FIXED-END MOMENTS AND FORCES FOR A PRISMATIC BEAM SUBJECTED TO DYNAMIC TRANSVERSE LOADING

John Francis Doyle

Assistant Professor, Department of Civil Engineering, University of Petroleum and Minerals, Dhahran, Saudi Arabia

INTRODUCTION

There are a variety of ways of evaluating fixed end moments and forces in cases of static loading [1] but none can be easily used for dynamic loading, which is usually dealt with by using equivalent static load. The fixed end moment for a transverse load W on a beam of length l is xWl where x is called the influence coefficient. The value of x is affected by such variables as axial load and beam geometry and a number of tables of influence coefficients are available in the literature [1]. In this paper formulae are developed for dynamic loading which is harmonic showing how the influence coefficient depends on the frequency of the load. The axial load effects are covered by the same formulae.

THE DIFFERENTIAL EQUATION OF A VIBRATING BEAM

The differential equation of a prismatic beam subjected to the dynamic transverse load $R(x, t)$, Figure 1, is given by [2]

$$EI\frac{\partial^4 Y}{\partial x^4}+P\frac{\partial^2 Y}{\partial x^2}+m\frac{\partial^2 Y}{\partial t^2}=R(x, t) \tag{1}$$

where E is Young's modulus, I the cross-sectional moment of inertia of the beam, P the axial compression, m the mass per unit length of the beam, and $Y=Y(x, t)$ the deflection at distance x along the beam, measured from end A, at time t.

$$\text{Let } R(x, t)=f(x)\,\mathrm{e}^{\mathrm{j}\omega t} \tag{2}$$

$$\text{then } Y(x, t)=y(x)\,\mathrm{e}^{\mathrm{j}\omega t} \tag{3}$$

is the steady state response.

Substitution of (2) and (3) into (1) gives

$$EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}+P\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-m\omega^2 y=f(x) \tag{4}$$

This equation has been solved for $f(x)=0$ and its eigenvalues give the natural frequencies with extension to natural frequencies of frames [3].

Using the Laplace transform, Equation (4) has the solution

$$y(x)=h(x)+g(x) \tag{5}$$

in which $h(x)$ is the complementary function and $g(x)$ the particular integral having the forms

$$h(x)=\zeta y(0)+\tau l y'(0)+\sigma l^2\left(y''(0)+\frac{P}{EI}y(0)\right)+$$

$$+\gamma\, l^3\left(y'''(0)+\frac{P}{EI}y'(0)\right), \tag{6}$$

0377-9211/81/020141-04\$01.00

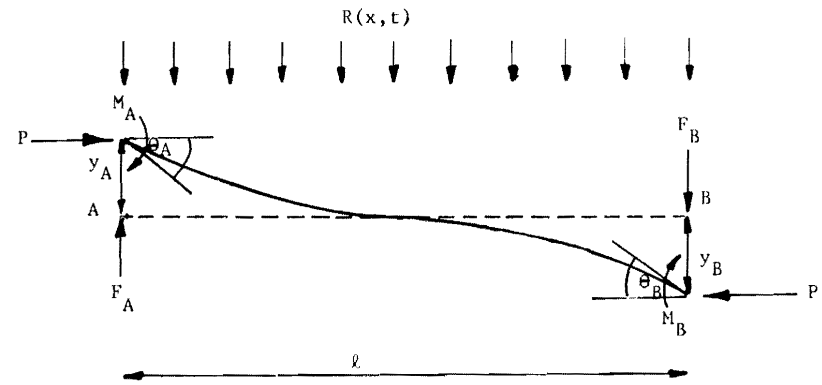

Figure 1

$$g(x)=\frac{l^3}{EI}\int_0^x f(u)\gamma(x-u)\mathrm{d}u=\frac{l^3}{EI}\int_0^x f(x-u)\gamma(u)\mathrm{d}u \tag{7}$$

where

$$\gamma=\gamma(x)=\frac{1}{\phi^2+\psi^2}\left(\frac{1}{\phi}\sinh\frac{\phi x}{l}-\frac{1}{\psi}\sin\frac{\psi x}{l}\right), \tag{8}$$

$$\sigma=l\gamma'(x),\ \tau=l^2\gamma''(x),\ \xi=l^3\gamma'''(x), \tag{9}$$

$$\phi^2=\left(\frac{\pi^4\rho^2}{4}+\lambda^4\right)^{1/2}-\frac{\pi^2\rho}{2}, \tag{10}$$

$$\psi^2=\left(\frac{\pi^4\rho^2}{4}+\lambda^4\right)^{1/2}+\frac{\pi^2\rho}{2}, \tag{11}$$

$$\lambda^4=\frac{m\omega^2l^4}{EI},\ \rho=P/P_e \text{ and } P_e=\frac{\pi^2EI}{l^2} \text{ is the Euler load} \tag{12}$$

If P is a tensile force then P in (1) is replaced by $-P$ and the solution is the same as the above except that ϕ and ψ are interchanged in (8).

For a fixed-end beam the end conditions are

$$y(0)=y'(0),\ y(l)=y'(l)=0 \tag{13}$$

and also

$$y''(0)=-\frac{l}{EI}M_A^F,\ y'''(0)=\frac{1}{EI}(F_A^F+P\theta_A) \tag{14}$$

where M_A^F is the fixed end moment at A and F_A^F is the fixed-end force at A.

Combining (5), (6), (7), (13) and (14) and solving for M_A^F and F_A^F gives

$$M_A^F=\frac{EI}{l^2\Delta}(\sigma(l)g(l)-l\gamma(l)g'(l)) \tag{15}$$

$$F_A^F=-\frac{EI}{l^3\Delta}(\tau(l)g(l)-l\sigma(l)g'(l)) \tag{16}$$

where

$$\Delta=\frac{1}{\phi\psi(\phi^2+\psi^2)^2}(2\phi\psi(1-\cosh\phi\cos\psi)+(\phi^2-\psi^2)\sinh\phi\sin\psi), \tag{17}$$

Example. Consider the case where $R(x,t)$ is a uniformly distributed load having amplitude of total load W. In this case

$$R(x,t)=\frac{W}{l}e^{j\omega t} \tag{18}$$

and (5) gives

$$g(x)=\frac{Wl^3}{EI(\phi^2+\psi^2)}\left[\frac{1}{\phi^2}\left(\cosh\frac{\phi x}{l}-1\right)+\frac{1}{\psi^2}\left(\cos\frac{\psi x}{l}-1\right)\right] \tag{19}$$

Substitution of (19) into (15) gives the fixed-end moment

$$M_A^F=\frac{Wl}{(\phi^2+\psi^2)^2\Delta}\left[(\cosh\phi-\cos\psi)\left(\frac{1}{\phi^2}(\cosh\phi-1)+\frac{1}{\psi^2}(\cos\psi-1)-\left(\frac{1}{\phi}\sinh\phi-\frac{1}{\psi}\sin\psi\right)\right]^2=\alpha Wl \tag{20}$$

which, in the limit as ϕ and ψ tend to zero, gives the established static load result

$$M_A^F=-\frac{1}{12}Wl.$$

The coefficient α of Wl in Equation (20) is called an influence coefficient.

Table 1

λ	Influence coefficients, α, for M_A^F			Influence coefficients, β, for F_A^F		
	$\rho=0$	$\rho=0.5$	$\rho=1$	$\rho=0$	$\rho=0.5$	$\rho=1$
0	−0.083	−0.083	−0.089	− 0.500	− 0.500	− 0.500
0.5	−0.083	−0.083	−0.096	− 0.500	− 0.500	− 0.500
0.1	−0.083	−0.089	−0.100	− 0.500	− 0.501	− 0.501
1.5	−0.084	−0.092	−0.103	− 0.504	− 0.505	− 0.505
2	−0.086	−0.094	−0.105	− 0.511	− 0.513	− 0.515
2.5	−0.090	−0.099	−0.112	− 0.529	− 0.534	− 0.540
3	−0.098	−0.110	−0.126	− 0.567	− 0.578	− 0.594
3.5	−0.115	−0.134	−0.162	− 0.648	− 0.683	− 0.726
4	−0.161	−0.205	−0.293	− 0.862	− 0.981	− 1.218
4.1	−0.180	−0.238	−0.372	− 0.949	− 1.120	− 1.513
4.2	−0.206	−0.289	−0.526	− 1.069	− 1.335	− 2.089
4.3	−0.243	−0.374	−0.961	− 1.245	− 1.705	− 3.713
4.4	−0.305	−0.564	−9.770	− 1.531	− 2.490	−36.605
4.5	−0.425	−1.225	1.086	− 2.066	− 5.267	3.932
4.6	−0.737	−4.442	0.491	− 3.428	18.557	1.710
4.7	−2.961	0.735	0.307	−13.874	2.973	1.023
4.8	1.219	0.398	0.218	5.550	1.500	0.690
4.9	0.515	0.253	0.165	2.118	0.946	0.494
5.0	0.242	0.185	0.131	1.230	0.657	0.365

Substituting (19) into (16) gives

$$F_A^F = \frac{1}{\phi^2\psi^2(\phi^2+\psi^2)\Delta}\,[\phi\sinh\phi\,(\cos\psi-1)+ \psi\sin\psi(\cosh\psi-1)]\,W=\beta W \qquad (21)$$

which reduces to $-\frac{1}{2}W$ in the static case and here the coefficient β of W is called an influence coefficient.

In accordance with normal practice [1] the coefficient of W in (20) is called the Influence Coefficient of the Fixed-end Moment, M_A^F, and similarly for the coefficient of W in (21). Table 1 gives some values of these influence coefficients shown graphically in Figure 2(a)–(f), overleaf.

CONCLUSION

The table shows that the dynamic effect is negligible for a low frequency load, as may be expected, but, when the frequency is increased the influence coefficients change markedly.

REFERENCES

[1] A. Ghali, and A. M. Neville, *Structural Analysis*, 2nd Edn., London, Chapman and Hall, 1978.
[2] J. F. Doyle, 'The Dynamic Analysis of Redundant Framed Structures', *Ph.D. Thesis*, Heriot–Watt University, Edinburgh, 1975.
[3] I. D. Armstrong, 'The Natural Frequencies of Multi Storey Frames', *The Structural Engineer*, **47**, (8) (1969), pp. 299–308.

Paper Received 22 March 1980; Revised 11 November 1980.

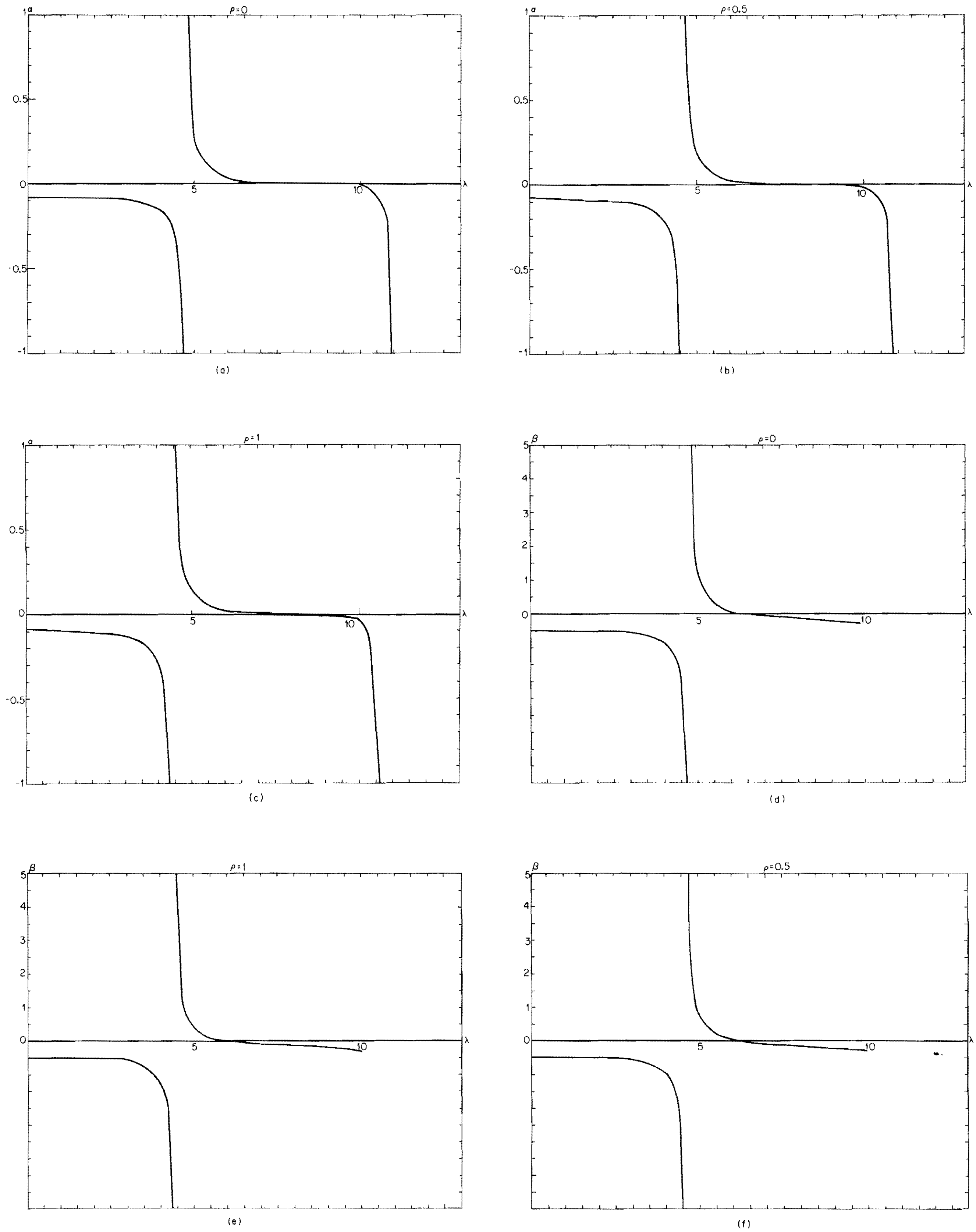

Figure 2