ON ALMOST SIMILAR OPERATORS

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الخلاصـة :

نُعرِّف في هذا البحث علاقة التشابه التقريبي بين المؤثرات على فضاء هلبرت . ونُثبت بعض الخصائص لهذه العلاقات ، كذلك نثبت أنَّ هناك صفات مشتركة بين المؤثرات المتشابهة تقريباً .

ABSTRACT

The almost similar relation between operators is defined. It is shown that some properties are shared by almost similar operators. Various results related to the almost similar relation are proved.

ON ALMOST SIMILAR OPERATORS

1. INTRODUCTION

Let H be a Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. If A and B are in $L(H)$, then A and B are called similar if there is an invertible operator N such that $A = N^{-1}BN$. We call A and B almost similar, A a.s. B, if there is an invertible operator N such that the following two conditions are satisfied:

$$A^*A = N^{-1}B^*BN,$$

$$A^* + A = N^{-1}(B^* + B)N.$$

If $A, B, C \in L(H)$, then it can be easily shown that the almost similar relation is an equivalence relation.

In this first section of this paper we show that some properties are shared by almost similar operators.

Proposition 1.1. Let $A, B, C \in L(H)$.

(i) If A a.s. O then $A = O$.

(ii) If A a.s. B, B is skew-adjoint then A is skew-adjoint.

(iii) If A a.s. C, C is isometric then A is isometric.

Proof. The proof is direct from the definition of the almost similar relation so we omit it.

Proposition 1.2. If $A \in L(H)$ and A a.s. I, then $A = I$.

Proof. Since A a.s. I, there is an invertible operator N such that

$$I^*I = N^{-1}A^*AN \tag{1.1}$$

$$I^* + I = N^{-1}(A^* + A)N. \tag{1.2}$$

From (1.1) and (1.2) above, we conclude that $A^*A = I$ and $A^* + A = 2I$. This implies that $A^*A + A^2 = 2A$. As $A^*A = I$, we get $A^2 - 2A + I = 0$. It is proved in ([5], p. 9) that the solution of the last equation is $A = I$, and for the sake of completeness, we outline the proof as it appeared in [5].

Let $x \in H$ then $(A-I)(A-I)x = 0$. Put $(A-I)x = y$. Thus we get $(A-I)y = 0$ and hence $Ay = y$ and $Ax = x + y$. By iteration we get $A^n x = x + ny$ for any natural number n. Hence

$$n||y|| = ||ny|| = ||A^n x - x|| \leq ||A^n x|| + ||x|| = ||x|| + ||x|| = 2||x||,$$

so that $n||y|| \leq 2||x||$ for all natural numbers n. Thus $||y|| \leq \frac{2}{n}||x|| \to 0$ as $n \to \infty$ and hence $y = 0$ and consequently $y = (A-I)x = 0$ for all $x \in H$. This implies that $Ax = x$ for all x and hence $A = I$.

Proposition 1.3. If $A, B \in L(H)$ such that A a.s. B and if A is compact, then so is B.

Proof. By assumption there is an invertible operator N such that $B^*B = N^{-1}A^*AN$. Since A is compact, $N^{-1}A^*AN$ is compact which implies that B^*B is compact. Thus, by ([4], p. 427), B is compact.

Definition 1.1. $A \in L(H)$ is called a θ-operator if $T^* + T$ commutes with T^*T. The class of all θ-operators in $L(H)$ is denoted by θ.

Proposition 1.4. If $A, B \in L(H)$ such that $B \in \theta$ and A a.s. B, then $A \in \theta$.

Proof. By assumption there is an invertible operator N such that $A^*A = N^{-1}B^*BN$ and $A^* + A = N^{-1}(B^* + B)N$. Thus we have:

$$(N^{-1}B^*BN)[N^{-1}(B^* + B)N] = A^*A(A^* + A) \tag{1.3}$$

and

$$[N^{-1}(B^* + B)N](N^{-1}B^*BN) = (A^* + A)A^*A. \tag{1.4}$$

From (1.3) we get:

$$N^{-1}B^*B(B^* + B)N = A^*A(A^* + A); \tag{1.5}$$

and from (1.4) we get

$$N^{-1}(B^* + B)B^*BN = (A^* + A)A^*A. \tag{1.6}$$

Since $B \in \theta$, the lefthand sides of (1.5) and (1.6) are equal, which implies that the righthand sides of (1.5) and (1.6) are equal. Thus $A \in \theta$.

Theorem 1.1. An operator $T \in L(H)$ is hermitian if and only if $(T + T^*)^2 \geq 4T^*T$.

Proof. (see [3], p. 316).

Remark 1.1. In the proof of the next proposition, we need a simpler version of Theorem 1.1, that is $T \in L(H)$ is hermitian if and only if $(T+T^*)^2 = 4T^*T$, and the proof of this is much simpler than the proof of the above theorem: If T is hermitian then $(T+T^*)^2 = 4T^2$ and also $4T^*T = 4T^2$. Now suppose that $(T+T^*)^2 = 4T^*T$ and let $T = A+Bi$ be the cartesian decomposition of T. Then $(T+T^*)^2 = 4A^2$ and $4T^*T = 4[(A^2+B^2)+(AB-BA)i]$. Thus we have $4A^2 = 4A^2 + 4B^2$ which implies that $B^2 = 0$. Since B is hermitian, $B = 0$ which implies that T is hermitian.

Proposition 1.5. If $A, B \in L(H)$ such that A a.s. B and B is hermitian, then A is hermitian.

Proof. Since A a.s. B, there is an invertible operator N such that $A^*A = N^{-1}B^*BN$, which implies that

$$4A^*A = N^{-1}(4B^*B)N. \tag{1.7}$$

Also, A a.s. B implies that $A^* + A = N^{-1}(B^* + B)N$, which implies that $[N^{-1}(B^* + B)N][N^{-1}(B^* + B)N] = (A^* + A)^2$. Thus

$$N^{-1}(B^* + B)^2N = (A^* + A)^2. \tag{1.8}$$

Since B is hermitian, we have $(B^* + B)^2 = 4B^*B$ and substituting from this in (1.8) we get

$$N^{-1}(4B^*B)N = (A^* + A)^2. \tag{1.9}$$

Now from (1.7) and (1.9) we have $(A^* + A)^2 = 4A^*A$, which implies, by Remark 1.1, that A is hermitian.

Definition 1.2. An operator $T \in L(H)$ is said to be partially isometric in case T^*T is a projection.

Proposition 1.6. If $A, B \in L(H)$ such that A a.s. B and A is partially isometric, then so is B.

Proof. A a.s. B implies that there is an invertible operator N such that $N^{-1}B^*BN = A^*A$. Since A is partially isometric, A^*A is a projection, which implies that $(N^{-1}B^*BN)(N^{-1}B^*BN) = N^{-1}B^*BN$. Thus we have $N^{-1}B^*BB^*BN = N^{-1}B^*BN$ which implies that $(B^*B)^2 = B^*B$. Thus B^*B is a projection which implies that B is partially isometric.

Proposition 1.7. If $A, B \in L(H)$ such that A a.s. B and A is a projection, then so is B.

Proof. A a.s. B implies that there exists an invertible operator N such that

$$A^*A = N^{-1}B^*BN \tag{1.10}$$

and

$$A^* + A = N^{-1}(B^* + B)N. \tag{1.11}$$

Since A is a projection, it is hermitian and this implies — proposition 1.5 — that B is hermitian. From (1.10) we get $A = N^{-1}B^2N$ and from (1.11) we get $A = N^{-1}BN$. This implies that $N^{-1}B^2N = N^{-1}BN$ which implies that B is a projection.

2. PROOFS OF RESULTS

In Section 2 of this paper we prove some results related to the almost similar relation.

Proposition 2.1. If $A, B \in L(H)$ such that A and B are unitarily equivalent, then A a.s. B.

Proof. By assumption there is a unitary operator U such that $A = U^*BU$ which implies that $A^* = U^*B^*U$. Thus $A^*A = U^*B^*UU^*BU = U^*B^*BU = U^{-1}B^*BU$, and $A^* + A = U^*B^*U + U^*BU = U^*(B^* + B)U = U^{-1}(B^* + B)U$. Thus A a.s. B.

Proposition 2.2. If $A, B \in L(H)$ such that A a.s. B and if A is hermitian, then A and B are unitarily equivalent.

Proof. By assumption there is an invertible operator N such that $A^* + A = N^{-1}(B^* + B)N$. Since A is hermitian and A a.s. B, then, by Proposition 1.5, B is hermitian. Thus we have $2A = N^{-1}2BN$ which implies that $A = N^{-1}BN$. This implies that A and B are similar and since both are normal, they are unitarily equivalent.

Proposition 2.3. If $A \in L(H)$ is normal, then A a.s. A^*.

Proof. Since A is normal then $A^*A = AA^*$. Thus we have $A^*A = AA^* = (A^*)^*A^* = I^{-1}(A^*)^*A^*I$. Also $A^* + A = A + A^*$ implies that $A^* + A = (A^*)^* + A^* = I^{-1}((A^*)^* + A^*)I$. Thus A a.s. A^*.

The following is a characterization of θ-operators in terms of the almost similar relation.

Proposition 2.4. If $A \in L(H)$ then $A \in \theta$ if and only if A a.s. B for some normal operator B.

Proof. Let $A \in \theta$ then, by ([2], p. 305), the operator $B = \frac{1}{2}(A^* + A + i\sqrt{4A^*A - (A^* + A)^2})$ is normal with $A^*A = B^*B$ and $A^* + A = B^* + B$. Thus $A^*A = I^{-1}B^*BI$ and $A^* + A = I^{-1}(B^* + B)I$. Hence A a.s. B.

Now let A a.s. T for some normal operator T, then there is an invertible operator N such that $A^*A = N^{-1}T^*TN$ and $A^* + A = N^{-1}(T^* + T)N$. Thus we have

$$A^*A(A^* + A) = N^{-1}T^*T(T^* + T)N; \tag{2.1}$$

$$(A^* + A)A^*A = N^{-1}(T^* + T)T^*TN. \tag{2.2}$$

Since T is normal, $T \in \theta$. Thus the righthand sides of (2.1) and (2.2) are equal which implies that $A^*A(A^* + A) = (A^* + A)A^*A$. Thus $A \in \theta$.

Proposition 2.5. If $T \in L(H)$ is invertible and T a.s. U for some unitary $U \in L(H)$, then T is unitary.

Proof. Since T a.s. U, there is an invertible operator N such that $T^*T = N^{-1}U^*UN = I$. This implies that $T^{*-1}T^*TT^{-1} = T^{*-1}T^{-1}$. Since $T^{*-1}T^*TT^{-1} = I$, $T^{*-1}T^{-1} = (TT^*)^{-1} = I$ which implies that $TT^* = I$. Hence T is unitary.

Next we give a characterization of isometric operators in terms of the almost similar relation.

Proposition 2.6. $A \in L(H)$ is isometric if and only if A a.s. U for some unitary operator U.

Proof. Let A be isometric, then $A \in \theta$. Thus, by Proposition 2.4, there is a normal operator N with A a.s. N. By Proposition 1.1(iii) N is isometric. Thus N is unitary.

Now, suppose that A a.s. U for some unitary U, then there is an invertible operator N with $N^{-1}A^*AN = U^*U = I$. This implies that $A^*A = NN^{-1} = I$. Thus A is isometric.

Let $T \in L(H)$ be unitary, then $T^*T = TT^*$. Thus $T^*T = I^{-1}(T^*)^*T^*I$. Also $T^* + T = T + T^*$ implies that $T^* + T = I^{-1}[(T^*)^* + T]I$. Hence T a.s. $T^* = T^{-1}$. However, if $T \in L(H)$ and T a.s. T^{-1}, then T is not necessarily unitary, as the following example shows:

Example 1. Consider the operator $T = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$ on the two-dimensional space R^2. Then it can be shown that $T^2 = I$ which implies that $T = T^{-1}$. Thus T a.s. T^{-1}. However $||T|| > 1$ which means that T is not unitary.

Proposition 2.7. If $A, B \in L(H)$ such that A a.s. B, then $(A + \lambda)$ a.s. $(B + \lambda)$ for all real λ.

Proof. By assumption there is an invertible operator N such that

$$A^* + A = N^{-1}(B^* + B)N \tag{2.3}$$

$$A^*A = N^{-1}B^*BN. \tag{2.4}$$

From (2.3) we have $A^* + A = N^{-1}B^*N + N^{-1}BN$ which implies that $A^* + A + 2\lambda = N^{-1}B^*N + N^{-1}BN + 2\lambda$. Thus we have $(A^* + \lambda) + (A + \lambda) = N^{-1}B^*N + N^{-1}BN + N^{-1}\lambda N + N^{-1}\lambda N$, which implies that $(A^* + \lambda) + (A + \lambda) = N^{-1}(B^* + \lambda)N + N^{-1}(B + \lambda)N$. Thus

$$(A + \lambda)^* + (A + \lambda) = N^{-1}[(B + \lambda)^* + (B + \lambda)]N. \tag{2.5}$$

From (2.5) we have

$$\lambda A^* + \lambda A + \lambda^2 = N^{-1}\lambda B^*N + N^{-1}\lambda BN + N^{-1}\lambda^2 N. \tag{2.6}$$

Adding (2.4) and (2.6) we get $A^*A + \lambda A^* + \lambda A + \lambda^2 = N^{-1}B^*BN + N^{-1}\lambda B^*N + N^{-1}\lambda BN + N^{-1}\lambda^2 N$ which implies that $(A^* + \lambda)(A + \lambda) = N^{-1}[(B^* + \lambda)(B + \lambda)]N$. Thus

$$(A + \lambda)^*(A + \lambda) = N^{-1}[(B + \lambda)^*(B + \lambda)]N. \tag{2.7}$$

From (2.5) and (2.7) we conclude that $(A + \lambda)$ a.s. $(B + \lambda)$.

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