EXPLICIT EXPRESSIONS OF THE RELATION BETWEEN THE SPECIFIC GENERATORS OF $R[\alpha] \cap R[\alpha^{-1}]$ AND THOSE OF $R[\alpha - a] \cap R[(\alpha - a)^{-1}]$ FOR $a \in R$

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الخلاصة :

هذا البحث يكمل عمل [5] بتسليط الضوء، عبر دليل تقني، حول العلاقة بين مُوَلّدات $R[\alpha^{-1}] \cap R[\alpha]$ ومُوَلّدات $R[(\alpha-a)^{-1}] \cap R[\alpha-a]$ ، حيث a عنصر في R.

ABSTRACT

This paper completes a former work [5] by making explicit, with a technical proof, the relation between the generators of $R[\alpha] \cap R[\alpha^{-1}]$ and those of $R[\alpha - a] \cap R[(\alpha - a)^{-1}]$, for $a \in R$.

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Let R be an integral domain with quotient field K and let $R[X]$ denote a polynomial ring over R. Let α be an element of an algebraic field extension of K, a an element of R, and $\varphi_{\alpha-a}(X)$ the monic minimal polynomial of $\alpha - a$ over K. Then the degree of $\varphi_{\alpha-a}(X)$ is unchanged for any $a \in R$. So let d be this constant value and write

$$\varphi_{\alpha-a}(X) := X^d + \eta_1^{(a)} X^{d-1} + \cdots + \eta_d^{(a)},$$

where $\eta_1^{(a)}, \ldots, \eta_d^{(a)} \in K$.

Let $\zeta_i^{(a)} := (\alpha - a)^i + \eta_1^{(a)}(\alpha - a)^{i-1} + \eta_2^{(a)}(\alpha - a)^{i-2} + \cdots + \eta_i^{(a)}$, $\eta_i = \eta_i^{(0)}$, and $\zeta_i = \zeta_i^{(0)}$, for $i = 1, \ldots, d-1$.

Let $\pi : R[X] \to R[\alpha]$ be the R-algebra homomorphism defined by $\pi(X) = \alpha$. Define

$$I_{[\alpha]} := \bigcap_{i=1}^{d} (R :_R \eta_i),$$

where $(R :_R \eta_i) := \{ c \in R \mid c\eta_i \in R \}$. Then $I_{[\alpha]}$ is an ideal of R. An element α is called an *anti-integral* element of degree d over R if $\operatorname{Ker}(\pi) = I_{[\alpha]}\varphi_\alpha(X)R[X]$ (*cf.*[4]).

Let $f : \mathbf{Z} \to R$ be a ring homomorphism sending $n \in \mathbf{Z}$ to $n \cdot 1 \in R$, where $\mathbf{Z}$ denotes the ring of integers and $F := \mathbf{Z}/\operatorname{Ker}(f)$, which is a subdomain of R.

To justify our title, recall the result in [1]: if α is an anti-integral element of degree d over R then $R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}$.

By use of this result, it is proved that $R[\alpha] \cap R[\alpha^{-1}] = R[\alpha - a] \cap R[(\alpha - a)^{-1}]$, for any $a \in R$, if α is an anti-integral element over R in [5]. In the proof of this equality, we have shown by induction that $\zeta_k^{(a)} - \zeta_k \in F[a]\zeta_{k-1} + \cdots + F[a]\zeta_1 + F[a] + F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$.

But by an explicit calculation, we prove that we may simplify this assertion showing that the part $F[a]\eta_1 + \cdots + F[a]\eta_{d-1}$ is not needed (Theorem 4).

Throughout this paper, we use the notation mentioned above unless otherwise specified.

Our notation is standard and unexplained terminology is referred to [2].

The following lemmas are seen in [5].

Lemma 1. (1) $\varphi_{\alpha-a}(X) = \varphi_\alpha(X + a)$.

(2) $K(\zeta_1, \ldots, \zeta_{d-1}) = K(\zeta_1^{(a)}, \ldots, \zeta_{d-1}^{(a)}) = K(\alpha)$.

Proof.

(1) For some $\lambda_1, \ldots, \lambda_d \in K$, we can write:

$$\varphi_\alpha(X) = (X - a)^d + \lambda_1(X - a)^{d-1} + \cdots + \lambda_d,$$

since $K[X] = K[X - a]$. Then $\varphi_{\alpha-a}(X) = X^d + \lambda_1 X^{d-1} + \cdots + \lambda_d = \varphi_\alpha(X + a)$.

(2) It is clear that $K(\zeta_1^{(a)}) = K(\alpha)$. □

Lemma 2.

$$\eta_k^{(a)} = \binom{d}{k} a^k + \eta_1 \binom{d-1}{k-1} a^{k-1} + \eta_2 \binom{d-2}{k-2} a^{k-2} + \cdots + \eta_{k-1} \binom{d-k+1}{1} a + \eta_k,$$

for $k = 1, \ldots, d$.

Proof. Compare the coefficients of $\varphi_\alpha(X + a) = \varphi_{\alpha-a}(X)$ (*cf.* Lemma 1(1)). □

Lemma 3. *For $k = 1, \ldots, d-1$,*

$$\begin{aligned} \zeta_{k+1} &= \alpha\zeta_k + \eta_{k+1}, \qquad \zeta_1 = \alpha + \eta_1 \\ \zeta_{k+1}^{(a)} &= (\alpha - a)\zeta_k^{(a)} + \eta_{k+1}^{(a)}, \quad \textit{and} \quad \zeta_1^{(a)} = (\alpha - a) + \eta_1^{(a)}. \end{aligned}$$

Proof. Trivial. □

It is well-known that

$$\binom{n+i}{i+1} = \binom{n+i-1}{i+1} + \binom{n+i-1}{i},$$

for $i \geq 0$.

The aim of this paper is to give the following explicit equality.

Theorem 4.

$$\zeta_k^{(a)} = \zeta_k + \binom{d-k}{1}a\zeta_{k-1} + \binom{d-k+1}{2}a^2\zeta_{k-2} + \binom{d-k+2}{3}a^3\zeta_{k-3} + \cdots$$

$$\cdots + \binom{d-3}{k-2}a^{k-2}\zeta_2 + \binom{d-2}{k-1}a^{k-1}\zeta_1 + \binom{d-1}{k}a^k,$$

i.e.,

$$\zeta_k^{(a)} = \sum_{i=0}^{k-1}\binom{d-k+i-1}{i}a^i\zeta_{k-i} + \binom{d-1}{k}a^k,$$

for $k = 1, \ldots, d-1$.

Proof. Let A_k denote the righthand side of the above equality, *i.e.*, $A_k := \zeta_k + \binom{d-k}{1}a\zeta_{k-1} + \binom{d-k+1}{2}a^2\zeta_{k-2} + \binom{d-k+2}{3}a^3\zeta_{k-3} + \cdots + \binom{d-3}{k-2}a^{k-2}\zeta_2 + \binom{d-2}{k-1}a^{k-1}\zeta_1 + \binom{d-1}{k}a^k$. We shall prove $\zeta_k^{(a)} = A_k$, by induction on k. For $k = 1$, $A_1 = \zeta_1 + \binom{d-1}{1}a = \zeta_1 + (d-1)a$ and $\zeta_1^{(a)} = (\alpha - a) + \eta_1^{(a)} = (\alpha - a) + \{\binom{d}{1}a + \eta_1\} = \alpha - a + da + \eta_1 = \alpha - a + da + (\zeta_1 - \alpha) = \zeta_1 + (d-1)a$, by Lemma 3. Hence $\zeta_1^{(a)} = A_1$. Assume that $\zeta_k^{(a)} = A_k$. We shall show that $\zeta_{k+1}^{(a)} = A_{k+1}$. By use of Lemmas 2 and 3, and by induction hypothesis, we have the following equality:

$$\begin{aligned} \zeta_{k+1}^{(a)} &= (\alpha - a)\zeta_k^{(a)} + \eta_{k+1}^{(a)} \\ &= (\alpha - a)\left\{\zeta_k + \binom{d-k}{1}a\zeta_{k-1} + \binom{d-k+1}{2}a^2\zeta_{k-2} + \cdots + \binom{d-2}{k-1}a^{k-1}\zeta_1 + \binom{d-1}{k}a^k\right\} + \eta_{k+1}^{(a)} \\ &= \alpha\zeta_k + \binom{d-k}{1}a\alpha\zeta_{k-1} + \binom{d-k+1}{2}a^2\alpha\zeta_{k-2} + \cdots + \binom{d-2}{k-1}a^{k-1}\alpha\zeta_1 + \binom{d-1}{k}a^k\alpha \\ &\quad -a\zeta_k - \binom{d-k}{1}a^2\zeta_{k-1} - \binom{d-k+1}{2}a^3\zeta_{k-2} - \cdots - \binom{d-2}{k-1}a^k\zeta_1 - \binom{d-1}{k}a^{k+1} + \eta_{k+1}^{(a)} \end{aligned}$$

$$
\begin{aligned}
&= (\zeta_{k+1} - \eta_{k+1}) + \binom{d-k}{1} a(\zeta_k - \eta_k) + \binom{d-k+1}{2} a^2(\zeta_{k-1} - \eta_{k-1}) + \cdots \\
&\quad \cdots + \binom{d-2}{k-1} a^{k-1}(\zeta_2 - \eta_2) + \binom{d-1}{k} a^k(\zeta_1 - \eta_1) - a\zeta_k - \binom{d-k}{1} a^2\zeta_{k-1} \\
&\quad - \binom{d-k+1}{2} a^3\zeta_{k-2} - \cdots\cdots - \binom{d-2}{k-1} a^k\zeta_1 - \binom{d-1}{k} a^{k+1} + \eta_{k+1}^{(a)} \\
&= \zeta_{k+1} + \binom{d-k}{1} a\zeta_k + \binom{d-k+1}{2} a^2\zeta_{k-1} + \cdots\cdots + \binom{d-2}{k-1} a^{k-1}\zeta_2 + \binom{d-1}{k} a^k\zeta_1 \\
&\quad - a\zeta_k - \binom{d-k}{1} a^2\zeta_{k-1} - \cdots\cdots - \binom{d-3}{k-2} a^{k-1}\zeta_2 - \binom{d-2}{k-1} a^k\zeta_1 - \binom{d-1}{k} a^{k+1} \\
&\quad - \binom{d-1}{k} a^k\eta_1 - \binom{d-2}{k-1} a^{k-1}\eta_2 - \cdots\cdots - \binom{d-k}{1} a\eta_k - \eta_{k+1} \\
&\quad + \left\{ \binom{d}{k+1} a^{k+1} + \binom{d-1}{k} a^k\eta_1 + \cdots + \binom{d-k+1}{2} a^2\eta_{k-1} + \binom{d-k}{1} a\eta_k + \eta_{k+1} \right\} \\
&= \zeta_{k+1} + \left\{ \binom{d-k}{1} - 1 \right\} a\zeta_k + \left\{ \binom{d-k+1}{2} - \binom{d-k}{1} \right\} a^2\zeta_{k-1} + \cdots \\
&\quad \cdots + \left\{ \binom{d-1}{k} - \binom{d-2}{k-1} \right\} a^k\zeta_1 + \left\{ \binom{d}{k+1} - \binom{d-1}{k} \right\} a^{k+1} \\
&= \zeta_{k+1} + \binom{d-k-1}{1} a\zeta_k + \binom{d-k}{2} a^2\zeta_{k-1} + \cdots\cdots + \binom{d-2}{k} a^k\zeta_1 + \binom{d-1}{k+1} a^{k+1} \\
&= A_{k+1},
\end{aligned}
$$

as desired. □

Remark 5. Theorem 4 shows that for $k = 1, \ldots, d-1$,

$$\zeta_k^{(a)} - \zeta_k \in F[a]\zeta_{k-1} + \cdots + F[a]\zeta_1 + F[a].$$

Corollary 6. *As R-submodules of $K[\alpha]$, $R + R\zeta_1 + \cdots + R\zeta_{d-1} = R + R\zeta_1^{(a)} + \cdots + R\zeta_{d-1}^{(a)}$, for any $a \in R$.*

Let H be an ideal of R and let $a \in R$. Put:

$$C_H^{(a)} := R + H\zeta_1^{(a)} + \cdots + H\zeta_{d-1}^{(a)},$$

and:

$$C_H := R + H\zeta_1 + \cdots + H\zeta_{d-1},$$

which are R-submodules of $K[\alpha]$.

Corollary 7. *$C_H = C_H^{(a)}$, for any $a \in R$.*

Proof. By Corollary 6, we have $R+R\zeta_1+\cdots+R\zeta_{d-1} = R+R\zeta_1^{(a)}+\cdots+R\zeta_{d-1}^{(a)}$, and hence $H+H\zeta_1+\cdots+H\zeta_{d-1} = H+H\zeta_1^{(a)}+\cdots+H\zeta_{d-1}^{(a)}$. Thus $C_H = R+H\zeta_1+\cdots+H\zeta_{d-1} = R+H\zeta_1^{(a)}+\cdots+H\zeta_{d-1}^{(a)} = C_H^{(a)}$. □

Remark 8. In [3], the following results are shown.

(1) Assume that α is an anti-integral element of degree $d \geq 2$ over a Noetherian domain R. If $H \subseteq I_{[\alpha]}$, then C_H is a subring of $R[\alpha]$.

(2) Assume that α is an anti-integral element of degree d over R. Then $R[\alpha] \cap R[\alpha^{-1}] = C_{I_{[\alpha]}}$ and $R[\alpha - a] \cap R[(\alpha - a)^{-1}] = C_{I_{[\alpha-a]}}$ (*cf.* [5], [1]). For this purpose, we showed that $I_{[\alpha]} = I_{[\alpha-a]}$ for any $a \in R$ and that $\alpha - a$ is an anti-integral element of degree d over R for any $a \in R$. So the main result in [5] mentioned in the beginning of this paper is obtained as a consequence of the more concrete argument above. But Theorem 4 seems to be interesting in itself.

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