos^2\phi}{X\cos^2(\theta+\phi)} \tag{9}$$

Since the angle θ is usually very small as shown earlier, then the following approximation can be safely used:

$$\cos(\theta+\phi) \simeq \cos\phi\ .$$

With the above approximation, Equation (9) becomes independent of θ, *i.e.* $A(\theta) = A$ given by Equation (10).

$$A = \frac{\omega_s R_r}{X}\ . \tag{10}$$

Since $\delta = \theta + \frac{\pi}{2}$, therefore

$$\frac{d\delta}{dt} = \frac{d\theta}{dt} \quad \text{and} \quad \frac{d^2\delta}{dt^2} = \frac{d^2\theta}{dt^2}\ .$$

Substituting Equations (7), and (10) and the above relations into Equation (3) yields the following linear second order differential equation:

$$\frac{d\theta^2}{dt^2} + 2\zeta\omega_n\frac{d\theta}{dt} + \omega_n^2\theta = \frac{P}{J}T_L \tag{11}$$

where the natural frequency and damping coefficient are respectively given by:

$$\omega_n = \sqrt{\left(\frac{PT_c}{J}\right)} = PV\sqrt{\left(\frac{3}{J\omega_s X}\right)} \tag{12}$$

$$\zeta = \frac{\omega_s R_r}{2\omega_n X} \tag{13}$$

and T_c is given by Equation (8).

3.2. Dynamic Response Due to a Sudden Change in the Load Torque

If the load torque T_L is changed by a unit step ΔT_L, then the normalized change in the angle θ has the following closed form solution:

$$\frac{\Delta\theta}{\theta_o} = \frac{\Delta T_L}{T_o}\{1 - \frac{1}{\cos\psi}\exp(-\zeta\omega_n t)\cos(\omega t - \psi)\} \tag{14}$$

where

$$\omega = \omega_n\sqrt{(1-\zeta^2)} \tag{15}$$

$$\psi = \sin^{-1}\zeta \tag{16}$$

and θ_o and T_o are the initial values of the load angle and electromagnetic torque respectively.

The normalized change in the electromagnetic torque can be related to that of the load angle θ by using Equation (7). It can be easily verified that:

$$\frac{\Delta T}{T_o} = \frac{\Delta\theta}{\theta_o}\ . \tag{17}$$

The rotor angular speed ω_r can be obtained by solving the familiar rotor dynamic equation as follows:

$$\frac{d\omega_r}{dt} = \frac{P}{J}(T - T_L) = \frac{P}{J}(\Delta T - \Delta T_L) \tag{18}$$

Substituting for the solution of ΔT from Equations (17) and (14) into the above equation and simplifying yields:

$$\frac{d\omega_r}{dt} = \frac{P\Delta T_L}{J\cos\psi}\cos(\omega t - \psi)\exp(-\zeta\omega_n t)\ . \tag{19}$$

The above first order differential equation can be integrated to give a closed form solution for the change in the rotor angular speed due to a sudden change in the load torque. The solution is:

$$\Delta\omega_r = -\frac{\omega_n \Delta T_L}{T_c\cos\psi}\{\sin(\omega t - 2\psi)\exp(-\zeta\omega_n t) + \sin 2\psi\}. \tag{20}$$

The magnitude of the rotor current can also be obtained in terms of the load angle θ as follows [6]:

$$|I_r| = \frac{V}{X}\sin\theta\ . \tag{21}$$

If an induction motor is subjected to a disturbance in its load torque, the changes in its electromagnetic torque, rotor speed, and magnitude of the rotor current can be obtained directly by evaluating Equations (14), (20), and (21) respectively.

4. DETAILED d/q MODEL

Accurate prediction of motors transients can be obtained using the well-established detailed d/q model. The equations which describe the induction motor in the synchronously rotating reference frame may be expressed as follows [7]:

$$[v] = [Z][i] \tag{22}$$

where

$$[v] = [v_{ds} \quad v_{qs} \quad 0 \quad 0]' \tag{23}$$

$$[i] = [i_{ds} \quad i_{qs} \quad i_{dr} \quad i_{qr}]' \tag{24}$$

and with $\omega = \omega_s - \omega_r$ and where p denotes d/dt operator, then

$$[Z] = \begin{bmatrix} R_s+L_sp & -\omega_s L_s & Mp & -\omega_s M \\ \omega_s L_s & R_s+L_sp & \omega_s M & Mp \\ Mp & -\omega M & R_r+L_rp & -\omega L_r \\ \omega M & Mp & \omega L_r & R_r+L_rp \end{bmatrix} \tag{25}$$

Equation (22) may be rewritten in state space form:

$$p[i] = [L]^{-1}\{[v]-[G][i]\} \tag{26}$$

where

$$[L] = \begin{bmatrix} L_s & 0 & M & 0 \\ 0 & L_s & 0 & M \\ M & 0 & L_r & 0 \\ 0 & M & 0 & L_r \end{bmatrix} \tag{27}$$

and

$$[G] = \begin{bmatrix} R_s & -\omega_s L_s & 0 & -\omega_s M \\ \omega_s L_s & R_s & \omega_s M & 0 \\ 0 & -\omega M & R_r & -\omega L_r \\ \omega M & 0 & \omega L_r & R_r \end{bmatrix}. \tag{28}$$

The electromagnetic torque is

$$T = \tfrac{3}{2} PM\{i_{qs} i_{dr} - i_{ds}\, i_{qr}\} \tag{29}$$

and the mechanical equation is given by:

$$\frac{d\omega_r}{dt} = \frac{P}{J}\{T - T_L\}. \tag{30}$$

5. RESULTS AND DISCUSSION

The closed form solution is tested by comparing its prediction of the dynamic response with that predicted by a detailed d/q model in a synchronously rotating reference frame. Two motors A and B are considered, the parameters of which are given in Table 1. The stators of both motors are Y-connected and their rotors are of the squirrel cage type.

Motor A is selected to be of a very low rating while motor B is of high rating. This choice is made to investigate the accuracy of the proposed model over a wide range of motor ratings and parameters. In the simulation, the motors are assumed to be running fully loaded and suddenly subjected to a step increase in the load torque. Various values of such step changes were investigated. Here the results will only be given for a 30% step increase as an example of the accuracy of the closed form solutions.

Figure 4 shows the normalized response of the change in the developed electromagnetic torque for motor A as obtained by the present simplified model and by the detailed d/q model. The normalized dynamic responses of the changes in the rotor current and that of the rotor angular speed are shown in Figures 5 and 6 respectively. The results of the dynamic performance of motor B are shown in Figures 7, 8, and 9 for the normalized change in the developed torque, rotor current and rotor angular speed respectively. The normalization is performed by dividing the dynamic response of the change of the variable by its initial value.

From these figures it can be seen very clearly that, despite its simplicity, the closed form solution gives excellent predictions compared to those obtained from the detailed d/q model for both small and large motors. From the results as well as the calculations, it was observed that the agreement between the two approaches is excellent for large motors. For small motors, in general, the torque obtained from the closed form solution is in excellent agreement with

Table 1. Motor Parameters.

Parameters	Motor A	Motor B
Rating (HP)	⅓	1550
# of poles	4	8
Voltage (V)	220	6600
$R_s\,(\Omega)$	7.2	0.162
$R_r\,(\Omega)$	3.0	0.123
$X_s\,(\Omega)$	14.4	2.444
$X_r\,(\Omega)$	14.4	2.049
$X_m\,(\Omega)$	157.0	55.446
J (kg m^2)	0.00615	305.910

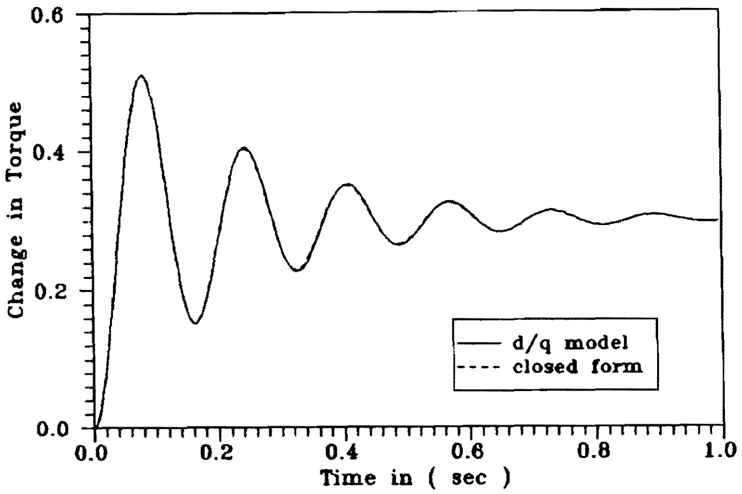

Figure 4. Normalized Change in the Developed Torque for the Large Motor.

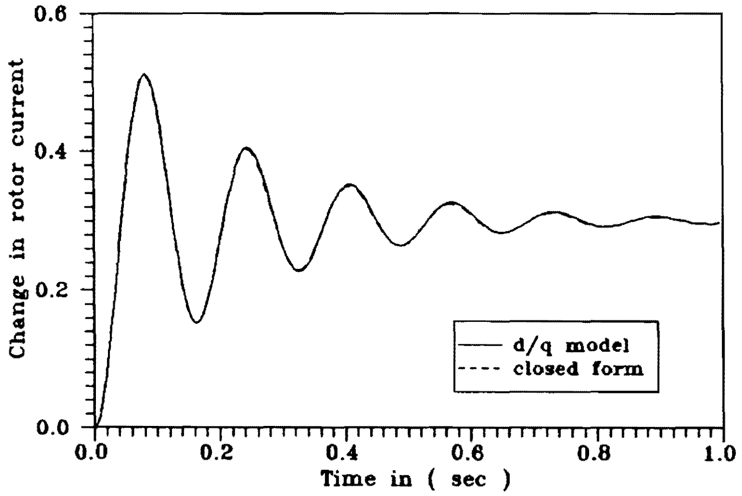

Figure 5. Normalized Change in the Rotor Current for the Large Motor.

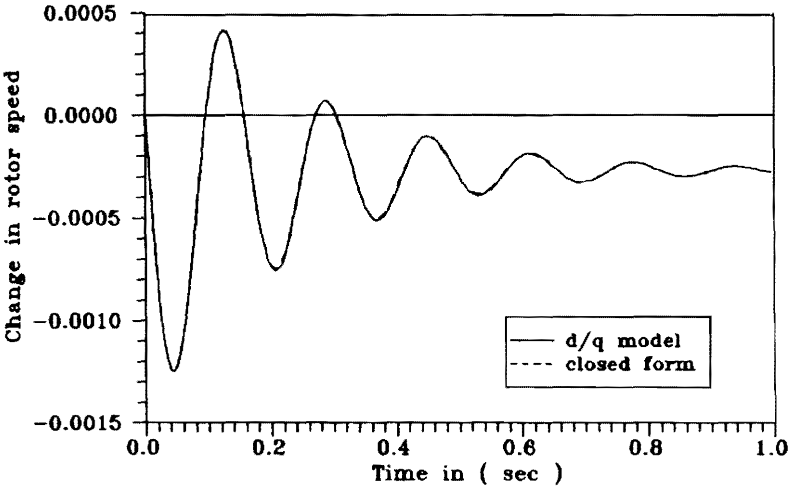

Figure 6. Normalized Change in the Rotor Speed for the Large Motor.

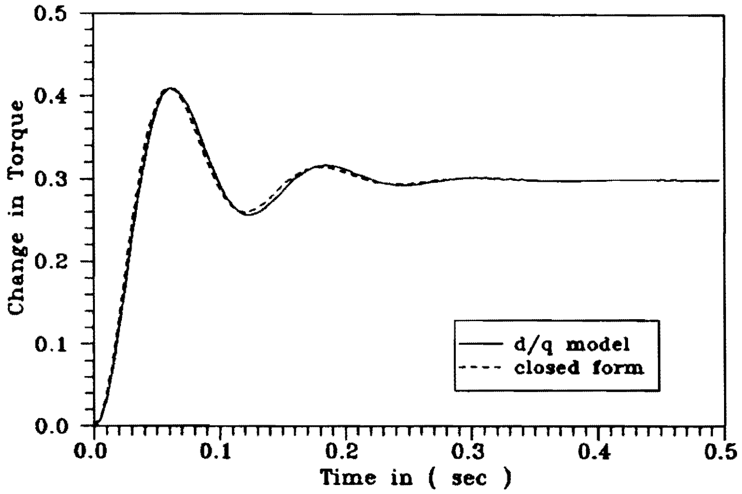

Figure 7. Normalized Change in the Developed Torque for the Small Motor.

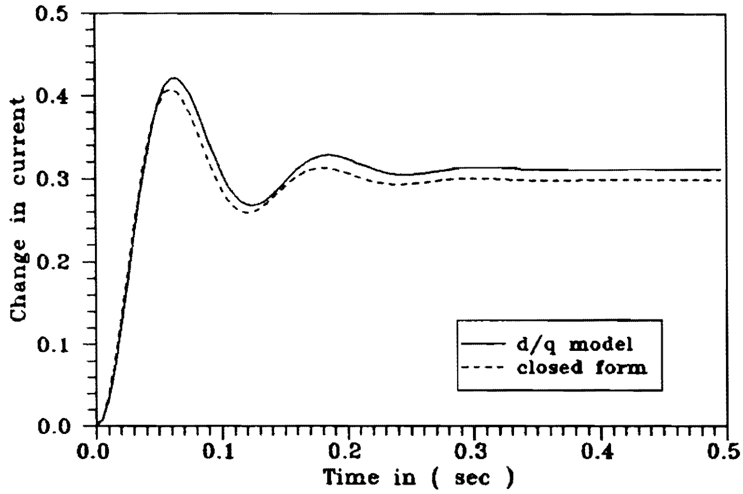

Figure 8. Normalized Change in the Rotor Current for the Small Motor.

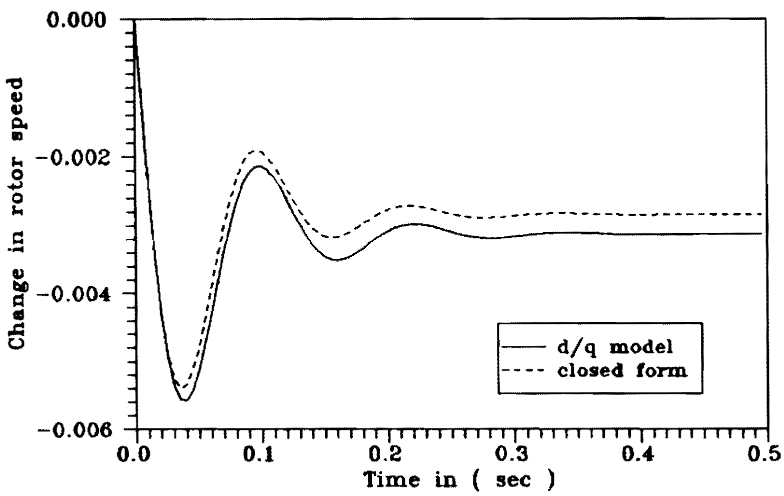

Figure 9. Normalized Change in the Rotor Speed for the Small Motor.

that obtained from the detailed d/q model. The current and speed, however, have very small differences between the two cases. This small departure, specially in the steady state, is due to the linearization process and the use of the approximate circuit of Figure 1. In that circuit the magnetizing reactance is moved to the motor terminals. For small motors where the stator resistance is relatively large, this simplification will have a slight effect in the steady state solution. Therefore, it can be stated in general, that the closed form solution provides very good simulation accuracy.

In this paper, only the dynamic behavior of induction motors due to a sudden disturbance in the load torque is obtained. The proposed approach can not be extended to obtain a closed form solution for the dynamic response of induction motors in the case of a disturbance in the terminal voltage and/or line frequency, because the angle θ is not small enough for the linearization to be performed. The variation of the angle θ for a 2% change in the terminal voltage, as obtained from the d/q model, is shown in Figure 10.

6. CONCLUSIONS

A simple closed form solution of the dynamic response of an induction machine has been obtained in this paper. The proposed solution can be used to obtain the dynamic responses of the developed torque, rotor speed, and magnitude of the rotor current of an induction motor at any time following a sudden change in the load torque with excellent accuracy irrespective of the motor horse power rating. It can also be concluded that, when the motor is subjected to a sudden change in the load torque, the normalized changes in the developed electromagnetic torque and rotor current are identical for large motors and nearly identical for small motors. When the transients die out, these normalized changes will approach that of the load torque.

The model fails to predict the dynamic response of an induction machine following an abrupt change in the terminal voltage and/or line frequency, because of the violation of the assumption that the model is based on.

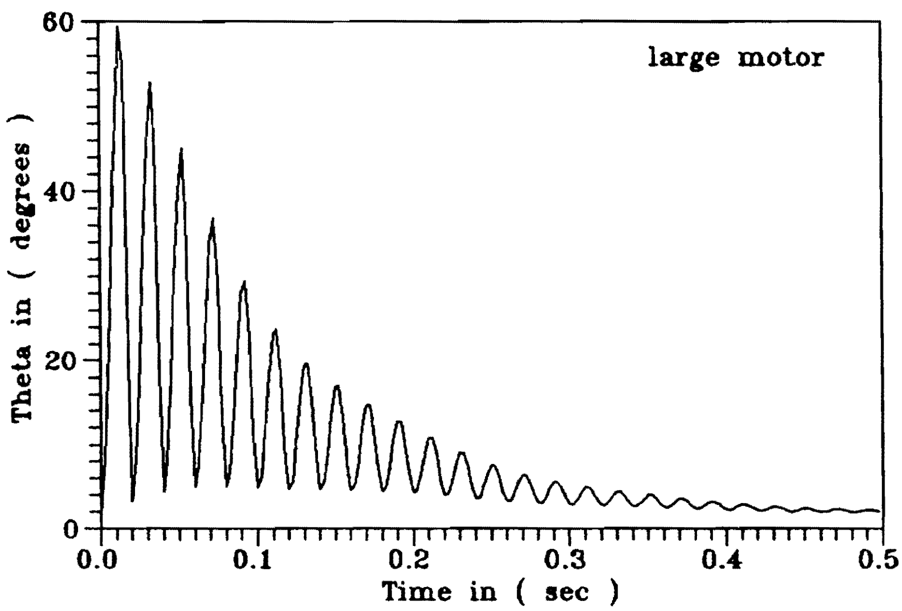

Figure 10. Variation of θ for a 2% Change in the Terminal Voltage.

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Paper Received 28 November 1987; Revised 16 April 1988.