SETS OF LENGTHS IN $V+XB[X]$ DOMAINS

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الخلاصة :

ندرس في هذا البحث خواص تعميلية في حلقات من نوع $V+XB[X]$ ، حيث B ترمز إلى الإقفال الصحيح لِ V داخل امتداد جبري منته لجسم الكسور الناتج عن V. نركز اهتمامنا على الحالة التي تكون فيها V حلقة تقييمية منفصلة وحيث إنَّ عنصرَها (الوحيد) غير القابل للاختزال P يتشعب في B. تحديداً، نحسب في هذه الحالة مجموعات الأطوال لعناصر $V+XB[X]$، وفي بعض الحالات، للمجموعات المعممة للأطوال.

ABSTRACT

In this paper, we investigate factorization properties in domains of type $V+XB[X]$, where B is the integral closure of V in a finite algebraic extension of the quotient field of V. We place particular emphasis on the case where V is a discrete valuation ring in which the unique up to associate irreducible element p of V ramifies in B. More precisely, we compute in this case the sets of lengths of the elements of $V+XB[X]$ and, in some cases, the generalized sets of lengths.

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1. INTRODUCTION AND NOTATIONS

Factorization properties of integral domains and monoids have been a frequent topic of the recent mathematical literature. One area of interest has been the factorization properties of polynomial rings ([3], [7], [9], [11], [15], [31]). Of particular interest have been polynomial rings of the form $A + XB[X]$ where $A \subseteq B$ is an integral extension of integral domains (see for example [10],[13],[23],[29],[30]). In [30], the second author of the present paper proved if $\mathbb{Z} \subset B$ is an integral extension, then the elasticity of $\mathbb{Z}+XB[X]$ is infinite. Also in [30], elasticity was investigated in the case where A is a (rank-one) discrete valuation domain of quotient field K, and B is the integral closure of A in a finite extension of K. In this paper, we continue the investigation begun in [30] of this latter case and consider the sets of lengths and generalized sets of lengths in these domains. While our work is computational in nature, we feel that it warrants further consideration due to the current state of the mathematical literature with respect to the theory of sets of lengths and generalized sets of lengths. Indeed, the development of these investigations centered about the study of Krull domains and monoids (see [16]). In such a structure, all problems involving factorizations of elements can be reduced to combinatorial problems on a *block monoid* (see [27]) which essentially consists of the *zero-sequences* from a given finite abelian group (see Section 3). In our current work, we show in a domain R which is not Krull that factorization problems may still be dependent on minimal zero-sequences.

We recall some basic definitions and results of factorization theory. We say that a domain R is *atomic* (see [22]) if each nonzero nonunit of R is a finite product of irreducible elements (or atoms) of R. Let $\mathcal{I}(R)$ represent the set of irreducible elements of R. In an atomic domain R, a nonzero nonunit may have several factorizations into irreducible elements, and two factorizations may have different lengths. Thus, we define R to be a *half-factorial domain* (or HFD, see [36] and [37]) if R is atomic and any two factorizations of a nonzero nonunit of R as products of irreducible elements have the same length. In order to measure how far an atomic domain R is from being an HFD, we define the *elasticity* of R (see [35]) as:

$$\rho(R) = \sup\left\{\frac{m}{n} \mid x_1 \dots x_m = y_1 \dots y_n \text{ where each } x_i, y_j \in \mathcal{I}(R)\right\}.$$

Clearly $1 \le \rho(R) \le \infty$ and all these intermediate values may occur (see [1, Theorem 3.2]). Also, $\rho(R) = 1$ if and only if R is a HFD. When $\rho(R)$ is a rational number m/n, we say that it is *realized* (*by a factorization*) if there exist some irreducible elements $x_1, \dots, x_r$, $y_1, \dots, y_s$ of R such that $x_1 \dots x_r = y_1 \dots y_s$ and $m/n = r/s$ (see [2], [8], [31]). An excellent source of information on elasticity is the survey article [5] by D.F. Anderson.

If R is an atomic domain and a a nonzero nonunit of R, then set:

$$L(a) = \{k|\exists\alpha_1, \dots, \alpha_k \in \mathcal{I}(R) \text{ with } a = \alpha_1 \dots \alpha_k\},$$

and

$$\mathcal{L}(R) = \{L(a)|a \text{ is a nonzero nonunit of } R\}.$$

$L(a)$ is called the *set of lengths of* a and $\mathcal{L}(R)$ is called the *set of lengths of* R (see [25] and [26]). For each positive integer n, set:

$$\mathcal{V}_R(n) = \{m|\exists x_1, \dots, x_n, y_1, \dots, y_m \in \mathcal{I}(R) \text{ such that } x_1 \dots x_n = y_1 \dots y_m\},$$

and denote by $\Phi_R(n)$ the cardinality of the set $\mathcal{V}_R(n)$. In [21], the sets $\mathcal{V}_R(n)$ are called *generalized sets of lengths.* The function $\Phi_R(n)$ and sets $\mathcal{V}_R(n)$ have been studied extensively in the papers [6], [7], [8], [17], [18], [19], [20], [21]. Note the following link between Φ_R and $\rho(R)$: if $\Phi_R(n)$ is infinite for some n, then $\rho(R)$ is infinite, but the converse does not hold in general [8].

Throughout the remainder of our work, let V be a discrete valuation ring with quotient field K. We denote by p the unique (up to units) irreducible element of V. Let L be a finite extension of K and B be the ring of integers of L over V. It is well-known that B is a principal ideal domain with only a finite number of irreducible elements (up to units) denoted by $\pi_1, \ldots, \pi_r$ ($r \geq 1$). In B, the element $p = \pi_1^{e_1} \ldots \pi_r^{e_r}$ can decompose in exactly one of three different ways.

(1) If $r > 1$, we say that p is *decomposed.*

(2) If $r = 1$ and $e_1 = 1$, we say that p is *totally inert* (in this case, pB is a prime ideal of B).

(3) If $r = 1$ and $e_1 > 1$, we say that p is *ramified* (in this case, $p = \pi_1^{e_1}$ where π_1 is the unique irreducible element - up to units - of B and e_1 is the ramification index).

We will also suppose throughout that B is a finitely generated V-module (then $B[X]$ is a finitely generated module on $V + XB[X]$). This condition is satisfied, for instance, when the extension L/K is separable or when K is of characteristic 0.

We consider in detail the case described above where the prime p ramifies in B and is neither inert nor decomposed. In Section 2, we compile some basic ring theoretic properties of the domain $V + XB[X]$ and show, in particular, that it is a weakly Krull domain of finite type. In Section 3, we consider the problem of computing the set of lengths of an element in $V + XB[X]$ and give a complete formula for their computation. Since elements which are units of B but not units of V can be used to construct easy examples of non-unique factorizations into irreducible elements in $V + XB[X]$ (if u is such a unit then $X \cdot X = (uX) \cdot (u^{-1}X)$), we find it somewhat surprising in our main result (Proposition 3.7) that the unit groups of V and B play no role. Moreover, our Proposition 3.7 is in line with the known structure of sets of lengths in weakly Krull domains and monoids (see Section 6 of [32] for details). In Section 4, we consider the problem of computing the generalized sets of lengths for $V + XB[X]$ and give a complete description of these sets in the case where $e = 2$ and 3. In Section 5, we discuss several open problems which the material in Sections 2 through 4 suggest.

If R is an integral domain, then $\mathcal{U}(R)$ will denote its group of units and R^* its set of nonzero elements. As usual, $\mathbb{Z}$ will denote the ring of integers, $\mathbb{N}$ the set of nonnegative integers and $\mathbb{N}_0$ the set of positive integers. For all integer $n > 1$, we set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and if $m \in \mathbb{Z}$, then set $\overline{m} = m + n\mathbb{Z}$. We also shall make use of the following notation, for every $a, b \in \mathbb{R}$:

$$[a, b] = \{k \in \mathbb{N} \mid a \leq k \leq b\}.$$

If $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots a_1X + a_0$ is an element of $V + XB[X]$, then we denoted by the *order of* $f(X)$ the smallest nonnegative integer i for which $a_i \neq 0$. For any further undefined terminology or notation, the interested reader is directed to [28].

2. BASIC RING-THEORETIC PROPERTIES OF $V + XB[X]$

We begin with some general results which hold no matter how p decomposes in B. Both parts of the next lemma are particular cases of Theorem 4.1.2 and Proposition 4.3.2 in [34].

Lemma 2.1. *Set $R = V + XB[X]$, then:*

(*a*) *$B[X]$ is the integral closure of R in its quotient field $L(X)$.*

(*b*) *R is Noetherian.*

Proof.

(*a*) Since $B[X]$ is integrally closed in $L(X)$, it suffices to show that each element of $B[X]$ is integral over R. Let $f \in B[X]$ and let a be the constant term of f. Since B is the integral closure of V in L, there exists a monic polynomial $g = b_0 + b_1X + \ldots + b_{r-1}X^{r-1} + X^r$ in $V[X]$ such that $g(a) = 0$. Let us consider

$$\phi = b_0 + \sum_{i=1}^{r-1} b_i \left(a^i - f(X)^i\right) + (a^r - f(X)^r) + \sum_{i=1}^{r-1} b_iT^i + T^r.$$

We have that $\phi \in R[T]$ is a monic polynomial with $\phi(f) = 0$. Thus, f is integral over R in $L(X)$.

(*b*) If $I = XB[X]$, then $R \subseteq B[X]$ share the ideal I. Moreover, $B[X]$ is Noetherian (since B is a principal ideal domain) and $B \simeq B[X]/I$ is a finitely generated V-module. As $V \simeq R/I$, we obtain from (or [24]), that R is Noetherian. □

Proposition 2.2. *If $R = V + XB[X]$, then:*

(*a*) *the Krull dimension of R is 2; and*

(*b*) *the prime ideal $XB[X]$ of R is of height one in R.*

Proof. From Lemma 2.1, the extension $R \subseteq B[X]$ is integral thus $\dim(R) = \dim(B[X])$. That is, $\dim(R) = 2$. Since $XB[X] \subset pV + XB[X]$, the prime ideal $XB[X]$ is not maximal in R, and hence $XB[X]$ is of height one. □

For an integral domain R, let $\mathfrak{X}(R)$ be the set of all prime ideals of height one. Following [4], we call R a *weakly Krull domain* if

$$R = \bigcap_{\mathfrak{p} \in \mathfrak{X}(R)} R_{\mathfrak{p}}$$

and, for all $a \in R^*$, we have $a \in \mathcal{U}(R_{\mathfrak{p}})$ for all but finitely many $\mathfrak{p} \in \mathfrak{X}(R)$.

A weakly Krull domain R is said of *finite type* if its integral closure $\overline{R}$ is a Krull domain and a finitely generated R-module ([32]).

Proposition 2.3. *$V + XB[X]$ is a weakly Krull domain of finite type.*

Proof. Set $R = V + XB[X]$ and let us consider the R-module $M = B[X]/R$. Then each prime ideal in $\mathrm{Ass}_R(M)$ contains $XB[X]$. Since $R \subseteq B[X]$ share the prime ideal $XB[X]$, the ideals of R containing $XB[X]$ correspond with the prime ideals of $R/XB[X] \simeq V$. Thus $XB[X] \in \mathrm{Ass}_R(M)$ and $\mathrm{Ass}_R(M) \subseteq \{XB[X], p + XB[X]\}$. By way of contradiction, let us suppose that $p + XB[X] \in \mathrm{Ass}_R(M)$. From Lemma 2.1, R is Noetherian, so we can write $p + XB[X] = \mathrm{ann}_R(f + R)$ with $f = a + Xg$ and $a \in B \setminus V$. Then $p \cdot a \in V$, whence $a \in B \cap K = V$. This yields a contradiction. Thus $\mathrm{Ass}_R(M)$ contains only $XB[X]$ which is of height one. From [33, Remark p. 352], R is weakly Krull. Moreover, from Lemma 2.1, the integral closure of R is $B[X]$ which is a Krull domain (since B is a PID) and it is a finitely generated R-module. □

3. FACTORIZATION OF ELEMENTS AND SETS OF LENGTHS

Unless otherwise specified, we assume throughout this section that $p = \pi^e$ ramifies with respect to B with index of ramification $e \geq 2$. Recall that if G is an additive abelian group and $g_1, \ldots, g_k \in G$, then we say that (see [21]):

(*a*) $\{g_1, \ldots, g_k\}$ is a *zero-sequence* if $g_1 + \ldots + g_k = 0$;

(*b*) $\{g_1, \ldots, g_k\}$ is a *zero-free sequence* if there does not exist a nonempty subset $I \subseteq \{1, \ldots, k\}$ such that $\sum_{i \in I} g_i = 0$; and

(*c*) $\{g_1, \ldots, g_k\}$ is a *minimal zero-sequence* if $g_1 + \ldots + g_k = 0$ and there does not exist a nonempty subset $I \subset \{1, \ldots, k\}$ such that $\sum_{i \in I} g_i = 0$.

Recall that if f is irreducible in $V + XB[X]$, then f is of order 0 or 1. Thus, the next theorem, which follows directly from [30, Lemmas 2.3 and 2.4], gives a complete classification of the irreducible elements in $V + XB[X]$.

Theorem 3.1. *Let $p = \pi^e$ be ramified over B and suppose that f is a nonconstant irreducible element of $R = V + XB[X]$.*

1. *If f has order 0 and is irreducible in $B[X]$, then f is prime in R.*

2. *If f has order 0 and is not irreducible in $B[X]$, then it factors in $B[X]$ as:*

$$f = u \cdot \prod_{i=1}^{k} (X\varphi_i + \pi^{e_i}),$$

where $\{\overline{e_1}, \ldots, \overline{e_k}\}$ is a minimal zero-sequence of $\mathbb{Z}_e$, u is a unit of V, the φ_i's are in $B[X]$ and each $(X\varphi_i + \pi^{e_i})$ is irreducible in $B[X]$.

3. *If f has order 1, then:*

$$f = (u \cdot X) \prod_{i=1}^{k} (X\varphi_i + \pi^{e_i}),$$

where $\{\overline{e_1}, \ldots, \overline{e_k}\}$ is a zero-free sequence of $\mathbb{Z}_e$, u is a unit of B, the φ_i's are in $B[X]$ and each $(X\varphi_i + \pi^{e_i})$ is irreducible in $B[X]$. Moreover, f is irreducible in $B[X]$ if and only if $f = u \cdot X$.

The behavior exhibited in Theorem 3.1 is quite different than what occurs in the case where p remains inert in B. In that case, the irreducible elements of $V + XB[X]$ remain irreducible in $B[X]$ (see the proof of Theorem 2.1 part (*ii*) in [30]). A similar, but more complex, version of Theorem 3.1 holds for the case where p splits in B. We illustrate that situation when f has order 0 with an example.

Example 3.2. Suppose that $p = \pi_1^{e_1} \cdots \pi_r^{e_r}$ splits in B (*i.e.* $r > 1$) with each $e_i > 1$. As in Theorem 3.1, if f is in $V + XB[X]$ and irreducible in $B[X]$ of order 0, then f is prime in $V + XB[X]$. If f has order 0 and is not irreducible in $B[X]$, then it factors in $B[X]$ as:

$$f = u \cdot \prod_{i=1}^{k} \left(X\varphi_i + \pi_1^{e_{i,1}} \cdots \pi_r^{e_{i,r}}\right),$$

where u is a unit of V, the φ_i's are in $B[X]$, each $\left(X\varphi_i + \pi_1^{e_{i,1}} \cdots \pi_r^{e_{i,r}}\right)$ is irreducible in $B[X]$, and:

$$\{(\overline{e_{1,1}}, \ldots, \overline{e_{1,r}}), \ldots, (\overline{e_{k,1}}, \ldots, \overline{e_{k,r}})\}$$

is a zero-sequence of $\mathbb{Z}_{e_1} \oplus \cdots \oplus \mathbb{Z}_{e_r}$ with the following properties:

1. there exists a $\ell \in \mathbb{N}$ such that $\sum_{i=1}^{k} e_{i,1} = \ell e_1$, $\sum_{i=1}^{k} e_{i,2} = \ell e_2$, ..., $\sum_{i=1}^{k} e_{i,r} = \ell e_r$; and
2. if I is a proper subset of $\{1, \ldots, k\}$ such that $\{(\overline{e_{j,1}}, \ldots, \overline{e_{j,r}})\}_{j \in I}$ is a zero-sequence in $\mathbb{Z}_{e_1} \oplus \cdots \oplus \mathbb{Z}_{e_r}$, then $\sum_{j \in I} e_{j,1} = he_1$ implies that $\sum_{j \in I} e_{j,d} \neq he_d$ for some $2 \leq d \leq r$.

In [30, Theorem 2.1], the elasticity of $R = V + XB[X]$ is computed for the cases where p is decomposed or totally inert in B (in the first case $\rho(R) = \infty$ and in the second $\rho(R) = 1$). We offer a slight improvement to this theorem by computing the elasticity when p ramifies.

Theorem 3.3. *If $R = V + XB[X]$ where $p = \pi^e$ ramifies in B, then $\rho(R) = \frac{e+1}{2}$ and the elasticity is realized.*

Proof. From [30, Theorem 2.1],

$$\frac{e+1}{e} \leq \rho(R) \leq \frac{e+1}{2}.$$

Hence, it suffices to show that we can write an equality between a product of two irreducible factors and a product of $e + 1$ factors. Set

$$f = X(X+\pi)^{e-1}, \; g = X\left(X + \pi^{e-1}\right)^{e-1}, \quad \text{and} \quad h = (X+\pi)\left(X+\pi^{e-1}\right).$$

By Theorem 3.1, f, g, h, and X are irreducible in $V + XB[X]$ and we have $fg = X^2h^{e-1}$, completing the argument. □

In the next example, we give an explicit formula for the set of lengths of an element of $V + XB[X]$ when $e = 2$.

Example 3.4. We consider the case where p is ramified and the index of ramification is $e = 2$. Set $R = V + XB[X]$, then (from Theorem 3.3) $\rho(R) = \frac{3}{2}$.

In $\mathbb{Z}/2\mathbb{Z}$, there is only one minimal zero sequence (which is $\{\bar{1}, \bar{1}\}$) and there is only one zero-free sequence (which is $\{\bar{1}\}$), thus (from Theorem 3.1) the irreducible elements of R are of 3 types:

(a) the elements of R which are irreducible in $B[X]$; moreover they are all prime in R, except X;

(b) polynomials of the form $(X\varphi_1 + \pi^{e_1})(X\varphi_2 + \pi^{e_2})$, with φ_1, φ_2 in $B[X]$ and $e_1, e_2 \equiv 1 \pmod 2$; and

(c) polynomials of the form $X(X\varphi + \pi^{e_1})$, with φ in $B[X]$ and $e_1 \equiv 1 \pmod 2$.

Let us consider $f \in R$ and its factorization in $B[X]$ (which is unique up to units, since B is a UFD),

$$f = X^r f_1 \dots f_s g_1 \dots g_t$$

where $r \geq 0$, $f_1, \dots, f_s$ are in R and of order 0, and $g_1, \dots, g_t$ are not in R and are also of order 0.

The polynomials $f_1, \dots, f_s$ are prime in R and thus appear in each factorization of f in R. Write such a factorization as:

$$f = f_1 \dots f_s X^\rho h_1 \dots h_u k_1 \dots k_v\,,$$

where $0 \leq \rho \leq r$, $h_1 \dots h_u$ are all of the form $(X\varphi_1 + \pi^{e_1})(X\varphi_2 + \pi^{e_2})$, and $k_1, \dots, k_v$ are all of the form $X(X\varphi + \pi^{e_1})$. Thus, we have $r = \rho + v$.

Since the polynomials $X\psi + \pi^{e_1}$ are irreducible in $B[X]$ and since $B[X]$ is a UFD, we obtain that the g_i's are all of the form $X\psi + \pi^{e_1}$. Thus, by a counting argument, we obtain $2u + v = t$ that is $2u = t - r + \rho$. It follows that:

$$L(f) \subseteq \{s + r + \frac{t - r + \rho}{2} \mid 0 \leq \rho \leq r \text{ and } \rho \equiv r - t \pmod 2\}.$$

Factoring f uniquely as a product of irreducibles in $B[X]$, we obtain:

$$f = X^r f_1 \dots f_s (X\varphi_1 + \pi^{e_1}) \dots (X\varphi_t + \pi^{e_t}), \tag{1}$$

where $e_1, \dots, e_t$ are odd integers. Choose an integer ρ such that $0 \leq \rho \leq r$ and $\rho \equiv r - t \pmod 2$, then set:

$$\begin{cases} k_1 = X(X\varphi_1 + \pi^{e_1}) \\ \vdots \\ k_{r-\rho} = X(X\varphi_{r-\rho} + \pi^{e_{r-\rho}}) \\ h_1 = (X\varphi_{r-\rho+1} + \pi^{e_{r-\rho+1}})(X\varphi_{r-\rho+2} + \pi^{e_{r-\rho+2}}) \\ \vdots \\ h_\sigma = (X\varphi_{t-1} + \pi^{e_{t-1}})(X\varphi_t + \pi^{e_t}), \end{cases}$$

where $\sigma = \frac{t-(r-\rho)}{2}$ (which is possible since $\rho \equiv r - t \pmod 2$). Then, the g_i's and the h_j's are irreducible elements of R. Thus, we obtain a factorization of f in R of length $s + r + \frac{t-r+\rho}{2}$ and therefore we have the equality. That is,

$$L(f) = \left\{ s + \frac{t + r + \rho}{2} \mid 0 \leq \rho \leq r \text{ and } \rho \equiv r - t \pmod 2 \right\}. \tag{2}$$

Thus $L(f) = \left[s + \frac{t+r}{2}, s + \frac{t+2r}{2}\right]$.

We note that if f is of order 0, then we obtain that $L(f) = \{s + \frac{t}{2}\}$ (since in this case $r = 0$). Thus, the *elasticity of the element* f in R is equal to 1. Thus, the elasticity of the domain R (which is equal to $\frac{3}{2}$) only depends on the elements of order 1. Moreover, a nice application of the set (2) can be made with respect to the asymptotic theory of factorizations (see [12]). For an f in R, set:

$$\ell(f) = \min L(f) \text{ and } \mathfrak{L}(f) = \max L(f).$$

The main theorem of [12] implies that both of the limits

$$\bar{\ell}(f) = \lim_{n\to\infty} \frac{\ell(f^n)}{n} \text{ and } \bar{\mathfrak{L}}(f) = \lim_{n\to\infty} \frac{\mathfrak{L}(f^n)}{n}$$

exist (although $\bar{\mathfrak{L}}(f)$ may be infinite). Writing f as in (1), we obtain for $n \in \mathbb{N}_0$ that:

$$\ell(f^n) = \lceil sn + \frac{nt+nr}{2} \rceil \text{ and } \mathfrak{L}(f^n) = \lfloor sn + \frac{nt+n2r}{2} \rfloor.$$

Hence,

$$\bar{\ell}(f) = \lim_{q\to\infty} \frac{\ell(f^{2q})}{2q} = s + \frac{t+r}{2},$$

and

$$\bar{\mathfrak{L}}(f) = \lim_{q\to\infty} \frac{\mathfrak{L}(f^{2q})}{2q} = s + r + \frac{t}{2}.$$

Before considering the general case, let us look more closely at the minimal zero-sequences and zero-free sequences of $\mathbb{Z}_e$. Given a finite sequence $\sigma = \{g_1, \ldots, g_s\}$ in $\mathbb{Z}_e$, set $\mid \sigma \mid = s$. If $\sigma = \{g_1, \ldots, g_s\}$ and $\tau = \{h_1, \ldots, h_l\}$ are two finite sequences in $\mathbb{Z}_e$, then call σ and τ *equivalent* if $\mid \sigma \mid = \mid \tau \mid$ and there exists a permutation γ of $\{1, \ldots, s\}$ such that $g_i = h_{\gamma(i)}$.

Definition 3.5. To each minimal zero-sequence $\sigma = \{g_1, \ldots, g_s\}$ of $\mathbb{Z}_e$, associate the function $\mu : \{1, \ldots, e-1\} \to \{0, \ldots, e\}$ such that for each $1 \le j \le e-1$, $\mu(j)$ is the number of i, with $1 \le i \le s$, such that $\bar{j} = g_i$.

Similarly, to each zero-free sequence $\tau = \{h_1, \ldots, h_l\}$ we associate the function $\nu : \{1, \ldots, e-1\} \to \{0, \ldots, e\}$ such that for each $1 \le j \le e-1$, $\nu(j)$ is the number of i, with $1 \le i \le l$, such that $\bar{j} = h_i$.

For example, in $\mathbb{Z}/4\mathbb{Z}$ the minimal zero-sequence $\sigma_1 = \{\bar{1}, \bar{1}, \bar{2}\}$ corresponds to the function $\mu_1 : \{1,2,3\} \to \{0,1,2,3,4\}$ such that $\mu_1(1) = 2$, $\mu_1(2) = 1$, and $\mu_1(3) = 0$, and the zero-free sequence $\tau_1 = \{\bar{2}, \bar{3}\}$ corresponds to the function $\nu_1 : \{1,2,3\} \to \{0,1,2,3,4\}$ such that $\nu_1(1) = 0$, $\nu_1(2) = 1$, and $\nu_1(3) = 1$.

Suppose that $\sigma_1, \sigma_2, \ldots, \sigma_{m(e)}$ is the complete set of nonequivalent minimal zero-sequences of $\mathbb{Z}_e$ and $\tau_1, \tau_2, \ldots, \tau_{n(e)}$ the complete set of nonequivalent zero-free sequences of $\mathbb{Z}_e$. Hence for each $1 \le i \le m(e)$,

$$\sum_{j=1}^{e-1} \mu_i(j) j \equiv 0 \pmod{e}$$

and for all $0 \le k_j < \mu_i(j)$ with $1 \le j \le e-1$, we have:

$$\sum_{j=1}^{e-1} j k_j \not\equiv 0 \pmod{e}.$$

Also, for each $1 \le i \le n(e)$ and every $0 \le k_j \le \nu_i(j)$ with $1 \le j \le e-1$, we have:

$$\sum_{j=1}^{e-1} j k_j \not\equiv 0 \pmod{e}.$$

Among the minimal zero-sequences $\sigma_1, \sigma_2, \ldots, \sigma_{m(e)}$, we can consider the sequences σ_i such that $\mu_i(j) = 0$ for all $1 \le j \le e-1$ but one, say j_0, and then $\mu_i(j_0)$ is the order of j_0 in the additive group $\mathbb{Z}_e$. We assume without loss of generality that these sequences are $\sigma_1, \ldots, \sigma_{e-1}$ where the index is the element of $\{1, \ldots, e-1\}$ for which the function does not vanish. For instance, in $\mathbb{Z}/4\mathbb{Z}$, they are the sequences $\{\bar{1}, \bar{1}, \bar{1}, \bar{1}\}$, $\{\bar{2}, \bar{2}\}$, and $\{\bar{3}, \bar{3}, \bar{3}, \bar{3}\}$. These sequences have the associated functions $\mu_1, \mu_2, \mu_3 : \{1,2,3\} \to \{0,1,2,3,4\}$ such that $\mu_1(1) = 4$, $\mu_1(2) = 0$, $\mu_1(3) = 0$, $\mu_2(1) = 0$, $\mu_2(2) = 2$, $\mu_2(3) = 0$, and $\mu_3(1) = 0$, $\mu_3(2) = 0$, $\mu_3(3) = 4$. The following result is clearly obtained from Theorem 3.1.

Lemma 3.6. *Set $R = V + XB[X]$ and suppose the index of ramification of p is e. The irreducible elements of R are of $1 + m(e) + n(e)$ types:*

(a) *the elements of R of order 0 which are irreducible in $B[X]$ (moreover, they are prime in R);*

(b) *for all $1 \leq i \leq m$, polynomials of the form:*

$$u \cdot \prod_{j \in E_i} \prod_{1 \leq k_j \leq \mu_i(j)} \left(X\varphi_{(j,k_j)} + \pi^{s(j,k_j)} \right),$$

where $E_i = \{j | \mu_i(j) \neq 0\}$, $\varphi_{(\alpha,\beta)} \in B[X]$, $s(j,k_j) \equiv j \pmod{e}$, and u is a unit of V;

(c) *for all $1 \leq i \leq n$, polynomials of the form:*

$$u \cdot X \prod_{j \in F_i} \prod_{1 \leq l_j \leq \nu_i(j)} \left(X\psi_{(j,l_j)} + \pi^{s(j,l_j)} \right),$$

where $F_i = \{j | \nu_i(j) \neq 0\}$, $\psi_{(\alpha,\beta)} \in B[X]$, $s(j,l_j) \equiv j \pmod{e}$, and u is a unit of B.

Now, we are able to generalize Example 3.4.

Proposition 3.7. *Set $R = V + XB[X]$ and suppose that the index of ramification of p is $e > 1$. For each j with $1 \leq j \leq e-1$, we denote by o_j the order of $\bar{j}$ in the additive group $\mathbb{Z}_e$ (thus $\mu_j(j) = o_j$). If $f \in R$ factors irreducibly in $B[X]$ as,*

$$f = X^r f_1 \cdots f_s \prod_{1 \leq j \leq e-1} \prod_{1 \leq i \leq t_j} \left(X\varphi^{(j,i)} + \pi^{s(i,j)} \right),$$

where $r \geq 0$, $f_1, \ldots, f_s$ are in R and of order 0, the $\varphi^{(j,i)}$'s are in $B[X]$, and $s(i,j) \equiv j \pmod{e}$, then:

$$L(f) = \left\{ r + s + \sum_{j=1}^{e-1} \frac{t_j - \sum_{a=e}^{m(e)} u_a \mu_a(j) - \sum_{b=1}^{n(e)} v_b \nu_b(j)}{o_j} + \sum_{j=e}^{m(e)} u_j \right\},$$

with the following conditions:

- $\sum_{b=1}^{n(e)} v_b \leq r$;
- *for all* $1 \leq j \leq e-1$, $\sum_{a=e}^{m(e)} u_a \mu_a(j) + \sum_{b=1}^{n(e)} v_b \nu_b(j) \leq t_j$; *and*
- *for all* $1 \leq j \leq e-1$, $\sum_{a=e}^{m(e)} u_a \mu_a(j) \equiv t_j - \sum_{b=1}^{n(e)} v_b \nu_b(j) \pmod{o_j}$.

Proof. The polynomials $f_1, \ldots, f_s$ are prime in R thus appear in each factorization of f in R. Write such a factorization:

$$f = X^\rho f_1 \cdots f_s \left[\prod_{1 \leq a \leq m(e)} \prod_{1 \leq i \leq u_a} g_{a,i} \right] \left[\prod_{1 \leq b \leq n(e)} \prod_{1 \leq i \leq v_b} h_{b,i} \right],$$

where for all $1 \leq a \leq m(e)$, the $g_{a,i}$'s are of type:

$$\prod_{j \in E_a} \prod_{1 \leq k_j \leq \mu_a(j)} \left(X\varphi_{(j,k_j)} + \pi^{s(j,k_j)} \right),$$

and, for all $1 \leq b \leq n(e)$, the $h_{b,i}$'s are of type:

$$X \prod_{j \in F_b} \prod_{1 \leq l_j \leq \nu_b(j)} \left(X\psi_{(j,l_j)} + \pi^{s(j,l_j)} \right).$$

We have the following relations:

- $0 \le \rho \le r$;
- $r = \rho + \sum_{1 \le b \le n(e)} v_b$; and
- for all $1 \le j \le e-1$, $t_j = \sum_{a=1}^{m(e)} u_a \mu_a(j) + \sum_{b=1}^{n(e)} v_b \nu_b(j)$, that is:

$$t_j = u_j \mu_j(j) + \sum_{a=e}^{m(e)} u_a \mu_a(j) + \sum_{b=1}^{n(e)} v_b \nu_b(j).$$

Consequently, $L(f) \subseteq \{r + s + \sum_{a=1}^{m(e)} u_a \mid t_j = u_j \mu_j(j) + \sum_{a=e}^{m(e)} u_a \mu_a(j) + \sum_{b=1}^{n(e)} v_b \nu_b(j)$ for all $1 \le j \le e-1$, and $\sum_{b=1}^{m(e)} v_b \le r\}$. Then:

$$L(f) \subseteq \left\{ r + s + \sum_{j=1}^{e-1} \frac{t_j - \sum_{a=e}^{m(e)} u_a \mu_a(j) - \sum_{b=1}^{n(e)} v_b \nu_b(j)}{o_j} + \sum_{j=e}^{m(e)} u_j \right\},$$

where:

- $\sum_{b=1}^{n(e)} v_b \le r$;
- for all $1 \le j \le e-1$, $\sum_{a=e}^{m(e)} u_a \mu_a(j) + \sum_{b=1}^{n(e)} v_b \nu_b(j) \le t_j$; and
- for all $1 \le j \le e-1$, $\sum_{a=e}^{m(e)} u_a \mu_a(j) \equiv t_j - \sum_{b=1}^{n(e)} v_b \nu_b(j) \pmod{o_j}$.

It is easy to obtain the other inclusion as in Example 3.4. □

Example 3.8. We apply the formula of Proposition 3.7 to the case where $e = 3$. Here the minimal zero-sequences of $\mathbb{Z}_3$ are $\sigma_1 = \{\overline{1}, \overline{1}, \overline{1}\}$, $\sigma_2 = \{\overline{2}, \overline{2}, \overline{2}\}$, and $\sigma_3 = \{\overline{1}, \overline{2}\}$, and the zero-free sequences are $\tau_1 = \{\overline{1}\}$, $\tau_2 = \{\overline{2}\}$, $\tau_3 = \{\overline{1}, \overline{1}\}$, and $\tau_4 = \{\overline{2}, \overline{2}\}$. By computing the appropriate values of $\mu_a(j)$ and $\nu_b(j)$ we obtain:

$$L(f) = \left\{ r + s + u_3 + \frac{t_1 - u_3 - v_1 - 2v_3}{3} + \frac{t_2 - u_3 - v_2 - 2v_4}{3} \right\},$$

where:

- $v_1 + v_2 + v_3 + v_4 \le r$,
- $u_3 + v_1 + 2v_3 \le t_1$,
- $u_3 + v_2 + 2v_4 \le t_2$,
- $u_3 \equiv t_1 - v_1 - 2v_3 \pmod 3$, and
- $u_3 \equiv t_2 - v_2 - 2v_4 \pmod 3$.

If f is of order 0, we obtain:

$$L(f) = \left\{ s + \frac{t_2 + t_1 + u_3}{3} \right\},$$

where $u_3 \equiv t_1 \pmod 3, u_3 \equiv t_2 \pmod 3$ and $0 \le u_3 \le \min\{t_1, t_2\}$.

4. GENERALIZED SETS OF LENGTHS

In this section, we consider the computation of the generalized sets of length $\mathcal{V}_R(n)$ in $R = V + XB[X]$ when p is ramified. We begin with two examples.

Example 4.1. Consider the case where the index of ramification is $e = 2$. From Theorem 3.3 we have that $\rho(R) = \frac{3}{2}$. We prove in the following that $\mathcal{V}_R(n) = \left[\frac{2n}{3}, \frac{3n}{2}\right]$ and thus, for each $n \geq 1$, $\mathcal{V}_R(n)$ is an interval.

First, let us note that if $m \in \mathcal{V}_R(n)$ then $n/m \leq 3/2$, $m/n \leq 3/2$, and $\frac{2n}{3} \leq m \leq \frac{3n}{2}$. By considering the two factorizations $[X(X+\pi)][X(X+\pi)] = X^2[(X+\pi)^2]$ and $[X(X+\pi)][X(X+\pi)][X(X+\pi)] = X^2[(X+\pi)^2][X(X+\pi)]$, we obtain that $\mathcal{V}_R(2) = [2,3]$ and $\mathcal{V}_R(3) = [2,4]$.

We prove by induction on n that $\mathcal{V}_R(n) = \left[\frac{2n}{3}, \frac{3n}{2}\right]$. The result holds for $n = 2, 3$. Suppose that it holds for all positive integers less than or equal to n and consider $\mathcal{V}_R(n+1)$. We distinguish two cases.

(1) If $n+1 = 2k$, we first prove that $\left\{n+1, \ldots, 3k = \frac{3(n+1)}{2}\right\} \subseteq \mathcal{V}_R(n+1)$. Indeed, consider $f = X(X+\pi)$ and the factorization

$$f^{n+1} = X^{2(k-p)}[(X+\pi)^2]^{k-p} f^{2p}$$

where $0 \leq p \leq k$. We obtain a factorization with $3k-p$ factors on the right where $2k \leq 3k - p \leq 3k$.

Let us prove that if $\frac{2(n+1)}{3} \leq a < n+1$, then $a \in \mathcal{V}_R(n+1)$. In this case, $\frac{2a}{3} < n+1 \leq \frac{3a}{2}$. Thus $n+1 \in \left[\frac{2a}{3}, \frac{3a}{2}\right] = \mathcal{V}_R(a)$ (by hypothesis). Consequently, $a \in \mathcal{V}_R(n+1)$.

(2) If $n+1 = 2k+1$, we first prove that $\left\{n+1, \ldots, \frac{3(n+1)}{2}\right\} \subseteq \mathcal{V}_R(n+1)$. Indeed, consider $f = X(X+\pi)$ and the factorization

$$f^{n+1} = X^{2(k-p)}[(X+\pi)^2]^{k-p} f^{2p+1}$$

where $0 \leq p \leq k$. We obtain a factorization with $3k-p+1$ factors on the right where $2k+1 \leq 3k-p+1 \leq 3k+1$.

Let us prove that if $\frac{2(n+1)}{3} \leq a < n+1$, then $a \in \mathcal{V}_R(n+1)$. In this case, $\frac{2a}{3} < n+1 \leq \frac{3a}{2}$. Thus $n+1 \in \left[\frac{2a}{3}, \frac{3a}{2}\right] = \mathcal{V}_R(a)$ (by hypothesis). Consequently, $a \in \mathcal{V}_R(n+1)$.

Hence, we have proved for each integer $n \geq 2$ that

$$\mathcal{V}_R(n) = \left[\frac{2n}{3}, \frac{3n}{2}\right].$$

Example 4.2. We consider the case where the index of ramification is $e = 3$. From Theorem 3.3 we have that $\rho(R) = 2$. We prove in the following that:

$$\mathcal{V}_R(n) = \begin{cases} \left[\frac{n}{2}, 2n\right] & \text{if} \quad n \equiv 0 \pmod 4 \\ \left[\frac{n}{2}, 2n-1\right] & \text{if} \quad n \equiv 1 \pmod 4 \\ \left[\frac{n}{2}+1, 2n\right] & \text{if} \quad n \equiv 2 \pmod 4 \\ \left[\frac{n}{2}, 2n-1\right] & \text{if} \quad n \equiv 3 \pmod 4, \end{cases}$$

and thus, for each $n \geq 1$, $\mathcal{V}_R(n)$ is an interval.

Since $\rho(R) = 2$, we can note that, for each $n \geq 2$,

$$\mathcal{V}_R(n) \subseteq \left[\frac{n}{2}; 2n\right].$$

It is easy to verify, by considering appropriate factorizations, that $\mathcal{V}_R(2) = [2,4]$, $\mathcal{V}_R(3) = [2,5]$, $\mathcal{V}_R(4) = [2,8]$, and $\mathcal{V}_R(5) = [3,9]$. The only non-trivial verifications in these computations consist of showing that $6 \notin \mathcal{V}_R(3)$ and $10 \notin \mathcal{V}_R(5)$. Let us show $6 \notin \mathcal{V}_R(3)$ (the other verification is similar). Suppose that $6 \in \mathcal{V}_R(3)$. Then we can write $f_1 f_2 f_3 = g_1 g_2 g_3 g_4 g_5 g_6$ with the f_i, g_j irreducible in R. From Theorem 3.1, each irreducible f_i or g_j is of the form $\prod_{i=1}^{k} (X\varphi_i + \pi^{e_i})$ where $\{\overline{e_1}, \ldots, \overline{e_k}\}$ is a minimal zero-sequence of $\mathbb{Z}_3$ and the φ_i's are in $B[X]$, or of the form $X \prod_{i=1}^{k} (X\varphi_i + \pi^{e_i})$ where $\{\overline{e_1}, \ldots, \overline{e_k}\}$ is a zero-free sequence of $\mathbb{Z}_3$ and the φ_i's are in $B[X]$. Moreover, in each case the factors are irreducible in the UFD $B[X]$.

Note that the number k of factors which appear in the factorization of f (or g) in $B[X]$ satisfies $1 \leq k \leq 3$ (and $k = 1$ if and only if $f = X$). Thus, the number s of factors of $f_1 f_2 f_3$ satisfies $3 \leq s \leq 9$ and the number t of factors of $g_1 g_2 g_3 g_4 g_5 g_6$ satisfies $6 \leq t \leq 18$. This equality implies that 3 elements among the g_j's are equal to X and the others are of order 0. Thus, the f_i's are of the form $X(X\varphi_1 + \pi^{e_1})(X\varphi_2 + \pi^{e_2})$ (with $e_1, e_2 \equiv 1 \pmod 3$) or of the form $X(X + \varphi_1 \pi^{e'_1})(X + \varphi_2 \pi^{e'_2})$ (with $e'_1, e'_2 \equiv 2 \pmod 3$), and the product of the g_j's is equal to a polynomial of the form $X^3[(X\varphi_1 + \pi^{\varepsilon_1})]^3[(X\varphi_2 + \pi^{\varepsilon_2}))]^3$ (with $\varepsilon_1, \varepsilon_2 \equiv 1 \pmod 3$) or of the form $X^3[(X\varphi_1 + \pi^{\varepsilon'_1})]^3[(X\varphi_2 + \pi^{\varepsilon'_2})]^3$ (with $\varepsilon'_1, \varepsilon'_2 \equiv 2 \pmod 3$). We obtain a contradiction and $6 \notin \mathcal{V}_R(3)$.

We have shown that the result holds for $n = 2, 3, 4$ and 5. Assume that it holds for all positive integers less than or equal to n and consider $\mathcal{V}_R(n+1)$. We have four cases to consider.

(1) If $n+1 \equiv 0 \pmod 4$ (that is $n \equiv 3 \pmod 4$), then $\mathcal{V}_R(n) = \left[\frac{n}{2}, 2n-1\right]$ and thus $\mathcal{V}_R(n+1) \supseteq \left[\frac{n}{2}+1, 2n\right]$. Moreover $\frac{n+1}{2} \equiv 0 \pmod 2$ and thus, by hypothesis, $2\frac{n+1}{2} \in \mathcal{V}_R\left(\frac{n+1}{2}\right)$. That is, $n+1 \in \mathcal{V}_R\left(\frac{n+1}{2}\right)$. It follows that $\frac{n+1}{2} \in \mathcal{V}_R(n+1)$ and therefore $\mathcal{V}_R(n+1) \supseteq \left[\frac{n+1}{2}, 2n\right]$. Finally, write $n+1 = 4t$ and consider the following factorizations for $k = 0$ or 1:

$$\left[X(X+\pi)^2\right]^{2t}\left[X(X+\pi^2)^2\right]^{2t} =$$

$$X^{4t-2k}\left[(X+\pi)(X+\pi^2)\right]^{4t-k}\left[X(X+\pi)\right]^k\left[X(X+\pi^2)\right]^k.$$

Then, we obtain that $8t, 8t-1 \in \mathcal{V}_R(4t)$. That is, $2n+1, 2n+2 \in \mathcal{V}_R(n+1)$. Since $\rho(R) = 2$, we obtain that $\mathcal{V}_R(n+1) = \left[\frac{n+1}{2}, 2n+1\right]$.

(2) If $n+1 \equiv 1 \pmod 4$ (that is $n \equiv 0 \pmod 4$), then $\mathcal{V}_R(n) = \left[\frac{n}{2}, 2n\right]$, and thus $\mathcal{V}_R(n+1)$ contains $\left[\frac{n}{2}+1, 2n+1\right] = \left[\frac{n+1}{2}, 2n+1\right]$. As previously mentioned, one can prove that $2n+2$ is not in $\mathcal{V}_R(n+1)$. Thus $\mathcal{V}_R(n+1) = \left[\frac{n+1}{2}, 2n+1\right]$.

(3) If $n+1 \equiv 2 \pmod 4$ (that is $n \equiv 1 \pmod 4$), then $\mathcal{V}_R(n) = \left[\frac{n}{2}, 2n-1\right]$ and thus $\mathcal{V}_R(n+1) \supseteq \left[\frac{n}{2}+1, 2n\right]$. That is, $\mathcal{V}_R(n+1) \supseteq \left[\frac{n+1}{2}+1, 2n\right]$. Write $n+1 = 4t+2$ and, for $k = 0$ or 1, note that:

$$\left[X(X+\pi)^2\right]^{2t+1}\left[X(X+\pi^2)^2\right]^{2t+1} = \left[(X+\pi)(X+\pi^2)\right]^{4t+2-k} X^{4t}\left[X(X+\pi)^k\right]\left[X(X+\pi^2)^k\right].$$

We obtain that $8t+4, 8t+3 \in \mathcal{V}_R(4t+2)$. That is, $2n+1$ and $2n+2$ are in $\mathcal{V}_R(n+1)$. Since $\rho(R) = 2$, we obtain that $\mathcal{V}_R(n+1) = \left[\frac{n+1}{2}+1, 2n+2\right]$.

(4) If $n+1 \equiv 3 \pmod 4$ (that is $n \equiv 2 \pmod 4$), then $\mathcal{V}_R(n) = \left[\frac{n}{2}+1, 2n\right]$ and thus $\mathcal{V}_R(n+1)$ contains $\left[\frac{n}{2}+2, 2n+1\right]$. Moreover $\frac{n}{2}+1 \equiv 1 \pmod 2$. Thus, by hypothesis, $2\left(\frac{n}{2}+1\right) - 1 \in \mathcal{V}_R\left(\frac{n}{2}+1\right)$. That is, $n+1 \in \mathcal{V}_R\left(\frac{n}{2}+1\right)$. It follows that $\frac{n}{2}+1 \in \mathcal{V}_R(n+1)$ and therefore $\mathcal{V}_R(n+1) \supseteq \left[\frac{n}{2}+1, 2n+1\right] = \left[\frac{n+1}{2}, 2n+1\right]$. As previously mentioned, one can prove that $2n+2$ is not in $\mathcal{V}_R(n+1)$. Thus $\mathcal{V}_R(n+1) = \left[\frac{n+1}{2}, 2n+1\right]$ and the argument is complete.

We have computed all the sets $\mathcal{V}_R(n)$ in the cases $e = 2$ or 3, but for larger values of e, this computation becomes more difficult. Note that a general computation of the generalized sets of length is not known even for Dedekind (or Krull) domains. The complete computation of the $\mathcal{V}_R(n)$'s is known for several specific class groups, but not in general (see [21]). Nevertheless, in the case of $R = V + XB[X]$ we are able for $n = 2$ to compute $\mathcal{V}_R(n)$ for each $e \geq 2$.

Lemma 4.3. *If p is ramified with index of ramification $e \geq 2$, then*

$$\mathcal{V}_R(2) = \{2, 3, ..., e+1\}.$$

Proof. For all $0 \leq k \leq e-1$, we have:

$$\left[X\left(X+\pi\right)^{e-1}\right]\left[X\left(X+\pi^{e-1}\right)^{e-1}\right]$$
$$= \left[\left(X+\pi\right)\left(X+\pi^{e-1}\right)\right]^{e-1-k}\left[X(X+\pi)^k\right]\left[X\left(X+\pi^{e-1}\right)^k\right],$$

that is, a product of 2 irreducible factors on the left and a product of $e+1-k$ irreducible factors on the right, for all $0 \leq k \leq e-1$, thus:

$$\mathcal{V}(2) \supseteq \{2, 3, ..., e+1\}.$$

Since $\rho(R) = \frac{e+1}{2}$, we have an equality. □

We can apply Lemma 4.3 to show the following.

Corollary 4.4. *If $R = V + XB[X]$ and p ramifies with index of ramification $e \geq 2$, then:*

$$\lim_{n \to +\infty} \frac{\Phi(n)}{n} = \frac{\rho(D)^2 - 1}{\rho(D)} = \frac{(e+1)^2 - 4}{2(e+1)}.$$

Proof. From Lemma 4.3, we have $\mathcal{V}_R(2) = \{2, 3, ..., e+1\}$. The proof now follows directly from [7, Theorem 3.2]. □

5. QUESTIONS AND PROBLEMS

Set $R = V + XB[X]$ and consider the case where p has index of ramification $e \geq 2$ (*i.e.*, $p = \pi^e$). Let $\mathfrak{X}(R)$ be the set of all height-one prime ideals in R. Then, denote by $P \subseteq \mathfrak{X}(R)$ the set of the height-one prime ideals $\mathfrak{P}$ of R such that $R_{\mathfrak{P}}$ is a DVR, and set $E = \mathfrak{X}(R) \setminus P$. From Proposition 2.3, R is a weakly Krull domain of finite type, and the set E is finite by [32, Lemma 6.3]. On the other hand, since $R \subseteq B[X]$ share the ideal $XB[X]$, the prime ideals of R verify the following two conditions (see [14]):

1. the prime ideals which contain $XB[X]$ correspond with the prime ideals of $V \simeq R/XB[X]$, and
2. for each prime $\mathfrak{P}$ not containing $XB[X]$, $R_{\mathfrak{P}} = (R \setminus \mathfrak{P})^{-1} B[X]$.

Since V is a DVR, the only prime ideals of V are (0) and pV. Thus, there are only two prime ideals of R which contain $XB[X]$, namely $XB[X]$ and $pV + XB[X]$. Since the height of $pV + XB[X]$ is 2, we only consider the case of the localization at $XB[X]$ (which is of height 1 from Proposition 2.2). Let us consider the element $f = \frac{pX+X^2}{\pi X+X^2}$ of $L(X)$. Then, it is easy to verify that neither f nor f^{-1} is in $R_{XB[X]}$, thus $R_{XB[X]}$ is not a valuation domain. Therefore, $XB[X] \in E$.

Moreover, from [32, p. 55] and as the integral closure of R is $B[X]$, for each prime ideal $\mathfrak{P}$ of R not containing $XB[X]$, we have $\mathfrak{P} \in P$ (that is $R_{\mathfrak{P}}$ is a DVR). Thus $\mathfrak{P} \notin E$ and we have proved that $|E| = 1$. We then deduce from [32, Remark p.57] and [32, Corollary 7.3] that,

- for all $F \subseteq \mathfrak{X}(R)$, every class of the t-class group $\mathrm{Cl}_t(R)$ of R contains t-invertible t-ideals $\mathfrak{a}$ such that $\mathfrak{a} \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in F$; and
- $\mathrm{Cl}_t(R)$ is generated by the t-classes containing primes.

These remarks allow us to raise the following question:

Question 1. Compute the t-class group of $V + XB[X]$.

Using [7, Theorem 3.2], Lemma 4.3 can be expanded to say:

$$\mathcal{V}_R(2k(e+1)) = \{4k, 4k+1, ..., k(e+1)^2\},$$

for any positive integer k. For each $n \geq 1$, let us note $\lambda(n) = \min \mathcal{V}_R(n)$ and $\mu(n) = \max \mathcal{V}_R(n)$. Using essentially the same argument presented in [21, Corollary 2.6], one can show $\mathcal{V}_R(2k)$ contains $[\lambda(2k-2)+2, \mu(2k)]$ for any positive integer k (this argument also needs the easily verified fact that $\mu(2k) = k(e+1)$).

Question 2. *Show that $\mathcal{V}_R(n)$ is an interval for all even values of n.*

If one could show that $\lambda(2k)$ equals either $\lambda(2k-2)+2$ or $\lambda(2k-2)+1$, then Question 2 would follow. If $\lambda(2k) = \lambda(2k-2)$, then one needs to show that $\lambda(2k-2)+1$ is also in $\mathcal{V}_R(2k)$.

The problem lies in the computation of $\mathcal{V}_R(3)$. We can see in [21] that this is a difficult problem for even Dedekind domains and Krull monoids. In our situation consider $e = 4$. Is $\mu(3) = 6$ or 7 ?

Question 3. *Compute $\mu(3)$ for an arbitrary e.*

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