SOME EXACT SOLUTIONS OF EQUATIONS OF MOTION OF AN ELECTRICALLY CONDUCTING FLUID MOVING IN A MAGNETIC FIELD

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الخلاصـة :

تعرض هذه الدراسة حلولاً مُـحَـدَّدة لمعادلات الجريان المستقر لسائل لزج غير قابل للانضغاط ذي موصلية كهربية مـحـدَّدة بوجود مجال مغناطيسي .

ABSTRACT

Some exact solutions of equations governing the steady motion of a viscous incompressible fluid of finite electrical conductivity in the presence of a magnetic field are determined.

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1. INTRODUCTION

In the present paper, the steady viscous incompressible plane flow problem of an electrically conducting fluid having finite electrical conductivity in the presence of a magnetic field is studied with the objective of obtaining some exact solutions. To achieve this objective, the basic flow equations are cast into a new form by introducing the streamfunction ψ, the magnetic flux function ϕ and the new independent variables r, α. The equations are then solved using an inverse method. In this inverse method, we select a form for the vorticity function ω and then determine the streamfunction ψ, the magnetic flux function ϕ and the energy function h from the corresponding differential equations.

We point out that the advantage of the new independent variables r, α is that the solutions which we get are not obtainable through techniques employed by the researchers in the study of MHD plane flows [1–7].

2. FLOW EQUATIONS

The basic non-dimensional equations governing the steady plane flow of a viscous incompressible fluid of finite electrical conductivity, in the presence of a magnetic field are,

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = -P_x + \frac{1}{Re}(u_{xx} + u_{yy}) - R_H H_2 (H_{2x} - H_{1y}) \tag{2}$$

$$\begin{aligned} uv_x + vv_y &= -P_y + \frac{1}{R_e}(v_{xx} + v_{yy}) + R_H H_1 (H_{2x} - H_{1y}) \\ uH_2 - vH_1 &= \frac{1}{R_\sigma}(H_{2x} - H_{1y}) + C_1 \end{aligned} \tag{3}$$

$$H_{1x} + H_{2y} = 0 \tag{4}$$

where u, v are the velocity components, H_1, H_2 the components of magnetic field vector $\mathbf{H}$, p the pressure, Re the Reynolds number, R_H the magnetic pressure number, R_σ the magnetic Reynolds number, and C_1 is an arbitrary constant.

Equations (1) and (4), respectively, imply the existence of the streamfunction ψ and magnetic flux function ϕ such that

$$\begin{aligned} u &= \psi_y, \quad v = -\psi_x \\ H_1 &= \phi_y, \quad H_2 = -\phi_x. \end{aligned} \tag{5}$$

The system of Equations (1–4), employing (5), transforms to the following system of partial differential equations

$$\begin{aligned}
-h_\eta &= \frac{1}{R_e}\,\omega_\xi + \omega\psi_\eta + \frac{R_H}{2}\left(\phi_{\xi\xi} + \phi_{\eta\eta}\right)\phi_\eta \\
-\,h_\xi &= -\frac{1}{R_e}\,\omega_\eta + \omega\psi_\xi + \frac{R_H}{2}\left(\phi_{\xi\xi} + \phi_{\eta\eta}\right)\phi_\xi \\
\psi_\xi\phi_\eta - \psi_\eta\phi_\xi &= \frac{1}{R_\sigma}\left(\phi_{\xi\xi} + \phi_{\eta\eta}\right) + C_1 \\
\psi_{\xi\xi} + \psi_{\eta\eta} + 2\omega &= 0
\end{aligned}$$

in the variables $\xi = x + y$ and $\eta = x - y$. In the above system of equations the energy function h is given by

$$h = p + \frac{1}{4}\left(\psi_\xi^2 + \psi_\eta^2\right).$$

Introducing the new independent variables r, α defined by

$$r = \sqrt{\xi^2 + \eta^2}, \qquad \alpha = \tan^{-1}(\eta/\xi)\,,$$

the above system of equations is replaced by the following system

$$-h_\alpha = \frac{1}{R_e}\,r\omega_r + \omega\psi_\alpha + \frac{R_H}{2r^2}\left(r^2\phi_{rr} + r\phi_r + \phi_{\alpha\alpha}\right)\phi_\alpha \tag{6}$$

$$-rh_r = -\frac{1}{R_e}\,\omega_\alpha + r\omega\psi_r + \frac{R_H}{2r}\left(r^2\phi_{rr} + r\phi_r + \phi_{\alpha\alpha}\right)\phi_r \tag{7}$$

$$r^2\phi_{rr} + r\left(1 + R_\sigma\psi_\alpha\right)\phi_r + \phi_{\alpha\alpha} - r\psi_r R_\sigma\phi_\alpha + 2C_1 R_\sigma r^2 = 0 \tag{8}$$

$$r^2\psi_{rr} + r\psi_r + \psi_{\alpha\alpha} + 2\omega r^2 = 0 \tag{9}$$

of four partial differential equations in four unknowns ψ, ω, ϕ, h as functions of r and α. In Equations (6–7), the energy function h is given by

$$h = p + \frac{1}{2}\left(\psi_r^2 + \frac{1}{r^2}\psi_\alpha^2\right). \tag{10}$$

Once a solution of this system is determined, the pressure p is found from the definition of the energy function h in (10).

3. SOLUTIONS

In this section, we determine the solutions of the system of Equations (6–9). Our strategy will be to specify ω, and calculate ψ from (9), and use this ψ to determine h and ϕ from (6–8).

(a) Irrotational Flows:

For this type of flows $\omega = 0$. Employing this in Equations (6–9), we get

$$h_\alpha = \frac{-R_H}{2r^2}\left(r^2\phi_{rr} + r\phi_r + \phi_{rr}\right)\phi_\alpha \tag{11}$$

$$h_r = \frac{R_H}{2r^2}\left(r^2\phi_{rr} + r\phi_r + \phi_{rr}\right)\phi_r \tag{12}$$

$$r^2\phi_{rr} + r\left(1 + R_\sigma\psi_\alpha\right)\phi_r + \phi_{\alpha\alpha} - r\psi_r R_\sigma\phi_\alpha + 2C_1 R_\sigma r^2 = 0 \tag{13}$$

$$r^2\psi_{rr} + r\psi_r + \psi_{\alpha\alpha} = 0. \tag{14}$$

A set of solutions of (14) is

$$\Psi = \begin{cases} A_1 + A_2\ln r + A_3\alpha, & \\ \left(A_4 r^{\sqrt{n}} + A_5 r^{-\sqrt{n}}\right)\left[A_6\cos\left(\sqrt{n}\alpha\right) + A_7\sin\left(\sqrt{n}\alpha\right)\right], & n > 0 \\ \left[A_8\cos\left(\sqrt{m}\ln r\right) + A_9\sin\left(\sqrt{m}\ln r\right)\right]\left(A_{10}e^{\sqrt{m}\alpha} + A_{11}e^{-\sqrt{m}\alpha}\right), & n = -m, \quad m > 0 \end{cases} \tag{15}$$

where $A_1, A_2, \ldots, A_{11}, n$ and m are arbitrary constants.

When $C_1 = 0$, a solution of (13) is $\phi = \psi$. Using this in (11–12), we find

$$h_\alpha = 0, \qquad h_r = 0.$$

This gives

$$h = C_2$$

where C_2 is an arbitrary constant.

Hence, for $C_1 = 0$, a solution of Equations (11–14), in the variables x, y, is

$$\Psi = \begin{cases} A_1 + \dfrac{A_2}{2}\ln\left(2x^2+2y^2\right) + A_3\tan^{-1}\left[(x-y/x+y)\right] \\ \left[A_4\left(2x^2+2y^2\right)^{\sqrt{n}} + A_5\left(2x^2+2y^2\right)^{-\sqrt{n}}\right]\left\{A_6\cos\left(\sqrt{n}\tan^{-1}\left[x-y/x+y\right]\right) + A_7\sin\left(\sqrt{n}\tan^{-1}\left[x-y/x+y\right]\right)\right\} \\ \left[A_8\cos\left(\sqrt{m}\ln\sqrt{2x^2+2y^2}\right) + A_9\sin\left(\sqrt{m}\ln\sqrt{2x^2+2y^2}\right)\right] \\ \qquad \times\left\{A_{10}\exp\left(\sqrt{m}\tan^{-1}[(x-y)/(x+y)]\right) + A_{10}\exp\left(-\sqrt{m}\tan^{-1}[(x-y)/(x+y)]\right)\right\} \end{cases}$$

$\phi = \psi, \qquad h = C_2.$

When $C_1 \neq 0$, we determine the solution of (13) as follows:

Assuming $\phi = \psi + f(r)$, the Equation (13) gives

$$r^2 f_{rr} + r\left(1 + R_\sigma \psi_\alpha\right) f_r + 2C_1 R_\sigma r^2 = 0.$$

A solution of this equation is

$$f(r) = C_4 + C_5 r^{1-C_3} - \frac{C_1 R_\sigma r^2}{1 + C_3} \tag{16}$$

provided

$$\psi_\alpha = \left(C_3 - 1\right)/R_\sigma . \tag{17}$$

Equation (14), utilizing (17), gives

$$\psi = C_6 + C_7 \ln r + \left(C_3 - 1\right)\alpha/R_\sigma \tag{18}$$

wherein C_6, C_7 are arbitrary constants.

Employing (16) and (18) in (11–12), we get

$$h_\alpha = -R_H \frac{(C_3-1)}{R_\alpha}\left[(1-C_3)^2 C_5 r^{-1-C_3} - \frac{4C_1R_\sigma}{1+C_3}\right]$$

$$h_r = -R_H \left\{(1-C_3)^2 C_5 r^{-1-C_3} - \frac{4C_1R_\sigma}{1+C_3}\right\}\left[\frac{C_7}{r} + C_5(1-C_3)r^{-C_3}\frac{-2C_1 r R_\sigma}{1+C_3}\right].$$

These give

$$h = \frac{4(C_3-1)}{1+C_3} R_H C_1 \alpha + \frac{4R_H C_1 R_\sigma}{1+C_3}\left(C_7 \ln r - \frac{C_1 R_\sigma r^2}{1+C_3}\right) + C_8$$

provided $C_5 = 0$. In above C_8 is an arbitrary constant.

Therefore, the expressions for ψ, ϕ, and h, in the physical plane, are

$$\psi = C_6 + \frac{C_7}{2}\ln\left(2x^2+2y^2\right) + \frac{1}{R_\sigma}(C_3-1)\tan^{-1}\left[\frac{x-y}{x+y}\right]$$

$$\phi = \psi + C_4 - 2C_1\left(x^2+y^2\right)R_\sigma/(1+C_3)$$

$$h = \frac{4(C_3-1)}{1+C_3}R_H C_1 \tan^{-1}\left[\frac{x-y}{x+y}\right] + \frac{4C_1R_HR_\sigma}{1+C_3}\left[\frac{C_7}{2}\ln\left(2x^2+2y^2\right) - \frac{-2C_1\left(x^2+y^2\right)}{1+C_3}R_\sigma\right] + C_8.$$

For $\phi = \psi + K(\alpha)$, Equation (13) gives

$$K_{\alpha\alpha} - r\psi_r R_\sigma K_\alpha + 2C_1 R_\sigma r^2 = 0.$$

A solution of this, for $C_1 = 0$, is

$$K = D_2 + D_3 \alpha^{D_1} \tag{19}$$

provided

$$r\psi_r R_\sigma = D_1 \tag{20}$$

D_1, D_2, D_3 being arbitrary constants.

Employing (20) in (14), we get

$$\psi = \frac{D_1}{R_\sigma} \ln r + D_4 \alpha + D_5 \tag{21}$$

where D_4 and D_5 are arbitrary constants. Equations (11–12) give

$$h = D_6$$

provided

$$D_1 = 1.$$

Hence for this case

$$\begin{aligned}
\psi &= \frac{D_1}{2R_\sigma} \ln\left(2x^2 + 2y^2\right) + D_4 \tan^{-1}\left(\frac{x-y}{x+y}\right) + D_5 \\
\phi &= \psi + D_3 \tan^{-1}\left(\frac{x-y}{x+y}\right) + D_2 \\
h &= D_6
\end{aligned}$$

where D_6 is an arbitrary constant.

(b) Rotational Flows:

For this type of flow, the vorticity ω is non-zero. Let us determine the solutions of Equations (6–9) for these flows employing some forms of ω.

(i) When $\omega = \omega_o$ (constant), the Equations (6–9) give:

$$-h_\alpha = \frac{R_H}{2r^2}\left(r^2\phi_{rr} + r\phi_r + \phi_{\alpha\alpha}\right)\phi_\alpha + \omega_o \psi_\alpha \tag{22}$$

$$-h_r = \frac{R_H}{2r^2}\left(r^2\phi_{rr} + r\phi_r + \phi_{\alpha\alpha}\right)\phi_r + \omega_o \psi_r \tag{23}$$

$$r^2\phi_{rr} + r\left(1 + R_\sigma \psi_\alpha\right)\phi_r + \phi_{\alpha\alpha} - r\psi_r R_\sigma \phi_\alpha + 2C_1 R_\sigma r^2 = 0 \tag{24}$$

$$r^2\psi_{rr} + r\psi_r + \psi_{\alpha\alpha} + 2\omega_o r^2 = 0. \tag{25}$$

For $\phi = \psi$, the Equation (24) and (25) give

$$r^2\psi_{rr} + r\psi_r + \psi_{\alpha\alpha} + 2C_1 R\sigma r^2 - 0 \tag{26}$$

and

$$\omega_o = C_1 R_\sigma .$$

The general solution of (26) is

$$\psi = A_3 - \frac{A_1}{2}\,\alpha^2 + A_2\alpha + A_4 \ln r + \frac{A_5}{2}\,(\ln r)^2 - \frac{C_1}{2}\,R_\sigma r^2 \tag{27}$$

where $A_1, \ldots, A_5$ are arbitrary constants.

Equations (22–23), employing $\phi = \psi$ and (27), give

$$\begin{aligned} h_\alpha &= (2R_H - 1)\,C_1 R_\sigma\,[-A_1\alpha + A_2] \\ h_r &= (2R_H - 1)\,C_1 R_\sigma \left[\frac{A_4}{r} - C_1 R_\sigma r + \frac{A_5}{r}\ln r\right]. \end{aligned} \tag{28}$$

Integration of (28) yields:

$$h = (2R_H - 1)\,C_1 R_\sigma \left[A_4 \ln r - \frac{C_1}{2}\,R_\sigma r^2 - \frac{A_1\alpha^2}{2} + A_2\alpha + \frac{A_5}{2}\,(\ln r)^2\right] + A_6$$

where A_6 is an arbitrary constant.

Therefore, a solution of (22–25), in the physical plane, is

$$\begin{aligned} \psi &= A_3 + A_2 \tan^{-1}\left(\frac{x-y}{x+y}\right) - \frac{A_1}{2}\left\{\tan^{-1}\left(\frac{x-y}{x+y}\right)\right\}^2 + \frac{A_4}{2}\ln\left(2x^2+2y^2\right) \\ &\quad + \frac{A_5}{8}\left[\ln\left(2x^2+2y^2\right)\right]^2 - C_1 R_\sigma\left(x^2+y^2\right) \\ \phi &= \psi \\ h &= (2R_H - 1)\,C_1 R_\sigma \left\{\frac{A_4}{2}\ln\left(2x^2+2y^2\right) - C_1 R_\sigma\left(x^2+y^2\right) + \frac{A_5}{8}\left[\ln\left(2x^2+2y^2\right)\right]^2 \right. \\ &\quad \left. - \frac{A_1}{2}\left[\tan^{-1}\left(\frac{x-y}{x+y}\right)\right]^2 - A_2 \tan^{-1}\left(\frac{x-y}{x+y}\right)\right\} + A_6. \end{aligned}$$

If we take $\phi = \psi + f(r)$, the Equations (24–25) give

$$r^2 f_{rr} + r\left(1 + R_\sigma \psi_\alpha\right) f_r = 2\left(\omega_o - C_1 R_\sigma\right) r^2$$
$$r^2 \psi_{rr} + r\psi_r + \psi_{\alpha\alpha} + 2C_1 R_\sigma r^2 = 0. \tag{29}$$

A solution of (29) is

$$\psi = A_7 + A_8 \ln r - \frac{r^2 \omega_o}{2} + \left(A_6 - 1\right)\alpha / R_\sigma \tag{30}$$

$$f = A_9 + A_{10} r^{1-A_6} + \frac{1}{1+A_6}\left(\omega_o - C_1 R_\sigma\right) r^2 \tag{31}$$

wherein $A_6, \ldots, A_{10}$ are arbitrary constants.

Equations (22–23), utilizing $\phi = \psi + f(r)$ and (30–31), give

$$-h_\alpha = \frac{\omega_o\left(A_6 - 1\right)}{R_\sigma} + \frac{R_H}{2r^2}\left[-2r^2\omega_o + r^2 f_{rr} + r f_r\right]\left(\frac{A_6 - 1}{R_\sigma}\right) \tag{32}$$

$$-h_r = \omega_o\left(\frac{A_8}{r} - \omega_o r\right) + \frac{R_H}{2r^2}\left[-2r^2\omega_o + r^2 f_{rr} + r f_r\right]\left(\frac{A_8}{r} - \omega_o r\right). \tag{33}$$

Integration of (32) yields

$$h = \frac{\omega_o\left(A_6 - 1\right)}{R_\sigma}\alpha + \frac{R_H}{2r^2 R_\sigma}\left(-2r^2\omega_o + r^2 f_{rr} + r f_r\right)\left[A_6 - 1\right]\alpha + K_1(r) \tag{34}$$

where $K_1(r)$ is an unknown function to be determined. This function $K_1(r)$ is determined from the fact that the expression for h_r obtained from (34) must be the same as that given by (33).

Differentiating (34) w.r.t. r, we get

$$h_r = \frac{-R_H}{2R_\sigma}\left(A_6 - 1\right)\alpha\left[\frac{1}{r^2}\left(-2r^2\omega_o + r^2 f_{rr} + r f_r\right)\right]_r + K_r. \tag{35}$$

Equations (33) and (35) give the same expression for h_r provided

$$\left[\frac{1}{r^2}\left(-2r^2\omega_o + r^2 f_{rr} + r f_r\right)\right]_r = 0 \tag{36}$$

and

$$K_r = -\omega_o\left(\frac{A_8}{r} - \omega_o r\right)\frac{-R_H}{2}\left[-C_1 R_\sigma - \omega_o + \left(\frac{\omega_o - C_1 R_\sigma}{r^2}\right)\right]\left(\frac{A_8}{r} - \omega_o r\right). \tag{37}$$

Equation (36) gives

$$r^2 f_{rr} + r f_r = \left(A_{11} + 2\omega_o\right) r^2 \tag{38}$$

where A_{11} is an arbitrary constant. The function $f(r)$ in (31) satisfies (38) provided

$$\begin{aligned} A_6 &= 1, \\ A_{11} &= -2C_1 R_\sigma. \end{aligned}$$

Integration of Equation (37) yields

$$K = \left[-\omega_o + \frac{R_H}{2}\left(\omega_o + C_1 R_\sigma\right)\right]\left(A_8 \ln r - \frac{\omega_o r^2}{2}\right) - \frac{R_H}{2}\left(\omega_o - C_1 R_\sigma\right)\left[\frac{-A_8}{2r^2} - \omega_o \ln r\right] + A_{12}$$

where A_{12} is an arbitrary constant.

Therefore, the expressions for ψ, ϕ, h are

$$\begin{aligned} \psi &= A_7 + \frac{A_8}{2}\ln\left(2x^2 + 2y^2\right) - \omega_o\left(x^2 + y^2\right) \\ \phi &= -C_1 R_\sigma\left(x^2 + y^2\right) + \frac{A_8}{2}\ln\left(2x^2 + 2y^2\right) + A_7^* \\ h &= \left[-\omega_o + \frac{R_H}{2}\left(\omega_o + C_1 R_\sigma\right)\right]\left\{\frac{A_8}{2}\ln\left(2x^2 + 2y^2\right) - \omega_o\left(x^2 + y^2\right)\right\} \\ &\quad - \frac{R_H}{2}\left(\omega_o - C_1 R_\sigma\right)\left[\frac{-A_8}{4\left(x^2 + y^2\right)} - \frac{\omega_o}{2}\ln\left(2x^2 + 2y^2\right)\right] + A_{12} \end{aligned}$$

where $A_7^* = A_7 + A_9$.

(ii) For $\omega = g(r)$, the solution of Equation (6–9), is

$$\psi = e_7 + \frac{e_8}{2}\ln\left(2x^2+2y^2\right) - \frac{e_1}{2}\left(x^2+y^2\right)\ln\left(2x^2+2y^2\right) + \left(e_1-e_2\right)\left(x^2+y^2\right) + \frac{1}{R_H-1}\tan^{-1}\left[\frac{x-y}{x+y}\right] + e_9$$

$$\phi = \frac{2R_\sigma}{R_e R_H}\left(R_H-1\right)\left\{\frac{e_1}{2}\left(x^2+y^2\right)\ln\left(2x^2+2y^2\right) + \left(e_2-\frac{e_1}{2}\right)\left(x^2+y^2\right) - \left(e_3+C_1R_HR_\sigma\right)\left(x^2+y^2\right)\right\} + e_6$$

$$\begin{aligned} h = & -\frac{e_1}{R_\sigma}\tan^{-1}\left(\frac{x-y}{x+y}\right) + \left[e_9 + \frac{1}{R_H-1}\tan^{-1}\left(\frac{x-y}{x+y}\right)\right] \\ & + \left\{\frac{-e_1}{2}\ln\left(2x^2+2y^2\right) - e_2 + e_3\right\}\left[e_9 + \frac{1}{R_H-1}\tan^{-1}\left(\frac{x-y}{x+y}\right)\right] + \frac{e_1e_2}{8}\left[\ln\left(2x^2+2y^2\right)\right]^2 \\ & + \frac{e_8}{2}\left(e_2-e_3\right)\ln\left(2x^2+2y^2\right) + \left\{\frac{e_2D_2 + D_1\left(e_2-e_3\right)}{2}\right\}\left[\ln\left(2x^2+2y^2\right) - 1\right]\left(x^2+y^2\right) \\ & + \frac{e_1D_1}{4}\left\{\left[\ln\left(2x^2+2y^2\right)\right]^2 - 2\ln\left(2x^2+2y^2\right) + 2\right\}\left(x^2+y^2\right) + \left(e_2-e_3\right)D_2\left(x^2+y^2\right) + e_{10} \end{aligned}$$

where $e_1 \ldots, e_{10}$ are the arbitrary constants and

$$D_1 = \left[\frac{2(R_H-1)}{R_H} - 1\right]e_1$$

$$D_2 = \left\{\frac{3e_1}{2} - 2e_2\left[1 - \frac{R_H-1}{R_H}\right] - \frac{2e_3(R_H-1)}{R_H} - 2C_1(R_H-1)R_\sigma\right\},$$

and e_1, e_3, C_1 satisfy

$$\left(\frac{5R_H-4}{R_H-1}\right)e_1 + 2e_2 = 2\left(1-R_H\right)C_1R_\sigma \ .$$

CONCLUSIONS

In the present work, we have determined some exact solutions of equations of motion of an electrically conducting fluid moving in a magnetic field.

REFERENCES

[1] H. Alfven, *Cosmical Electrodynamics*, Oxford: Clarendon Press, 1950.

[2] G. Kingston and R. Talibot, "The Solutions to a Class of Magnetohydrodynamic Flows with Orthogonal Magnetic and Velocity Distributions", *Z. Angew. Math. Phys.*, **20** (1969) pp.956–965.

[3] V. I. Nath and O. P. Chandna, "On Plane Viscous Magnetohydrodynamic Flows", *Quart. Appl. Math.*, **31** (1973), pp.351–362.

[4] O. P. Chandna and H. Toews, "Plane Constantly-Inclined MHD Flow with Isometric Geometry", *Quart. Appl. Math.*, **35** (1977), pp.331–337.

[5] R. M. Barron and O. P. Chandna, "Hodograph Transformations and Solutions in Constantly-Inclined MHD Plane Flows", *Eng. Math.*, **15** (1981), pp.221–220.

[6] O. P. Chandna, C. E. Stoeros, and M. K. Swaminathan, "Some Solutions for Finitely Conducting Viscous MHD Plane Flows", *Quart. Appl. Math.*, July 1984, pp.143–158.

[7] O. P. Chandna and N. Rudraiah, "Solutions for Steady Plane Aligned Electromagneto-Gasdynamics", *Quart. Appl. Math.*, April 1985, pp.85–95.

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