WALLMAN COMPACTIFICATION FOR BITOPOLOGICAL SPACES

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الخلاصة :

يهدف هذا البحث إلى دراسة تصميت (ولمان) من النوع bi $\lambda_\gamma-T_0$ والنوع bi $\lambda_\gamma-R_0$ والتي من أهم خصائصها: أن تصميت ولمان للفراغات الثنائية التوبولوجية يكون شبه bi $\lambda_\gamma-T_2$ اذا واذا فقط كان الفراغ الثنائي التوبولوجي (X, τ_1, τ_2) هو فراغ من النوع bi λ_γ المعتاد المحدود وأيضاً دراسة نوع جديد من التمدد على تصميت ولمان للفراغات الثنائية التوبولوجية.

ABSTRACT

The purpose of the present study is to construct a Wallman compactification for the larger classes which are bi $\lambda_\gamma-T_0$ and bi $\lambda_\gamma-R_0$. Some characterizations are given; also one of our main results is that the Wallman compactification of a bitopological space (X, τ_1, τ_2) is semi bi λ_γ-T_2 iff (X, τ_1, τ_2) is finite bi λ_γ-normal. A near type of extension over Wallman compactification of bispaces is also studied.

Key Words and Phrases: $(j,i)\lambda_\gamma$-open ($(j,i)\lambda_\gamma$-closed) set, $(j,i)\lambda_\gamma\lambda_\gamma^*$-homeomorphism, $(j,i)\lambda_\gamma$-compact, bi λ_γ-normal, Wallman compactification for bispaces, and bi λ_γ-continuous extension.

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WALLMAN COMPACTIFICATION FOR BITOPOLOGICAL SPACES

1. INTRODUCTION

For brevity we refer to a bitopological space (X, τ_1, τ_2) (see [1]) as a bispace. Throughout the present paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bispaces and $f: X \to Y$ represents a function. For a T_1-space (X,τ), consider Wallman compactification $(\chi,(X^*,\omega))$ [2] consisting of the set X^* of all ultraclosed filters on X. The topology ω on X^* generated by $\{U^* : U \in \tau\}$, where $U^* = \{F \in X^* : U \in \tau\}$ and the dense embedding $\chi : X \to X^*$ defined by setting $\chi(x) = \rho(x) = \{A \subseteq X, x \in A\}$. In 1980 [2] Asha Singal and Sunder Lal studied a Wallman type compactification for pairwise T_1 spaces. In 1980 [3] Dvalishvili constructed the Wallman compactification of the completely regular bispaces. We denote the closure (interior) operator with respect to (w.r.t.) the topologies τ_i $(i = 1,2)$ by $\mathrm{cl}_{\tau_i}(\mathrm{int}_{\tau_i})$ respectively. In 1979, Kasahara [4] defined an operation α on a topology τ on a non-empty set X to be a function of τ onto the power set $P(X)$ such that $G \subseteq G^\alpha$, for every $G \in \tau$, where G^α denotes the value of α at G. The family of all operations α is denoted by $O_{\tau(X)}$. In 1983, Abd El-Monsef *et al.* [5] generalized Kasahara's operation by introducing an operation on the power set $P(X)$ of a topological space (X,τ). A function $\Delta : P(X) \to P(X)$ (resp. $\delta : P(X) \to P(X)$) is said to be an operation on $P(X)$ of type I [5] (resp. of type II) [5], if $\mathrm{int}_\tau(A) \subseteq A^\Delta$ (resp. $\mathrm{cl}_\tau(A) \supseteq A^\delta$), for every $A \in P(X)$, where $A^\Delta(A^\delta)$ denotes the value of $\Delta(\delta)$ at A. The family of all operations of type I (resp. of type II) is denoted by $O_{P(X)}$ (resp. $O'_{P(X)}$).

2. BI λ_γ-CLOSED FILTER

Let $\{i \neq j, i,j = 1,2\}$, always.

Definition 2.1 [6]. A function $\lambda_\gamma : P(X) \to P(X)$ is called a (j,i) operation on $P(X)$ of a bispace (X, τ_1, τ_2), if λ_γ is an operation on $P(X)$ of type I and also of type II with respect to (X,τ_j) and (X, τ_i) respectively; *i.e.*, $\mathrm{int}_{\tau_j}(A) \subseteq A^{\lambda_\gamma}$ (resp. $\mathrm{cl}_{\tau_i}(A) \supseteq A^{\lambda_\gamma}$) for every $A \in P(X)$, where A^{λ_γ} denotes the value of λ_γ at A.

Definition 2.2 [6]. A subset A of a bispace (X, τ_1, τ_2) is called a (j,i) λ_γ-open set, if $A \subseteq A^{\lambda_\gamma}$. A is (j,i) λ_γ-closed set if $X \backslash A \subseteq (X \backslash A)^{\lambda_\gamma}$ or $A \supseteq X \backslash (X \backslash A)^{\lambda_\gamma}$.

It is easy to get corresponding statements for $(j,i)\lambda_\gamma$-closed in bispaces. In a bispace (X, τ_1, τ_2), the class of (j,i) λ_γ-open ((j,i) λ_γ-closed) will be denoted by (j,i) $\lambda_\gamma O(X)$((j,i) $\lambda_\gamma C(X)$).

Definition 2.3 [6]. Let A be a subset of a bispace (X, τ_1, τ_2). Then the intersection of all (j,i) λ_γ-closed sets containing A is called (j,i) λ_γ-closure of A and is denoted by (j,i) λ_γ-cl(A).

Definition 2.4 [7]. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (j,i) λ_γ-continuous if the inverse image of each σ_j-open set in Y is (j,i) λ_γ-open set in X.

Definition 2.5 [7]. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) if the image of every τ_j-open (τ_j-closed) set in X is a (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) set in Y, where $\lambda^*_\gamma: P(Y) \to P(Y)$.

Definition 2.6 [7]. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(j,i)\lambda_\gamma\lambda^*_\gamma$ continuous, if the inverse image of each (j,i) λ^*_γ-open set in Y is (j,i) λ_γ-open in X.

Definition 2.7 [7]. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $(j,i)\lambda_\gamma\lambda^*_\gamma$-open [$(j,i)$ $\lambda_\gamma\lambda^*_\gamma$-closed], if the image of each $(j,i)\lambda_\gamma$-open ((j,i) λ_γ-closed) set in X is a (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) set in Y.

Definition 2.8 [7]. Two bispaces X and Y are called (j,i) $\lambda_\gamma\lambda^*_\gamma$-homeomorphic equivalent, if there exists a bijective function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ such that f is (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous and $(j,i)\lambda_\gamma\lambda^*_\gamma$-open such function f is called $(j,i)\lambda_\gamma\lambda^*_\gamma$-homeomorphism.

Definition 2.9 [8]. A family $\beta_i(\beta_j)$ of $(i,j)\lambda_\gamma$-closed $((j,i)\lambda_\gamma$-closed) subsets of a bispace (X, τ_1, τ_2) is called $(i,j)\lambda_\gamma$-closed $((j,i)\lambda_\gamma$-closed) filter if:

(*i*) $\varnothing \notin \beta_i(\beta_j)$;

(*ii*) if $A, B \in \beta_i(\beta_j)$, implies $A \cap B \in \beta_i(\beta_j)$;

(*iii*) if $B \supseteq A \in \beta_i(\beta_j)$, implies $B \in \beta_i(\beta_j)$.

Definition 2.10 [8]. Let (X, τ_1, τ_2) be a bispace, $\beta_i(\beta_j)$ is (i,j) λ_γ-closed $((i,j)$ λ_γ-closed) filter, then $\beta = \beta_i \times \beta_j$ is called a bi λ_γ-closed filter on X if $(A,B) \in \beta$, implies $A \cap B \neq \varnothing$, for each $A \in \beta_i$, $B \in \beta_j$, where β_i, β_j are called the families of first and second coordinates respectively of the bi λ_γ-closed filter β.

If $\Phi = \Phi_i \times \Phi_j$ is another bi λ_γ closed filter, then we say $\Phi \geq \beta$ if $\beta_j \subseteq \Phi_j$. It is clear that $\geq$ is a partial order relation in the collection of all bi λ_γ-closed filters.

A maximal bi λ_γ-closed filter is called a bi λ_γ-ultraclosed filter (*i.e.* a bi λ_γ-closed filter that is not contained in any other bi λ_γ-closed filter).

Lemma 2.1 [8].

(*i*) For each bi λ_γ-closed filter Φ there is a bi λ_γ-ultraclosed filter Q containing Φ.

(*ii*) If $\Phi = \Phi_i \times \Phi_j$ is a bi λ_γ-ultraclosed filter and there is $A_j \in (j,i)$ λ_γ-closed sets in X such that $\Phi_j \cup A_j$ is a bi λ_γ-closed filter and $A_j \cap A \neq \varnothing$ for each $A \in \Phi_j$, then $A_j \in \Phi$.

(*iii*) If $U, V \in (j,i)$ λ_γ $C(X)$, Φ is a bi λ_γ-ultraclosed filter, then $U \cup V \in \Phi$ implies $U \in \Phi$ or $V \in \Phi$.

(*iv*) If Φ_1, Φ_2 are bi λ_γ-ultraclosed filters on X, $\Phi_1 \neq \Phi_2$, there exists a (i,j) λ_γ-closed set C_1, a (j,i) λ_γ-closed set C_2, such that $C_1 \in \Phi_1$, $C_2 \in \Phi_2$, $C_1 \cap C_2 = \varnothing$.

Definition 2.11 [9]. If $E \subseteq X$ is finite joint closed, then there is an open dual family $\{(U_\propto, V_\propto)$: $\propto \in A$: so $E = X \backslash \cup \{U_\propto \cap V_\propto) : \propto \in A\}$ and this family is finite.

Definition 2.12. A bispace (X, τ_1, τ_2) is called finite bi λ_γ-normal if for any finite joint closed set E and any (j,i) λ_γ-closed set F with $E \cap F = \varnothing$, there exist $U \in (i,j)\lambda_\gamma O(X)$, $V \in (j,i)$ $\lambda_\gamma O(X)$ with $U \cap V = \varnothing$, and $E \subseteq U$, $F \subseteq V$ $(E \subseteq V, F \subseteq U)$.

For $x \in X$, $\beta_x = \beta_{ix} \times \beta_{jx}$, where $\beta_{ix}(\beta_{jx})$ is a family of all subsets of $\beta_i(\beta_j)$ containing x such that if $(A,B) \in \beta_x$ then $A \cap B \neq \varnothing$, for each $A \in \beta_{ix}$, $B \in \beta_{jx}$.

Proposition 2.1 [8]. $\beta_x = \beta_{ix} \times \beta_{jx}$ is a bi λ_γ-ultraclosed filter.

Definition 2. 13 [8]. A bispace (X, τ_1, τ_2) is called:

(*i*) **bi λ_γ-R_o** if each (j,i) λ_γ open set O and each $x \in O$, (j,i) λ_γ-cl$\{x\} \subseteq O$;

(*ii*) **bi λ_γ-T_o** if for each two distinct points x, y of X, there exists a (i,j) λ_γ-open set U such that $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$;

(*iii*) **bi λ_γ R_1** if for each two distinct points x, y of X such that (j,i) λ_γ-cl$\{x\} \neq (j,i)$ λ_γ-cl$\{y\}$, there exists a $(i,j)\lambda_\gamma$-open set U and a $(j,i)\lambda_\gamma$-open set V such that (j,i) λ_γ-cl$\{x\} \subseteq U$, (j,i) λ_γ-cl$\{y\} \subseteq V$, $U \cap V = \varnothing$;

(*iv*) **bi λ_γ T_1** if for two distinct points x, y of X, there exists a $(i,j)\lambda_\gamma$-open set U containing x to which y does not belong and a $(j,i)\lambda_\gamma$-open set V containing y to which x does not belong;

(*v*) **bi λ_γ-T_2** if for each two distinct points x, y of X, there exists a $(i,j)\lambda_\gamma$-open set U and a $(j,i)\lambda_\gamma$-open set V such that $x \in U$, $y \in V$, $U \cap V = \varnothing$.

Proposition 2.2. If $x \in X$, then $\cap \{\beta_{ix} \cup \beta_{jx}\} = \{x\}$.

3. WALLMAN COMPACTIFICATION FOR BISPACES

Definition 3.1 [7]. A bispace (X, τ_1, τ_2) is called (j,i) λ_γ-compact if every (j,i) λ_γ-open cover X has a finite subcover.

A bispace which is bi λ_γ-R_k and bi λ_γ-T_k is called semi bi-λ_γ-T_{k+1}, $k \in \{0,1,2\}$.

Definition 3.2 [7]. A subset A of a bispace (X, τ_1, τ_2) is called $(j,i)\lambda_\gamma$ dense if for any $(j,i)\lambda_\gamma$-open set G such that $G \cap A \neq \emptyset$.

Definition 3.3 [7]. A bispace (X, τ_1, τ_2) is called $(j,i)\lambda_\gamma\lambda_\gamma^*$-embeddable in a bispace (Y,s,t) if there exists a $(j,i)\lambda_\gamma\lambda_\gamma^*$-homeomorphism from (X, τ_1, τ_2) onto a $(j,i)\lambda_\gamma$ dense subpace (Y,s,t).

Definition 3.4 [8]. If $\gamma X = \{\beta : \beta$ is a bi λ_γ-ultraclosed filter on $X\}$, K_i (resp. K_j) be a (i,j) λ_γ-closed (resp-(j,i) λ_γ-closed) set. We let $K_i^* = \{\beta \in \gamma X, (K_i, X) \in \beta\}$, $K_j^* = \{\beta \in \gamma X, (X, K_j) \in \beta\}$, Then we can define $C_i = \{K_i^*, K_i$ is (i,j) λ_γ-closed$\}$, $C_j = \{K_j^*, K_j$ is (j,i) λ_γ-closed$\}$.

Proposition 3.1 [8]. $C_i(C_j)$ is a base for the closed subsets of a topology P_1 (a topology p_2) on γX.

Definition 3.5 [8]. A bispace $(X^*, \tau_1^*, \tau_2^*)$ is called a (i,j) λ_γ-compactification of a bispace (X, τ_1, τ_2) if there exists a (j,i) λ_γ-embedding function from (X, τ_1, τ_2) onto $(X^*, \tau_1^*, \tau_2^*)$ and if $(X^*, \tau_1^*, \tau_2^*)$ is (j,i) λ_γ-compact.

Theorem 3.1 [8]. A bispace $(\gamma X, P_1, P_2)$ is (j,i) λ_γ-compactification of a bispace (X, τ_1, τ_2).

We called $(\gamma X, P_1, P_2)$ its Wallman compactification.

Theorem 3.2 [8]. Let (X, τ_1, τ_2) be a bispace and $(\gamma X, P_1, P_2)$ its Wallman compactification. Then the following statements hold:

(*i*) for each $(i,j)\lambda_\gamma$-closed set K in X, we have $K_i^* = \text{cl}_{p_i} f(K)$ and with a corresponding result for $(j,i)\lambda_\gamma$-closed sets;

(*ii*) for $(i,j)\lambda_\gamma$-closed sets K_1, K_2 we have $(K_1 \cap K_2)^* = K_1^* \cap K_2^*$, $(K_1 \cup K_2)^* = K_1^* \cup K_2^*$, and with the corresponding results for $(j,i)\lambda_\gamma$-closed sets;

(*iii*) for each $(i,j)\lambda_\gamma$-closed set $K \subseteq X$ and a $(j,i)\lambda_\gamma$-closed set $T \subseteq X$ we have $K_i^* \cap T_i^* \neq \emptyset$ iff $K \cap T \neq \emptyset$.

Proposition 3.2 [8]. The Wallman compactification $(\gamma X, P_1, P_2)$ is bi λ_γ-T_1.

Theorem 3.3. Let (X, τ_1, τ_2) be a bispace and $(\gamma X, P_1, P_2)$ its Wallman compactification. Then $(\gamma X, P_1, P_2)$ is semi bi λ_γ-T_2 iff (X, τ_1, τ_2) is finite bi λ_γ-normal.

Proof. Let $\delta, \beta \in \gamma X$ and $\delta \neq \beta$. This implies $\delta \not\subseteq \beta$ and $\beta \not\subseteq \delta$. If $\delta \not\subseteq \beta$, then there exist $(A_1, A_2) \in \delta$ such that $(A_1, A_2) \notin \beta$. This gives that $(A_1, X) \notin \beta$ or $(X, A_2) \notin \beta$. Hence there exist $B_1 \in \tau_1$-closed, $B_2 \in \tau_2$-closed such that $(B_1, B_2) \in \beta$ and $A_1 \cap B_1 \cap B_2 = \emptyset$ or $A_2 \cap B_1 \cap B_2 = \emptyset$, $E = B_1 \cap B_2 = X\backslash[(X\backslash B_1) \cup (X\backslash B_2)]$ is a finite closed set, and $A_1 \cap E = \emptyset$ or $A_2 \cap E = \emptyset$. Then we have from hypothesis $(i,j)\lambda_\gamma$-open and $(j,i)\lambda_\gamma$-open sets which are disjoint and contain respectively A_1, E (or E, A_2). From these sets we can obtain the desired (i,j) λ_γ-open and $(j,i)\lambda_\gamma$-open sets in γX which verifies that $(\gamma X, P_1, P_2)$ is bi λ_γ-T_2. In the same way, if $\delta, \beta \in \gamma X$, $\delta \notin \text{cl}_{p_1}\beta$ (resp. $\delta \notin \text{cl}_{p_2}\beta$) then we have a $(i,j)\lambda_\gamma$-open set U and a $(j,i)\lambda_\gamma$-open set V such that $\delta \in U$, $\beta \in V$ (resp. $\delta \in V$, $\beta \in U$) and $U \cap V = \emptyset$. This means $(\gamma X, P_1, P_2)$ is semi-bi λ_γ-T_2. Conversely, let $E \cap F \subseteq X$, $E \cap F = \emptyset$, where F is an $(i,j)\lambda_\gamma$-closed set, $E = X\backslash \cup (U_\alpha \cap V_\alpha)$, where U_α is a τ_i-open set, $V\alpha$ is a τ_j-open set, $\alpha = 1, 2 ...... n$. We can write $E = \cup(F_k \cap K_k)$ where F_k are τ_i-closed sets and K_k are τ_j-closed sets, $k = 1,2... m$. It follows that $F_k \cap K_k \cap F = \emptyset$, for each $k = 1,2... m$. By Theorem 3.2 (*i*), we get $\text{cl}_{p_1}(F_k) \cap \text{cl}_{p_2}(K_k) \cap \text{cl}_{p_1}(F) = \emptyset$. If $\delta \in \text{cl}_{p_1}(F_k) \cap \text{cl}_{p_2}(K_k)$ and $\Im \in \text{cl}_{p_1}(F)$, then $\delta \notin \text{cl}_{p_1}\Im$.

Since (X, τ_1, τ_2) is semi bi λ_γ-T_2, we have $\delta \in U \in (i,j)$ $\lambda_\gamma O(\gamma X)$, $\Im \in V \in (j,i)$ $\lambda_\gamma O(\gamma X)$ and $U \cap V = \emptyset$. We know that $\text{cl}_{p_1}(F_k) \cap \text{cl}_{p_2}(K_k)$ and $\text{cl}_{p_1}(F)$ are $(j,i)\lambda_\gamma$-compact ($(i,j)\lambda_\gamma$-compact). It follows that $\text{cl}_{p_1}F_k \cap \text{cl}_{p_2}K_k$ and $\text{cl}_p F$ can be covered respectively by $(j,i)\lambda_\gamma$-open ((i,j) λ_γ-open) sets which are disjoint. The inverse images of these sets separate E, F. Hence $(\gamma X, P_1, P_2)$ is bi λ_γ-normal. A similar proof holds when F is (j,i) λ_γ-closed.

Theorem 3.4. Let $(X^*, \tau_1^*, \tau_2^*)$ be a bi λ_γ-T_1 compactification of a bispace (X, τ_1, τ_2). Then $(X^*, \tau_1^*, \tau_2^*)$ and $(\gamma X, P_1, P_2)$ are $(j,i)\lambda_\gamma\lambda_\gamma^*$-homeomorphic iff the following conditions are satisfied (f_1 is the (j,i) $\lambda_\gamma\lambda_\gamma^*$ embedding function from (X, τ_1, τ_2) to $(X^*, \tau_1^*, \tau_2^*)$).

(*i*) $(\mathrm{cl}_{\tau_1^*} f(K)$: K is a (i,j) λ_γ-closed set in $X)$ is a base for the closed subsets of the topology τ_1^* and with a similar result for the topology τ_2^*.

(*ii*) Let K_1, K_2 be a (i,j) λ_γ-closed sets in X and F_1, F_2 be (j,i) λ_γ-closed sets in X. Then:

$$\mathrm{cl}_{\tau_1^*} f_1 (K_1 \cap K_2) = \mathrm{cl}_{\tau_1^*} f_1 (K_1) \cap \mathrm{cl}_{\tau_1^*} f_1 (k_2);$$

$$\mathrm{cl}_{\tau_2^*} f_1 (F_1 \cap F_2) = \mathrm{cl}_{\tau_2^*} f_1 (F_1) \cap \mathrm{cl}_{\tau_2^*} f_1 (F_2).$$

(*iii*) Let K be a (i,j) λ_γ-closed set and F a (j,i) λ_γ-closed set in X. Then $F \cap K \neq \varnothing$ iff $\mathrm{cl}_{\tau_1^*} f_1(K) \cap \mathrm{cl}_{\tau_2^*} f_1(F) \neq \varnothing$.

Proof. We assume that the conditions are satisfied. We want to see that $(X^*, \tau_1^*, \tau_2^*)$ and $(\gamma X, P_1, P_2)$ are $(j,i)\lambda_\gamma\lambda_\gamma^*$-homeomorphic. We define a function $g: (\gamma X, P_1, P_2) \to (X^*, \tau_1^*, \tau_2^*)$ in the following way. For each $\mathfrak{I} \in \gamma X$, $\{\mathrm{cl}_{\tau_2^*} f(E), \mathrm{cl}_{\tau_1^*} f(F)\}_{(E,F) \in \mathfrak{I}}$. Define a bi λ_γ-closed filter $\mathfrak{I}_1$ which has a cluster point x in $(X^*, \tau_1^*, \tau_2^*)$; this point is unique. Otherwise, if $y \in \cap_{(E_1,F_1) \in \mathfrak{I}_1} E_1 \cap F_1$ and $y \neq x$, $\mathfrak{I} \subseteq \mathfrak{I}_1$, then we get a contradiction from conditions (*ii*) and (*iii*), because $(X^*, \tau_1^*, \tau_2^*)$ is bi λ_γ-T_1 and $\mathfrak{I}$ is a bi λ_γ-ultraclosed filter. This leads us to set $g(\mathfrak{I}) = x$, so defining the function g it is clear that $g^{-1}(\mathrm{cl}_{\tau_1^*} f_1(K)) = (\mathrm{cl}_P f_1(K))$ and $g^{-1}(\mathrm{cl}_{\tau_2^*} f_1(K)) = (\mathrm{cl}_{p_2} f_1(K))$. From the conditions and these equations we can get (j,i) $\lambda_\gamma\lambda_\gamma^*$-continuity of g. Let $x_1 \in X^*$. We define a bi λ_γ-closed filter $\beta_1 = \{(F, K)$: $F \subseteq X$ is (j,i) λ_γ-closed, $K \subseteq X$ is (i,j) λ_γ-closed, $x_1 \in (\mathrm{cl}_{\tau_1^*} f_1 (K)) \cap (\mathrm{cl}_{\tau_2^*} f_1 (F))$. If $\mathfrak{I}_1$ is a bi λ_γ-ultraclosed filter containing β_1 then $g(\mathfrak{I}_1) = y_1$, implies that $x_1 = y_1$. It follows that g is onto.

We now see that g is a one-to-one function. Let $\mathfrak{I}, \beta \in \gamma X$ and $g(\beta) = g(\mathfrak{I}) = x$. If $\beta = \mathfrak{I}$ then, from maximality of $\mathfrak{I}$ and β, there exist $B_1 \in \tau_1$-closed, $B_2 \in \tau_2$-closed such that $(B_1, B_2) \in \beta$, $(F_1, F_2) \in \mathfrak{I}$ and $B_1 \cap B_2 \cap F_1 \cap F_2 = \varnothing$. The third condition gives us $\mathrm{cl}_{\tau_2^*} f_1 (B_1) \cap \mathrm{cl}_{\tau_2^*} f_1(F_1) \cap \mathrm{cl}_{\tau_1^*} f_1(F_2) \cap \mathrm{cl}_{\tau_1^*} f_1(B_2) = \varnothing$. But from the definition of g we have $x \in \mathrm{cl}_{\tau_2^*} f_1(B_1) \cap \mathrm{cl}_{\tau_2^*} f_1(F_1) \cap \mathrm{cl}_1^* f_1(F_2) \cap \mathrm{cl}_{\tau_1^*} f_1(B_2) \neq \varnothing$; this is a contradiction. It can be shown that $g(K_i^*) = \mathrm{cl}_{\tau_2^*} f_1(K)$, $g(F_j^*) = \mathrm{cl}_{\tau_2^*} f_1(F)$, for each $K \subseteq X$ is an (i,j) λ_γ-closed set and $F \subseteq X$ is (j,i) λ_γ-closed. This means that g is a $(j,i)\lambda_\gamma\lambda_\gamma^*$-closed function (*i.e.* g^{-1} is $(j,i)\lambda_\gamma\lambda_\gamma^*$ continuous). Finally it is clear that $g_o f = f_1$. Theorem 4. 2 gives the necessity.

Proposition 3.3. Let $(X^*, \tau_1^*, \tau_2^*)$ be a semi bi λ_γ-T_2 bispace which is a compactification of (X, τ_1, τ_2). Then $\wp_{\tau_1^*} = \{\mathrm{cl}_{\tau_1^*} F$: F is $(i,j)\lambda_\gamma$-closed in $X\}$ is a base for the closed subsets of the topology τ_1^* and in the same way $\wp_{\tau_2^*} = \{\mathrm{cl}_{\tau_2^*} F$: F is (j,i) λ_γ-closed in $X\}$ is a base for the closed subsets of the topology τ_2^*.

Proof. Let $F_1 \subseteq X^*$ be a τ_1^*-closed set, $x \in X^*$ and $x_1 \notin F_1$, $(X^*, \tau_1^*, \tau_2^*)$ is semi-$\lambda_\gamma T_2$ and *i* F_1 is (i,j) λ_γ-compact. It follows that $x_1 \in X^* \backslash F_1 = U_1$ is τ_1^*-open, $F_1 \subseteq V_1 \in (j,i)$ λ_γ-open in X^* such that $U_1 \cap V_1 = \varnothing$. Let $F = X \backslash [X^* \backslash \mathrm{cl}_{\tau_1^*} V_1 \cap X]$. Since $\mathrm{cl}_{\tau_1^*} V_1 \subseteq X^* \backslash U_1$, implies $U_1 \cap \mathrm{cl}_{\tau_1^*} V_1 = \varnothing$, hence $U_1 \cap \mathrm{cl}_{\tau_1^*} F = \varnothing$, implies $x \notin \mathrm{cl}_{\tau_1^*} F$.

Let us assume that F_1 is not a subset of $\mathrm{cl}_{\tau_1^*} F$. Then we can take $y \in F_1$ such that $y \notin \mathrm{cl}_{\tau_1^*} F$. There exists a τ_1^*-open set $U_2 = X^* \backslash \mathrm{cl}_{\tau_1^*} F$ such that $y_1 \in U_2$ and $U_2 \cap F = \varnothing$. We know that $y_1 \in F_1 \subseteq V_1$. Then $y_1 \in U_2 \cap V_1 \neq \varnothing$ so $y_1 \in \mathrm{cl}_{\tau_1^*} V_1$, but $y_1 \notin F$, implies $y_1 \in X \backslash F = X^* \backslash (\mathrm{cl}_{\tau_1^*} V_1 \cap X) = X^* \backslash \mathrm{cl}_{\tau_1^*} V_1$, implies $y_1 \notin \mathrm{cl}_{\tau_1^*} V_1$. We have a contradiction. Hence we have $F_1 \subseteq \mathrm{cl}_{\tau_1^*} F$.

Definition 3.6 [1]. A bispace (X, τ_1, τ_2) is called pairwise regular if for any τ_i-closed set F, $x \notin F$ there exist two disjoint upon sets U, V, such that $x \in U$, $F \subseteq V$.

Proposition 3.4. Let $(X^*, \tau_1^*, \tau_2^*)$ be a semi bi λ_γ-T_2 space which is a compactification of (X, τ_1, τ_2). Let ρ be a (j,i) $\lambda_\gamma\lambda_\gamma^*$ continuous function on X^* to γX such that $\rho(x) = \{x\}$, for each $x \in X$. Also, f_1, f_2 are (j,i) $\lambda_\gamma\lambda_\gamma^*$-embedding functions on X to respectively $X^*, \gamma X$. If $F \subseteq X$ is a (i,j) λ_γ-closed set, then $\rho^{-1}(\mathrm{cl}_1^*(f_1(F))) \subseteq (\mathrm{cl}_{\tau_1^*}(f_1(F))$.

Proof. If there exists a member $x \in \rho^{-1}(\mathrm{cl}_{\tau_1^*}(f_1(F)))$ and $x \notin (\mathrm{cl}_{\tau_1^*}(f_2(F)))$, then $\rho(x) = \beta = \mathrm{cl}_{\tau_1^*}(f_1(F))$ and $(X, F) \in \beta$. It can be shown that $(X^*, \tau_1^*, \tau_2^*)$ is pairwise regular; then there exists $U_1 \in \tau_1^*$, $V_1 \in \tau_2^*$ such that $x \in U_1$, $\mathrm{cl}_{\tau_1^*}(f_2(F)) \subseteq V_1$, $U_1 \cap V_1 = \varnothing$. We can obtain a τ_2-closed set $K = (X \cap (X^* \backslash V_1))$ in X such that $\mathrm{cl}_{\tau_1^*}(f_2(F)) \subseteq V_1 \subseteq X^* \backslash U_1$ and $(X^* \backslash U_1) \cap X = F_1$ is a τ_1-closed set in X. Then we have $F \subseteq X\backslash K \subseteq F_1$, so $F \cap (X^*\backslash V_1) = \varnothing$, $K \cap F_1 = \varnothing$, implies $(X, F) \notin \beta$ and $(K, X) \notin \beta$, $\beta \in \gamma X\backslash K_{\tau}^*$, $x \in \rho^{-1}(\gamma X\backslash K_{\tau_2^*})$ is a (2,1) λ_γ-open set in X^* (because $\rho(x) = \beta$, ρ is $(j,i)\lambda_\gamma\lambda_\gamma^*$ continuous). On the other hand, $x \in X^*\backslash\mathrm{cl}_\tau^* F_1$, then there exists a member $z \in \rho^{-1}(\gamma X\backslash K_{\tau_2^*}) \cap ((X^*\backslash\mathrm{cl}_\tau^* F) \cap X)$, but X is $(j,i)\lambda_\gamma$-dense subset of X^* and γX. Thus $\rho^{-1}(\gamma X\backslash K_{\tau_2^*}) \cap X \neq \varnothing$, $(X^*\backslash\mathrm{cl}_\tau^* F_1) \cap X) \neq \varnothing$ and $\rho^{-1}(\gamma X\backslash K_{\tau_2^*}) \cap (X^*\backslash\mathrm{cl}_{\tau_1^*} F_1) \cap X \neq \varnothing$. It follows that $z \notin K \cup F_1 = X$ and this a contradiction $(x \notin \mathrm{cl}_{\tau_1^*}(f_1(F))$. Hence the proposition is proved.

Theorem 3.5. If a bispace (X, τ_1, τ_2) is finite bi λ_γ normal then $(\gamma X, P_1, P_2)$ is the projectively largest semi bi λ_γ-T_2 compactification of (X, τ_1, τ_2).

Proof. Let ρ be a (j,i) $\lambda_\gamma\lambda_\gamma^*$-continuous function from any semi bi λ_γ-T_2 compactification $(X^*, \tau_1^*, \tau_2^*)$ of (X, τ_1, τ_2) to $(\gamma X, P_1, P_2)$. It will be enough to show that ρ is a $(j,i)\lambda_\gamma\lambda_\gamma^*$-homeomorphism.

(*i*) Let $\beta \in \gamma X$ and define $\delta = \{(\mathrm{cl}_{\tau_2^*}F, \mathrm{cl}_{\tau_1^*}K) : (F, K) \in \beta\}$. δ has a cluster point $x_1 \in X$; let $\rho(x_1) = \beta$. If $\rho(x_1) = \beta_1$ then, $\beta \neq \beta_1$ we have a member $(F, K) \in \beta$, $(F, K) \notin \beta_1$. Then $\beta_1 \notin (\gamma X\backslash F_{\tau_2^*})$ or $\beta_1 \notin (\gamma X\backslash K_{\tau_1^*})$. It follows that $x_1 \in \rho^{-1}(\gamma X\backslash F_\tau^*)$ is a $(2,1)\lambda_\gamma$-open set in X^* or $x_1 \in \rho^{-1}(\gamma X\backslash K_\tau^*)$ is a $(1,2)\lambda_\gamma$-open set in X^*, $(F, K) \in \beta$ gives that $x_1 \in (\mathrm{cl}_{\tau_2^*}F \cap \mathrm{cl}_{\tau_1^*}K)$ and $\rho^{-1}(\gamma X\backslash F_{\tau_2^*}) \cap K \neq \varnothing$, $\rho^{-1}(\gamma X\backslash K_{\tau_1^*}) \cap F \neq \varnothing$. But these are impossible. It means $\beta_1 = \beta$ and ρ is onto.

(*ii*) Let $x_1, x_2 \in X^*$, $\rho(x_1) = \rho(x_2) = \beta$ and $x_1 \neq x_2$. We have $x_1 \notin \mathrm{cl}_\tau^* X_2$ or $x_2 \notin \mathrm{cl}_\tau^* X_2$ (where X_2 is another compactification of X). Since X is bi λ_γ-T_1, there exists a $(j,i)\lambda_\gamma$-closed set F in X by Proposition 3.3 such that $x_1 \notin \mathrm{cl}_{\tau_1^*} F$, $\mathrm{cl}_{\tau_2^*} X_2 \subseteq \mathrm{cl}_{\tau_1^*} F$. It follows by Proposition 3.4 that $\rho^{-1}(\mathrm{cl}_{p_1} F) \subseteq \mathrm{cl}_{\tau_1^*} F$. Then $x_1 \notin \rho^{-1}(\mathrm{cl}_{p_1} F)$, $\beta = \rho(x_1) = \rho(x_2) \in \mathrm{cl}_{p_1} F$; on the other hand $x_2 \in \mathrm{cl}_{\tau_1^*} X_2 \subseteq \mathrm{cl}_\tau^* F$ gives that $\rho(x_2) \in \mathrm{cl}_{p_1} F$; this is a contradiction. It means that ρ is one to one.

(*iii*) Let F be a (i,j) λ_γ-closed set in X^* and $\rho(F)$ not be $(i,j)\lambda_\gamma$-closed in γX. Take $\beta \in \mathrm{cl}_{p_1}\rho(F)$ and $\beta \notin \rho(F)$, *i.e.*, $\mathrm{cl}_{p_1}\rho(F) \not\subseteq \rho(F)$. There exists a (i,j) λ_γ-closed set in X; by Proposition 4.3 we have $\rho^{-1}(\beta) \notin \mathrm{cl}_\tau^* K$ and $F \subseteq \mathrm{cl}_\tau^* K$. It follows that $\rho(F) \subseteq \mathrm{cl}_{p_1}\rho(K)$ and $\beta \in \mathrm{cl}_{p_1}\rho(F) \subseteq \mathrm{cl}_{p_1}\rho(K)$, implies $\rho^{-1}(\beta) \in \rho^{-1}(\mathrm{cl}_{p_1} K)$. On the other hand, $\rho^{-1}(\beta) \notin \rho^{-1}(\mathrm{cl}_{\tau_1^*} K)$ contradicts Proposition 3.4. It means that ρ is a $(i,j)\lambda_\gamma$-closed function. Hence the theorem is proved.

Remark 3.1.

(*i*) If $i = j$ then we return to the ordinary case of Wallman compactification as in [10].

(*ii*) If every τ_j-closed set is a τ_i-open set in bispace (X, τ_1, τ_2), then we return to the pairwise case of Wallman compactification as in [3].

4. THE BI λ_γ-CONTINUOUS EXTENSION OVER THE WALLMAN COMPACTIFICATION FOR BISPACES

Definition 4.1 [8]. A bispace (X, τ_1, τ_2) is called bi λ_γ-regular if for each τ_j-closed set F, $x \notin F$, there exist two disjoint sets U, V such that $x \in V \in (j,i)\ \lambda_\gamma O(X)$, $F \subseteq U \in (i,j)\ \lambda O(X)$.

Lemma 4.1. Let (X, τ_1, τ_2) be bi λ_γ-regular. If $x, y \in X$, $x \in A \in \tau_1$-open and $y \notin A$ then there exist two disjoint sets U, V such that $x \in U \in (i,j)\lambda_\gamma O(X)$, $y \in V \in (j,i)\ \lambda_\gamma O(X)$, $\mathrm{cl}_{\tau_2} U \cap \mathrm{cl}_{\tau_1} V = \varnothing$.

The following theorem is a generalization of the extension problem in Engelking [11] for bispace.

Theorem 4.1. Let A be a (j,i) λ_γ-dense subspace of X and f is a $(j,i)\lambda_\gamma\lambda_\gamma^*$-continuous function of A to a semi bi λ_γ-$T_2(j,i)\lambda_\gamma$-compact space Y. The function f has a $(j,i)\lambda_\gamma\lambda_\gamma^*$-continuous extension f' over X iff for every pair $F \in (i,j)\lambda_\gamma C(Y)$, $K \in (j,i)\ C(Y)$, $f \cap K = \varnothing$, then the inverse images $\mathrm{cl}_{\tau_2}(f^{-1}(K)) \cap \mathrm{cl}_{\tau_1}(f^{-1}(F)) = \varnothing$.

Proof. Let f' be an extension of f and $F \in (i,j)\lambda_\gamma C(Y)$, $K \in (j,i)\ C(Y)$, $F \cap K = \varnothing$, then $\varnothing = f'^{-1}(F \cap K) = f'^{-1}(F) \cap f'^{-1}(K) = \mathrm{cl}_{\tau_1}(f'^{-1}(F)) \cap \mathrm{cl}_{\tau_2}(f^{-1}(K))$ and this gives that $\mathrm{cl}_{\tau_1}(f^{-1}(F)) \cap \mathrm{cl}_{\tau_2}(f^{-1}(K)) = \varnothing$. We shall prove that the condition is sufficient.

Let $x \in X$, denote by $\beta(x)$ the family of all (j,i) λ_γ-neighborhood of x and define $\Im(x) = \{(\mathrm{cl}_t f(U \cap A), (\mathrm{cl}_s f(V \cap A)\}$, $(U,V) \in \beta(x)$. $\Im(x)$ is a base for a bi λ_γ-closed filter in Y. Then it has a cluster point z in (Y,s,t). We shall show that z is unique and define $f'(x) = z$. At first if $f'(x) \in S \in (i,j)\lambda_\gamma\, O(Y)$, then $\cap\, \mathrm{cl}_t\, f(U \cap A) \subseteq S$ and if $f'(x) \in T \in (j,i)\lambda_\gamma\, O(Y)$, then $\cap\, \mathrm{cl}_s(f(V \cap A)) \subseteq T$. If we take a member $y \in \cap\, \mathrm{cl}_t(f(U \cap A))$, $y \notin S$, there exists $S_1 \in (i,j)\lambda_\gamma O(Y)$, $T_1 \in (j,i)\lambda_\gamma O(Y)$, by Lemma 4.1, such that $f'(x) \in S_1$, $y \in T_1$, $\mathrm{cl}_t S_1 \cap \mathrm{cl}_s T_1 = \varnothing$, $\mathrm{cl}_{\tau_2}(f^{-1}(S_1)) \cap \mathrm{cl}_{\tau_1}(f^{-1}(T_1)) = \varnothing$. By assumption, $x \in X$ gives that $x \in X \backslash \mathrm{cl}_{\tau_2}(f^{-1}(S_1))$ or $x \in X \backslash \mathrm{cl}_{\tau_1}(f^{-1}(T_1))$.

(*i*) If $x \in X \backslash \mathrm{cl}_{\tau_1}(f^{-1}(T_1))$ then $y \in \mathrm{cl}_t(f(A \cap X \backslash \mathrm{cl}_{\tau_1}(f^{-1}(T_1))$ and $T_1 \cap (f(A \cap X \backslash \mathrm{cl}_{\tau_1}(f^{-1}(T_1)) \neq \varnothing$. But this is impossible.

(*ii*) If $x \in X \backslash \mathrm{cl}_{\tau_2}(f^{-1}(S_1))$ then $f'(x) \in \mathrm{cl}_s(f(A \cap X \backslash \mathrm{cl}_{\tau_2}(f^{-1}(S_1))$ and $S_1 \cap (f(A \cap X \backslash \mathrm{cl}_{\tau_2}(f^{-1}(S_1)) \neq \varnothing$. This is also impossible. Hence $\cap_{u \in \beta u(x)} \mathrm{cl}_s f(U \cap A) \subseteq S$ and converse is similar: $x \in \mathrm{cl}_t z \cap \mathrm{cl}_s z \subseteq \cap_{v \in \beta v(x)} \mathrm{cl}_s(f(V \cap A)) \cap \cap_{u \in \beta u(x)} \mathrm{cl}_t(f(U \cap A))$, since z is a cluster point of $\Im(x)$. Let us take a member $y_0 \in \cap_{v \in \beta v(x)} \mathrm{cl}_s(f(V \cap A)) \cap \cap_{u \in \beta u(x)} \mathrm{cl}_t(f(U \cap A))$ and $y_0 \in \mathrm{cl}_s\{z\}$.

We know that X is preseparated; then there exists $G \in (j,i)\, \lambda_\gamma O(Y)$, $H \in (i,j)\lambda_\gamma\, O(Y)$, such that $y_0 \in G$, $z \in H$, and $G \cap H = \varnothing$. It can be written that $\cap_{v \in \beta v(x)} \mathrm{cl}_s(f(V \cap A)) \subseteq H$ for $x \in H$; it follows that $y \in G \cap H \neq \varnothing$, which is a contradiction. This means that $y \in \mathrm{cl}_s\{z\}$. In the same way we have $y_0 \in \mathrm{cl}_s\{z\}$. Hence $\cap_{v \in \beta v(x)} \mathrm{cl}_s(f(V \cap A)) \cap \cap_{u \in \beta u(x)} \mathrm{cl}_t(f(U \cap A)) = \mathrm{cl}_t z \cap \mathrm{cl}_s z = \{z\}$ by bi λ_γ-T_1 of X. We shall show that f is $(j,i)\lambda_\gamma\lambda^*_\gamma$-continuous. Let S be a (i,j) λ_γ-neighborhood of $f(x)$ then $f(x) \in \cap\, (\mathrm{cl}_t\, f(U \cap A)) \subseteq S$. We have for $U_1, \ldots U_n \in \beta_u(x)$, $\mathrm{cl}_t f(U_1 \cap A) \cap \ldots \mathrm{cl}_t f(U_n \cap A) \subseteq S$. Since $Y \backslash S$ is (i,j) λ_γ-compact, $U = U_1 \ldots U_n \in \beta_u(x)$ gives that $\mathrm{cl}_t\, f\,(U \cap A)) \subseteq S$. We have $f(x) \in \mathrm{cl}_t\, f\,(U \cap A)) \subseteq S$ for each $x \in U$. This means that $f'(U) \subseteq S$ and f' is (i,j) $\lambda_\gamma\lambda^*_\gamma$-continuous. In the same way f is $(i,j)\lambda_\gamma\lambda^*_\gamma$-continuous. Hence the theorem is proved.

Theorem 4.2. Let (X, τ_1, τ_2) be a semi bi λ_γ-T_1 space and (Y,s,t) be a semi bi $\lambda_\gamma T_2(j,i)\lambda_\gamma$-compact space. If f is a $(j,i)\lambda_\gamma\lambda^*_\gamma$-continuous function on X to Y then it has a (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous extension over $(\gamma X, P_1, P_2)$ to (Y,s,t).

Proof. Let F be an (i,j) λ_γ-closed set, K a (j,i) λ_γ-closed set in Y and $F \cap K = \varnothing$. Then $f^{-1}(F) \cap f^{-1}(K) = \varnothing$ and by Theorem 3.2, $f^{-1}(F)^*_i \cap f^{-1}(K)^*_j = \varnothing$ since f is (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous. Hence if h is a (j,i) $\lambda_\gamma\lambda^*_\gamma$-embedding function on X to $\gamma\, X$ then $\varnothing = \mathrm{cl}_{p_1}(h(f^{-1}(F)) \cap \mathrm{cl}_{p_2}(h(f^{-1}(K)) \subset f^{-1}(F)^*_i \cap f^{-1}(K)^*_j$. Hence the proof is completed by Theorem 4.1.

We come now to an important property of the Wallman compactification on bitopological spaces.

Definition 4.2. Let (X, τ_1, τ_2), (Y,s,t) are bispaces and f be a function on X to Y. If f satisfies the following conditions:

(*i*) f is (i,j) $\lambda_\gamma\lambda^*_\gamma$-continuous;

(*ii*) f is (i,j) $\lambda_\gamma\lambda^*_\gamma$-closed;

(*iii*) $f^{-1}(y) \subseteq X$ is (i,j) $\lambda_\gamma\lambda^*_\gamma$-compact for each $y \in Y$, then f is called the bi $\lambda_\gamma\lambda^*_\gamma$-perfect function.

Theorem 4.3. Let (X, τ_1, τ_2), (Y,s,t) be finite bi λ_γ-normal semi bi λ_γ-T_1 bispaces. Let $(\gamma X, \tau^*_1, \tau^*_2)$, $(\gamma Y, s^*, t^*)$ be respectively their Wallman compactification, and f be a bi $\lambda_\gamma\lambda^*_\gamma$-perfect function on X to Y. Then there exists a (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous extension f' of f on γX to γY and we have $f^{-1}(g(Y)) = h(X)$, where g is an embedding function on Y to $\gamma(Y)$ and h is a (j,i) $\lambda_\gamma\lambda^*_\gamma$-embedding function on X to $\gamma\, X$.

Proof. Let $F \subseteq \gamma Y$ be (i,j) λ_γ-closed, $K \subseteq \gamma\, Y$ be (j,i) λ_γ-closed and $F \cap K = \varnothing$, $(g_o f)^{-1}(F) \cap (\, g_o f)^{-1}(K) = \varnothing$ gives that $\mathrm{cl}_{\tau^*_1}(\, g_o f)^{-1}(F) \cap \mathrm{cl}_{\tau^*_2}(g_o f)^{-1}(K) = \varnothing$, since $(g_o f)$ is $(i,j)\lambda_\gamma\lambda^*_\gamma$-continuous, and by Theorem 3.2. It follows that, by Theorem 3.3. and Theorem 4.2, $(g_o f)$ has (i,j) $\lambda_\gamma\lambda^*_\gamma$-continuous extension f on $\gamma\, X$ to γY. To show that $f^{-1}(g(Y)) = h(X)$, let $y \in Y$ and $g(y) = \beta_y$. If $f(\beta) = \beta_y$ for $\beta \in \gamma X$, then we must find an $x \in X$ such that $\beta = \beta_x$. We have $\mathrm{cl}_{\tau^*_1}(h(f^{-1}(\mathrm{cl}_s\,(y))) = ((f^{-1}(\mathrm{cl}_s(y))^*_i$ and $\beta = \beta_y$. We have $\mathrm{cl}_{\tau^*_2}(h(f^{-1}(\mathrm{cl}_t(y))) = ((f^{-1}(\mathrm{cl}_s(y))^*_j$, since $(f^{-1}(\mathrm{cl}_s(y))$ is an (i,j) λ_γ-closed set in X, and $(f^{-1}(\mathrm{cl}_t(y)$ is a (j,i) λ_γ-closed set in X and by Theorem 3.2. This gives that $\beta \in (f^{-1}(\mathrm{cl}_s(y))^*_i \cap (f^{-1}(\mathrm{cl}_t(y))^*_j$. On the contrary, let $\beta \notin (f^{-1}(\mathrm{cl}_s(y))^*_i$. From Theorem 3.3 $(\gamma X, \tau^*_1, \tau^*_2)$ is semi bi λ_γ-T_2, then there exists $G, H \subseteq \gamma X$ such that $\beta \in G \in (i,j)\lambda_\gamma\, O(\gamma X)$, $(f^{-1}(\mathrm{cl}_s(y))^*_i \subseteq H \in (j,i)\lambda_\gamma O\,(\gamma\, X)$, $G \cap H = \varnothing$. We have $\beta \notin (\gamma X) \backslash G \in (i,j)$ $\lambda_\gamma C\,(\gamma X)$. By Proposition 4.3, there exists an

$(i,j)\lambda_\gamma$-closed set in X such that $\beta \notin F_i^*$, $\gamma X \backslash G \subseteq F_i^*$, $h^{-1}(H) \in (j,i)\,\lambda_\gamma O(X)$, which implies that $f(X \backslash h^{-1}(H)) \in (j,i)\lambda_\gamma\, C(Y)$ since f is $(j,i)\,\lambda_\gamma\lambda_\gamma^*$-closed. Let $T = \gamma\, X \backslash [f(X \backslash h^{-1}(H))]_j^* \in (j,i)\lambda_\gamma O(\gamma Y)$. If $\beta_y \notin T$ then $\beta_y \in [f(X \backslash h^{-1}(H))]_j^*$ and this gives that $y \in (f(X \backslash h^{-1}(H))$ such that $y = f(z)$. But $x \notin h^{-1}(H)$ gives that $h(z) \notin H$ and $h(z) \notin ((f^{-1}(\mathrm{cl}_s(y))_i^*$ hence $z \notin f^{-1}(\mathrm{cl}_s(y))$ and $f^{-1}(z) \notin (\mathrm{cl}_s(y))$. But this conflicts with $y \in f(z)$. This gives that $\beta_y \in T$, $\gamma X \backslash G \subseteq F_i^*$ gives $X \backslash h^{-1}\,(G) \subseteq F$ and $(X \backslash F) \subseteq h^{-1}(G)\,(X \backslash F) \cap h^{-1}(H) \backslash = \varnothing$, since $G \cap H = \varnothing$. This means that $f \cup (X \backslash h^{-1}(H)) = X$. Now we have $((X \backslash h^{-1}(H), X) \in \beta$ or $(X, F) \in \beta$ since β is maximal, $F \in (i,j)\lambda_\gamma\, C(X)$ and $(X \backslash h^{-1}(H)) \in (j,i)\,\lambda_\gamma\, C(X)$. If $(X, F) \in \beta$ then $\beta \in F_i^*$. But this is a contradiction. Then we have $((X \backslash h^{-1}(H)), X) \in \beta$. It means that $\beta \in [X \backslash h^{-1}(H)]_j^* = \mathrm{cl}_{\tau_2^*}\,(h(X \backslash h^{-1}(H))$. The $(j,i)\,\lambda_\gamma\lambda_\gamma^*$-continuity of f' and $f'(\beta) = \beta_y$ gives that $\beta \in f'^{-1}(T) \in (j,i)\,\lambda_\gamma\, O(\gamma X)$. Then $f^{-1}(T) \cap h(X \backslash h^{-1}(H) \neq \varnothing$ and $x \in X \backslash h^{-1}(H), f(\beta_x) = g_o f(x) \in T$. We have $h(x) \notin H$, $g(f(x)) \in \gamma\, X \backslash [f(X \backslash h^{-1}(H))]_j^*$ and then $f(x) \notin f(X \backslash h^{-1}(H))$, $x \notin X \backslash h^{-1}(H)$, $h(x) \in H$. But this is a contradiction. Thus $\beta \in ((f^{-1}(\mathrm{cl}_s(y))_i^*$. It can be shown that $\beta \in (f^{-1}(\mathrm{cl}_t(y))_j^*$ in the same way. Then we have $(f^{-1}(\mathrm{cl}_s(y)) f^{-1}(\mathrm{cl}_t(y)) \in \beta$ and for each $(F, K) \in \beta$, $F \cap K \cap f^{-1}(\mathrm{cl}_s(y)) \cap f^{-1}(\mathrm{cl}_t(y)) = F \cap K \cap f^{-1}(\mathrm{cl}_s(y) \cap \mathrm{cl}_t(y)) = F \cap K \cap f^{-1}(y) \neq \varnothing$ since (Y, s, t) is semi bi λ_γ-T_1 there exists an $x \in X$ such that $x \in F \cap K \cap f^{-1}(y)$ for all $(F, K) \in \beta$, since $f^{-1}(y)$ is $(j,i)\,\lambda_\gamma$-compact. This gives $\beta \subseteq \beta_x$. The maximality of β gives that $\beta = \beta_x$ and then $f(x) = y$ completes the proof.

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