STEINITZ PROPERTIES IN TRIVIAL EXTENSIONS OF COMMUTATIVE RINGS

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الخلاصة :

يدرس هذا البحث إمكانية نقل خاصية « ستينتز Steinitz » ومفاهيم أخرى مرتبطة بها إلى الامتدادات البديهية للحلقات التبادلية. نثبت كذلك أن هذه المفاهيم ليست محلية .

ABSTRACT

In this work, we study the transfer of Steinitz, semi-Steinitz, and weakly semi-Steinitz properties in trivial extensions. We also show that semi-Steinitz and weakly semi-Steinitz properties are not local properties.

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1. INTRODUCTION

All rings considered below are commutative with unit, and, by a proper ideal, we mean a nonzero ideal distinct from the whole ring. A ring A is a Steinitz ring if any linearly independent subset of a free A-module F can be extended to a basis of F by adjoining elements of a given basis. We say that an ideal I of A is T-nilpotent ("T" for transfinite) if for any sequence $\{x_i\}$ of elements of I, there is an integer n such that: $\prod_{i=1}^{n} x_i = 0$. In [7, Theorem 2], Chwe and Neggers showed that Steinitz rings are precisely the local rings with T-nilpotent maximal ideals. Simultaneously, in [12], Lenzing obtains the same result and shows, in addition, that A is a Steinitz ring if and only if A satisfies the "weaker" property, that is, any linearly independent subset of a free A-module F can be extended to a basis of F. On the other hand, in [5, Proposition (5.4)], Cox and Pendleton showed that Steinitz rings are exactly the local rings A such that every flat A-module is free.

We say that A is a semi-Steinitz ring if any linearly independent finite subset of a finitely generated free A-module F can be extended to a basis of F, by adjoining elements of a given basis of F. In [13, Theorem 2.1], Nashier and Nichols showed that A is a semi-Steinitz ring if and only if A is local and every finitely generated proper ideal of A has a non-zero annihilator.

A ring A is said to be weakly semi-Steinitz if every linearly independent finite subset of a finitely generated free A-module F can be extended to a basis of F. We say that A is a Hermite ring if for every $a_1, \ldots, a_n$ in A such that $\sum_{i=1}^{n} a_i A = A$, the row $[a_1, \ldots, a_n]$ can be completed to an invertible square matrix (*cf.* [11, I.4.6]). It is not difficult to see that a ring A is Hermite if and only if it is so modulo its Jacobson radical. On the other hand, Nashier and Nichols [13, Theorem 2.2] established that A is a weakly semi-Steinitz ring if and only if A is a Hermite ring and satisfies the (CH)-property (*i.e.*, every finitely generated proper ideal of A has a non-zero annihilator).

Let A be a ring and E an A-module. Let $R = A\alpha E$ be the set of pairs (a, e), $a \in A$, $e \in E$, with pairwise addition and multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. This is a commutative ring with unity $(1, 0)$, called the trivial extension of A by E. An ideal J of R is of the form $J = I\alpha E'$, where I is an ideal of A and E' is a A-submodule of E such that $IE \subseteq E'$. If J is finitely generated then so is I (*cf.* [10, Theorem 25.1]). Further, $Spec(R) = \{P\alpha E / P \in Spec(A)\}$ (*cf.* [10, Theorem 25.1]).

In Section 2, we examine the transfer of Steinitz, Hermite, semi-Steinitz, and weakly semi-Steinitz properties to trivial extensions. In the third section, we show that the properties: semi-Steinitz, weakly semi-Steinitz, and the (CH)-property are not local properties (Example 3.1). Also, we show that for a ring A, if A_M satisfies the (CH)-property for each maximal ideal M, then so does A (Proposition 3.2). Finally, we show that for a Noetherian ring A, if A_M is a weakly semi-Steinitz ring for each maximal ideal M, then so is A (Theorem 3.3).

2. STEINITZ PROPERTIES IN TRIVIAL EXTENSIONS

We first examine the transfer of Steinitz and Hermite properties to trivial extensions.

Theorem 2.1. *Let A be a commutative ring, E an A-module, and R the trivial extension ring of A by E. Then:*

(1) *R is a Steinitz ring if and only if so is A.*

(2) *R is a Hermite ring if and only if so is A.*

Proof.

(1) Assume that R is a Steinitz ring. Hence, R is a local ring and A is also a local ring (*cf.* [10, Theorem 25.1]). Let m be the maximal ideal of A, so $M := m\alpha E$ is the maximal ideal of R. It remains to show that m is a T-nilpotent ideal (*cf.* [7, Theorem 2]).

Let $(a_i)_i$ be a sequence of elements of m. Hence $((a_i, 0))_i$ is a sequence of elements of the maximal ideal M which is T-nilpotent. Therefore, there exists a positive integer n such that $0 = \prod_{i=1}^{n}(a_i, 0) = (\prod_{i=1}^{n} a_i, 0)$, so $\prod_{i=1}^{n} a_i = 0$ and m is a T-nilpotent ideal of A.

Conversely, assume that A is a Steinitz ring, then A is a local ring. Let m be its T-nilpotent maximal ideal. Hence R is a local ring with maximal ideal $M = m\alpha E$. We claim that M is a T-nilpotent ideal. Indeed, let $((a_i, e_i))_i$ be a sequence of elements of M. Then $(a_i)_i$ is a sequence of elements of m and therefore there exists a non-negative integer n such that $\prod_{i=1}^{n} a_i = 0$ since m is a T-nilpotent ideal. So, $\prod_{i=1}^{n}(a_i, e_i) = (\prod_{i=1}^{n} a_i, e) = (0, e)$, where $e \in E$. Thus we may assume that $a_1 = 0$.

Since m is T-nilpotent, there exists a positive integer q such that $\prod_{i=2}^{q} a_i = 0$. Then $\prod_{i=2}^{q}(a_i, e_i)$ is of the form $(0, e')$ for some $e' \in E$. Hence, $\prod_{i=1}^{q}(a_i, e_i) = (0, e_1)(0, e') = (0, 0) = 0_R$ and M is T-nilpotent. Therefore, R is a Steinitz ring (*cf.* [7, Theorem 2]).

(2) By [10, Theorem 25.1], $J(R) = J(A)\alpha E$, and hence $R/J(R) = (A\alpha E)/(J(A)\alpha E) \cong A/J(A)$. Therefore, R is a Hermite ring if and only if A is a Hermite ring (since a ring is Hermite if and only if it is so modulo its Jacobson radical).

Corollary 2.2. *Let K be a field, E a K-vector space, and R the trivial extension of K by E. Then R is a Steinitz ring.*

Proof. It follows from Theorem 2.1.(1) since K is a Steinitz ring.

Example 2.3. *Any trivial extension ring A of K by a K-vector space E is a Steinitz ring (*cf. *Corollary 2.2). Therefore, by Theorem 2.1.(1), the trivial extension ring R of A by any A-module E_A is a Steinitz ring.*

Remark 2.4. *Even if a ring is not semi-Steinitz, it could have a semi-Steinitz trivial extension. Indeed, let A be a local domain which is not a field, M its maximal ideal, $E = A/M$, and R the trivial extension ring of A by E. The ring R is semi-Steinitz by [2, Proposition 5] since R is a local ring. On the other hand, A does not satisfy the (CH)-property since A is a domain which is not a field. Therefore, A is not weakly semi-Steinitz and hence not semi-Steinitz.*

Theorem 2.5. *Let A be a ring, E an A-module, and R the trivial extension of A by E. Then:*

(1) *(a) If A is weakly semi-Steinitz, then so is R.*
    (b) If A is semi-Steinitz, then so is R.

(2) *Assume that either E is a submodule of a free A-module or $E = A/P$, where P is a prime ideal of A with non-zero annihilator. Then:*
    (a) R is semi-Steinitz if and only if so is A.
    (b) R is weakly semi-Steinitz if and only if so is A.

The proof of this theorem relies mainly on the following Lemma.

Lemma 2.6. *Let A, E, R be as in Theorem 2.5. Then:*

(1) *If A satisfies the (CH)-property, then so does R.*

(2) *Assume that either (a) E is a submodule of a free A-module, or (b) $E = A/P$, where P is a prime ideal of A with non-zero annihilator. Then A satisfies the (CH)-property if and only if so does R.*

Proof.

(1) Let $J = I\alpha E'$ be a finitely generated proper ideal of R, where I is a finitely generated proper ideal of A and E' is an A-submodule of E such that $IE \subseteq E'$. Since A satisfies the (CH)-property, there exists a non-zero element a of A such that $aI = 0$. Two cases are then possible:

Case 1: $aE \neq 0$. Let e be an element of E, such that $ae \neq 0$, and $b = (0, ae) \in R - \{0\}$. Hence, $bJ = (0, ae)(I\alpha E') = 0$ since $aI = 0$.

Case 2: $aE = 0$. Let $b = (a, 0) \in R - \{0\}$. Since $aI = 0$ and $aE' \subseteq aE = 0$, so $bJ = (a, 0)(I \propto E') = aI \propto aE' = 0$. It follows that J has a non-zero annihilator and therefore R satisfies the (CH)-property.

(2)(a) Assume that E is a submodule of a free A-module F. If A is a (CH)-ring, then so is R by (1). Conversely, assume that R is a (CH)-ring. We wish to show that A is a (CH)-ring. Let I be a finitely generated proper ideal of A. Then $J := (I\alpha 0)R$ is a finitely generated proper ideal of R. Hence, there exists a non-zero element $(a, e) \in R$ such that $0 = (a, e)J = (a, e)(I\alpha 0)R$, since R satisfies the (CH)-property. Therefore, $aI = 0$ and $eI = 0$. Two cases are then possible:

Case 1: $a \neq 0$. Then I has a non-zero annihilator since $aI = 0$.

Case 2: $a = 0$. In this case, $eI = 0$ and $e \neq 0$ since $(a, e) \neq 0$.

On the other hand, $e \in E \subseteq F$ is a free A-module, then e is of the form: $e = \sum_{i=1}^{n} a_i b_i$, where $B = \{b_1, \dots, b_n\}$ is a subset of a basis of F and $a_i \in A$ for each $i = 1, \dots, n$. It follows that, $0 = eI = \sum_{i=1}^{n}(a_i I)b_i$ and then $a_i I = 0$ for each $i = 1, \dots, n$. Now, let $j \in \{1, \dots, n\}$ be such that $a_j \neq 0$ (possible since $e = \sum_{i=1}^{n} a_i b_i \neq 0$). Therefore, $a_j I = 0$ and $a_j \neq 0$.

(2)(b) Assume that $E = A/P$, where P is a prime ideal of A with non-zero annihilator. If A is a (CH)-ring, then so is R by (1). Conversely, assume that R is a (CH)-ring. We wish to show that A is a (CH)-ring. Let I be a finitely generated proper ideal of A. Then, J $(:= (I\alpha 0)R)$ is a finitely generated proper ideal of R and so there exists a non-zero element $(a, e + P)$ of R such that $(a, e + P)J = 0$. Hence, $aI = 0$ and $eI \subseteq P$. Two cases are possible:

Case 1: $a \neq 0$. Then I has a non-zero annihilator since $aI = 0$.

Case 2: $a = 0$. Since $(0, e + P) \neq 0$, then $e \notin P$ and $I \subseteq P$ since $eI \subseteq P$ and P is prime. Therefore, since P has a non-zero annihilator, then so is I.

Proof of Theorem 2.5. The proof follows by combining Theorem 2.1.(2), Lemma 2.6, and the fact that R is a local ring if and only if so is A (*cf.* [10, Theorem 25.1]).

The following is an example of a semi-Steinitz ring which is not a Steinitz ring.

Example 2.7. *Let A be a local domain which is not a field, M its maximal ideal, $E = A/M$, and R the trivial extension ring of A by E. The ring R is semi-Steinitz by [2, Prop.5] since R is a local ring. However, R is not a Steinitz ring by Theorem 2.1.(1) since A is not a Steinitz ring (since M is not a T-nilpotent ideal of A).*

The following is an example of a weakly semi-Steinitz ring which is not a semi-Steinitz ring.

Example 2.8. *Let K be a field, A a weakly semi-Steinitz ring, and $B := K \times A$ the direct product of K by A. Let $M = 0 \times A$ and R be the trivial extension ring of B by B/M. Then:*

(1) *M is a maximal ideal of B (since $B/M \cong K$) and $(1,0)M = 0_B$.*

(2) *By using [13, Theorem 2.2], it is easy to see that B is weakly semi-Steinitz since K and A are weakly semi-Steinitz.*

(3) *R is weakly semi-Steinitz by Theorem 2.5.(2) since B is weakly semi-Steinitz.*

(4) *Since B is not local then B is not semi-Steinitz by [13, Theorem 2.1]. Therefore, by Theorem 2.5.(2), R is not semi-Steinitz.*

Remark 2.9. *Let A be a non-local domain, $E = A/P$, where P is a prime ideal of A, and let R be the trivial extension of A by E. Then A and R are not weakly semi-Steinitz.*

Proof. We shall show that A and R do not satisfy the (CH)-property. This is clear for A. On the other hand, first note that A contains a non-invertible element b such that $b \notin P$. Let $J = R(b,0)$. Then J is a proper ideal of R. We claim that J has no non-zero annihilator. Indeed, let $(a, e+P) \in R$ such that $(a, e+P)J = 0$, where $a, e \in A$. But $(a, e+P)(b,0) = (ab, eb+P)$; so $ab = 0$ and $eb \in P$. So, $a = 0$ (since A is a domain and $b \neq 0$) and $e \in P$ (since $P \in Spec(A)$ and $b \notin P$). Therefore, $(a, e+P) = 0$ and R is not a (CH)-ring.

Next, we give a new characterization *via* trivial extensions:

Proposition 2.10. *Let A be a Noetherian ring, E an A-module, and R the trivial extension ring of A by E. Then A is a weakly semi-Steinitz ring if and only if R is a weakly semi-Steinitz ring and $ae \neq 0$ for every non-zero-divisor a of A and every non-zero element e of E.*

Proof. Assume that A is a weakly semi-Steinitz ring. By Theorem 2.5.(1)(a), R is a weakly semi-Steinitz ring. On the other hand, let a be a non zero-divisor of A and e a non-zero element of E. Since A is a weakly semi-Steinitz ring, a is a unit, hence $ae \neq 0$.

Conversely, since A is a Noetherian ring, it suffices to show that every non-zero-divisor of A is a unit (*cf.* [13, Corollary 2.5]). Let a be a non-zero-divisor of A, then, $(a,0)$ is a non-zero-divisor of R, since $(a,0)(x,e) = (0,0)$ implies $ax = 0$ and $ae = 0$, hence $x = 0$ and $e = 0$. Since R is a weakly semi-Steinitz ring, then $(a,0)$ is a unit in R and therefore a is a unit in A.

3. LOCAL–GLOBAL QUESTIONS

We first show that semi-Steinitz and weakly semi-Steinitz properties are not local properties.

Example 3.1. *Let A be a local domain which is not a field, M_0 its maximal ideal, $E = A/M_0$, and R the trivial extension ring of A by E. Then:*

(1) *R is a semi-Steinitz ring (and hence a weakly semi-Steinitz ring).*

(2) *R_P is not a weakly semi-Steinitz ring (and hence not a semi-Steinitz ring), for every non-maximal prime ideal $P \neq 0 \alpha E$.*

Proof.

(1) R is a semi-Steinitz ring (*cf.* Example 2.7), so R is also a weakly semi-Steinitz ring.

(2) Let $P(\neq 0\alpha E)$ be a non-maximal prime ideal of R, that is, $P = P_0\alpha E$, where $P_0(\neq 0)$ is a prime ideal of A such that $P_0 \neq M_0$ by [10, Theorem 25.1] and the fact that R is a local ring with a maximal ideal $M := M_0\alpha E$. Our aim is to show that R_P does not satisfy the (CH)-property and this suffices to show that R_P is neither a weakly semi-Steinitz ring nor a semi-Steinitz ring (*cf.* [13, Theorem 2.1 and Theorem 2.2]).

Let $I = R(a,0)$ be a finitely generated proper ideal of R, where a is non-zero element of P_0, and set $J := I_P$. Then J is a finitely generated ideal of R_P. We claim that J is a proper ideal of R_P. Indeed, $J = I_P \subseteq PR_P(\neq R_P)$. On the other hand, let $(b,y) \notin P(= P_0\alpha E)$. So, $b \neq 0$ and hence $ab \neq 0$, since A is a domain and $a \neq 0$. Therefore, $(b,y)(a,0) = (ab, ay) \neq 0$ for each $(b,y) \notin P$; so $(a,0)/1 \neq 0_{R_P}$ and then $J \neq 0$. It remains to show that J has no non-zero annihilator. Let $(c,z)/1 \in R_P$ such that $((c,z)/1)J = 0_{R_P}$. We claim that $(c,z)/1 = 0_{R_P}$. Indeed, since $((c,z)/1)J = 0_{R_P}$, then $((c,z)/1)((a,0)/1) = 0_{R_P}$. Hence, there exists $(d,e) \notin P$ such that $(d,e)(c,z)(a,0) = 0_R$. So $acd = 0_A$. On the other hand, A is a domain, $a \neq 0$ and $d \neq 0$ (since $(d,e) \notin P(= P_0\alpha E)$), thus $c = 0_A$. In addition, $(t,0)(0,z) = (0,tz) = 0_R$ for each $t \in M_0 - P_0$. Therefore, $(c,z)/1 = (0,z)/1 = 0_{R_P}$ since $(t,0) \notin P$. Hence, J has no non-zero annihilator and R_P does not satisfy the (CH)-property.

Proposition 3.2. *Let A be a ring. If A_M satisfies the (CH)-property for each maximal ideal M, then so does A.*

Proof. Let $I := \sum_{i=1}^{n} Ax_i$ be a finitely generated proper ideal of A. Let M be a maximal ideal such that $I \subseteq M$. Then, IA_M is a finitely generated ideal of A_M. Two cases are then possible:

Case 1: $IA_M = 0$. So, for each $i = 1,\ldots,n$, $x_iA_M = 0$. So there exist $s_i \notin M$ such that $s_ix_i = 0$. Set $s = \prod_{i=1}^{n} s_i(\notin M)$. For each $i = 1,\ldots,n$, $sx_i = 0$, thus $sI = 0$. Therefore, I has a non-zero annihilator, since $s \neq 0$ ($s \notin M$).

Case 2: $IA_M \neq 0$. The ideal IA_M is a finitely generated proper ideal of A_M, so there exists a non-zero element a/u of A_M such that $(a/u)IA_M = 0_{A_M}$, where a is a non-zero element of A and $u \notin M$. By the same proof as in Case 1, there exists $s \notin M$ such that $saI = 0_A$. But, $sa \neq 0_A$ since $(a/u) \neq 0_{A_M}$. Therefore, I has a non-zero annihilator, and A satisfies the (CH)-property.

It is well-known that a ring A is weakly semi-Steinitz if and only if it is a Hermite ring and satisfies the (CH)-property (*cf.* [13, Theorem 2.2]). Hence, by Proposition (3.2), one may consider the following question:

Question: Let A be a commutative ring such that A_P is a weakly semi-Steinitz ring for each prime ideal P. Is A a weakly semi-Steinitz ring?

If A is Noetherian ring, we give an affirmative answer to this question.

Theorem 3.3. *Let A be a Noetherian ring. If A_M is a weakly semi-Steinitz ring for each maximal ideal M, then so is A.*

Proof. Let A be a Noetherian ring. To show that A is a weakly semi-Steinitz ring, it suffices to show that every non-zero-divisor of A is a unit (*cf.* [13, Corollary 2.5]). Let a be a non-zero-divisor of A. Assume that a is not a unit in A and let M be a maximal ideal of A such that $a \in M$. Hence, $(a/1) \in MA_M$ and so $(a/1)$ is not a unit in A_M. On the other hand, we claim that $(a/1)$ is not a zero divisor in A_M.

Indeed, let $(b/s) \in A_M$ such that $(a/1)(b/s) = 0_{A_M}$, where $b \in A$ and $s \notin M$. Hence, there exists $u \notin M$ such that $uab = 0_A$, so $ub = 0_A$ since a is not a zero-divisor in A. Therefore, $(b/s) = 0_{A_M}$ and $(a/1)$ is not a zero-divisor in A_M. But A_M is a weakly semi-Steinitz ring, so $(a/1)$ is a unit in A_M, a contradiction. It follows that a is a unit in A and hence A is a weakly semi-Steinitz ring.

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