EXTENSIONS $R[a\alpha] \cap R[(a\alpha)^{-1}]$ OF A NOETHERIAN DOMAIN R

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الخلاصـة :

لتكن R حلقة «نـوتيرية» و α عنصراً فوق–بدائياً بدرجة d على R . لتكن R' مجموعة عناصر a (غير الصفر) من R حيث يكون $a\alpha$ عنصراً فوق–بدائياً على R. ضع $B_a = R[a\alpha] \cap R[(a\alpha)^{-1}]$، حيث a عنصر في R'. لنفترض أن d أكبر من أو تساوي ٣ وأن a و b عنصران في R'. في هذا البحث نُثبت أن $B_a = B_b$ إذا وإذا فقط كان a و b مرتبطين. هذا الاجراء يوسع مجال تطبيق [٢، Theorem ٣] الى حالةٍ تكون d فيها أكبر من أو تساوي ٣.

ABSTRACT

Let R be a Noetherian domain and α a super-primitive element of degree d over R. Let R' be the set of non-zero elements a of R such that $a\alpha$ is a super-primitive element over R. Set $B_a = R[a\alpha] \cap R[(a\alpha)^{-1}]$, for $a \in R'$. Assume that $d \geq 3$ and a, b are elements of R'. In this paper we prove that $B_a = B_b$ if and only if a and b are associate. This extends [2, Theorem 3] when $d \geq 3$.

Keywords: anti-integral element, super-primitive element.

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Let R be a Noetherian domain with quotient field K, and $R[X]$ the polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K, and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K and deg $\varphi_\alpha = d$, and write:

$$\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d, \qquad \text{with} \qquad \eta_1, \ldots, \eta_d \in K.$$

We define $I_{[\alpha]} := \bigcap_{i=1}^d I_{\eta_i}$ and $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$, where $I_{\eta_i} = \{c \in R; c\eta_i \in R\}$ and $(1, \eta_1, \ldots, \eta_d)$ is the R-module generated by $1, \eta_1, \ldots, \eta_d$. An element α is called an anti-integral element of degree d over R if $\operatorname{Ker} \pi = I_{[\alpha]}\varphi_\alpha(X)R[X]$. An element α is said to be a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset p$, for every $p \in \mathrm{Dp}_1(R)$, where $\mathrm{Dp}_1(R) = \{p \in \mathrm{Spec}R; \mathrm{depth}R_p = 1\}$.

Our notation is standard; for further explanation, refer to [4].

Let α be an anti-integral element of degree d over R and set $\zeta_i = \alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_{i-1}\alpha + \eta_i$, for $1 \le i \le d-1$.

Lemma 1. [1, Theorem 1 and Example] *The following equality holds:*

$$R[\alpha] \cap R[\alpha^{-1}] = R \oplus I_{[\alpha]}\zeta_1 \oplus \cdots \oplus I_{[\alpha]}\zeta_{d-1}.$$

The following Lemma does not need the assumption that α is an anti-integral element over R.

Lemma 2. *Let a be a non-zero element of R. Then:*

$$I_{[a\alpha]} = \cap_{i=1}^d I_{a^i\eta_i}.$$

Proof. Since $\varphi_{a\alpha}(X) = X^d + a\eta_1 X^{d-1} + \cdots + a^d\eta_d$, we get the required result. □

Let α be a super-primitive element of degree d over R and R^* the unit group of R. Let R' be the set of non-zero elements a's of R such that $a\alpha$ is a super-primitive element over R. It is easily verified that R' contains R^*. Let a be a non-zero element of R. Then set: $B_a = R[a\alpha] \cap R[(a\alpha)^{-1}]$. Note that super-primitive elements are anti-integral elements by [7, Theorem 1.12].

Lemma 3. *Let a be an element of R'. Then the following equality holds:*

$$B_a = R \oplus I_{[a\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1}.$$

Proof. Note that $\varphi_{a\alpha}(X) = X^d + a\eta_1 X^{d-1} + \cdots + a^d\eta_d$. Set $\zeta_{a\alpha,i} = (a\alpha)^i + a\eta_1(a\alpha)^{i-1} + \cdots + a^i\eta_i$, for $i = 1, 2, \ldots, d-1$. Then

$$\zeta_{a\alpha,i} = a^i(\alpha^i + \eta_1\alpha^{i-1} + \cdots + \eta_i) = a^i\zeta_i.$$

Hence we reach the conclusion. □

The following result is a direct consequence of Lemma 3:

Proposition 4. *Let a and b be elements of R'. Then $B_a = B_b$ if and only if $I_{[a\alpha]}a^i = I_{[b\alpha]}b^i$, for $i = 1, 2, \ldots, d-1$.*

Let α be a super-primitive element of degree d over R. Then $J_{[\alpha]p} = R_p$, for every element p of $\mathrm{Dp}_1(R)$. Hence $I_{[\alpha]p}$ is an invertible ideal of R_p. This implies that $I_{[\alpha]p}$ is a principal ideal of R_p by [3, Theorem 59].

In the case $d = 1$, it is known that $B_a = R$, for $a \in R'$ by [7, Remark 1.6.] and [5, (2.5) Theorem] because $a\alpha$ is an anti-integral element over R. Hence $B_a = B_b$ always holds, for $a, b \in R'$.

In the case $d \geq 2$, we quote [2, Theorem 3]: Let α be a super-primitive element of degree $d \geq 2$ over R. Let a and b be elements of R. Assume that a and b are non-zero divisors on $R/I_{[\alpha]}$. Then the following statements are equivalent:

(*i*) $B_a = B_b$;

(*ii*) $a = ub$ for some $u \in R^*$.

The condition that a is a non zero-divisor on $R/I_{[\alpha]}$ assures us of super-primitivity of $a\alpha$ by [2, Lemma 2].

When $d \geq 3$, we extend [2, Theorem 3] showing that we may omit the assumption that a and b are non zero-divisors of $R/I_{[\alpha]}$.

Association in R' is an equivalence relation of R'. The quotient set of this equivalence relation is denoted by R'/R^*. The symbol $cl(a)$ denotes the equivalence class of $a \in R'$.

Theorem 5. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. Assume that $d \geq 3$ and a, b are elements of R'. Then the following assertions hold:*

(1) *$B_a = B_b$ if and only if b/a is in R^*.*
(2) *A mapping of $\{B_a; a \in R'\}$ into R'/R^* defined by $B_a \mapsto cl(a)$ is well-defined and a one-to-one correspondence.*

Proof.

(1) Assume that $B_a = B_b$. Then we get $I_{[a\alpha]}a = I_{[b\alpha]}b$ and $I_{[a\alpha]}a^2 = I_{[b\alpha]}b^2$, by Proposition 4, since $d \geq 3$. Hence:

$$I_{[b\alpha]}b^2 = I_{[a\alpha]}a^2 = (I_{[a\alpha]}a)a = (I_{[b\alpha]}b)a = I_{[b\alpha]}ab.$$

This shows that $I_{[b\alpha]}a = I_{[b\alpha]}b$. Since $b\alpha$ is a super-primitive element over R, $I_{[b\alpha]p}$ is a principal ideal of R_p, for every $p \in \mathrm{Dp}_1(R)$. Therefore $I_{[b\alpha]p} = cR_p$, for some $c \in R_p$. Then $caR_p = cbR_p$. This means that b/a and a/b are in R_p. Note that $R = \cap_{p \in \mathrm{Dp}_1(R)} R_p$ by [6, (33.8)]. Hence b/a and a/b are in R, that is, b/a is in R^*.

Conversely, assume that b/a is in R^*. Then there exists an element u of R^* satisfying $b = ua$. By Lemma 2 we see that $I_{[b\alpha]} = I_{[a\alpha]}$. Hence, by Lemma 3, we get

$$\begin{aligned} B_b &= R \oplus I_{[b\alpha]}b\zeta_1 \oplus \cdots \oplus I_{[b\alpha]}b^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[a\alpha]}ua\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}u^{d-1}a^{d-1}\zeta_{d-1} \\ &= R \oplus I_{[a\alpha]}a\zeta_1 \oplus \cdots \oplus I_{[a\alpha]}a^{d-1}\zeta_{d-1} \\ &= B_a. \end{aligned}$$

(2) is clear from the assertion (1). □

The next result clarifies a part of the set R':

Proposition 6. *Let R be a Noetherian domain and α a super-primitive element of degree d over R. Let a be an element of R such that a is a non-zero divisor of R/I_{η_i}, for $i = 1, 2, \ldots, d$. Then a is in R'.*

Proof. First we show that $I_{[\alpha]} = I_{[a\alpha]}$. It is obvious that $I_{\eta_i} \subset I_{a^i\eta_i}$. Let x be an element of $I_{a^i\eta_i}$. Then we see that $a^i x$ is in I_{η_i}. Since a^i is a non-zero divisor of R/I_{η_i}, we obtain the fact x is in I_{η_i}. Hence $I_{\eta_i} = I_{a^i\eta_i}$, for $i = 1, 2, \ldots, d$. This implies that $I_{[\alpha]} = I_{[a\alpha]}$, by Lemma 2. From the argument above, we have $J_{[a\alpha]} = I_{[a\alpha]}(1, a\eta_1, \ldots, a^d\eta_d) = I_{[\alpha]}(1, a\eta_1, \ldots, a^d\eta_d)$. Let p be an element of $\mathrm{Dp}_1(R)$ and assume that $J_{[a\alpha]} \subset p$. Then $I_{[\alpha]} \subset p$. Since α is a super-primitive element over R, $I_{[\alpha]p}$ is a principal ideal of R_p. Hence pR_p is a prime divisor of $I_{[\alpha]p}$. Then p is a prime divisor of $I_{[\alpha]}$, since the set of zero-divisors of $R_p/I_{[\alpha]p}$ is contained in the set of zero-divisors of $R/I_{[\alpha]}$. Therefore p is a prime divisor of I_{η_i} for some i. Since a is not a zero-divisor of R/I_{η_i}, we see that a is not in p. This implies that:

$$\begin{aligned} J_{[\alpha]p} &= I_{[\alpha]p}(1, \eta_1, \ldots, \eta_d) \\ &= I_{[\alpha]p}(1, a\eta_1, \ldots, a^d\eta_d) \\ &= I_{[a\alpha]p}(1, a\eta_1, \ldots, a^d\eta_d) \\ &= J_{[a\alpha]p}. \end{aligned}$$

Since $J_{[a\alpha]p} \subset pR_p$, we get $J_{[\alpha]p} \subset pR_p$. This contradicts the condition $J_{[\alpha]} \not\subset p$. □

Let M be an R-module. Then $R - \mathrm{Ass}_R(M)$ denotes the set of elements c's of R such that c is not in any p of $\mathrm{Ass}_R(M)$. Note that:

$$\{a \in R; a \text{ is a non-zero divisor of } R/I_{\eta_i}\} = R - \mathrm{Ass}_R(R/I_{\eta_i}).$$

By Proposition 6, we have the following:

Corollary 7. $R - \cup_{i=1}^{d} \mathrm{Ass}_R(R/I_{\eta_i}) \subset R'$.

Let $\overline{R}$ be the integral closure of R in its quotient field K. Let $(R : \overline{R})$ be the conductor ideal of R in $\overline{R}$.

Lemma 8. *Assume that $\overline{R}$ is a finite R-module. Let p be an element of $\mathrm{Dp}_1(R)$. If $p \supset (R : \overline{R})$, then p is a prime divisor of $(R : \overline{R})$.*

Proof. If $\mathrm{ht}p = 1$, then p is a prime divisor of $(R : \overline{R})$, since $(R : \overline{R}) \neq (0)$. If $\mathrm{ht}p > 1$, then [9, Proposition 1.10] implies that p is also a prime divisor of $(R : \overline{R})$. □

Proposition 9. *Assume that $\overline{R}$ is a finite R-module. Then*

$$R - \mathrm{Ass}_R(R/(R : \overline{R})) \subset R'.$$

Proof. Let a be an element of $R - \mathrm{Ass}_R(R/(R : \overline{R}))$. Since super-primitivity is a local-global property, we have only to prove that $a\alpha$ is a super-primitive element over R_p, for every $p \in \mathrm{Dp}_1(R)$. Let p be an element of $\mathrm{Dp}_1(R)$. If $p \not\supset (R : \overline{R})$, then $R_p = \overline{R}_p$, and R_p is a normal domain. Therefore $a\alpha$ is a super-primitive element over R_p by [7, Theorem 1.13]. If $p \supset (R : \overline{R})$, then p is a prime divisor of $(R : \overline{R})$, by Lemma 8, and a is in $R - \mathrm{Ass}_R(R/(R : \overline{R})$. Hence a is not in p. So a is a unit of R_p. Since α is a super-primitive element over R_p, we obtain the fact $a\alpha$ is also a super-primitive element over R_p. □

Let α be an element of a finite algebraic field extension of K. It is said that α is of degree d if $[K(\alpha) : K] = d$. We say that α is an ultra-primitive element of over R if $\mathrm{grade}(I_{[\alpha]} + (R : \overline{R})) > 1$ where we define $\mathrm{grade}(R) = \infty$. For the definition of an ultra-primitive element, see also [2] and [8].

Lemma 10. *Assume that $\overline{R}$ is a finite R-module and α a ultra-primitive element over R. Then $a\alpha$ is a super-primitive element over R, for every $a \in R$.*

Proof. Let $a \in R$. Let p be an element of $\mathrm{Dp}_1(R)$. Then either $p \not\supset (R : \overline{R})$ or $p \not\supset I_{[\alpha]}$. If $p \not\supset (R : \overline{R})$, then R_p is normal. Recall [7, Theorem 1.13]: Assume that R is a Krull domain, then every element α which is algebraic over R is a super-primitive element over R. Thus $a\alpha$ is a super-primitive element over R_p. Hence $J_{[a\alpha]} \not\subset p$. If $p \not\supset I_{[\alpha]}$, then $p \not\supset I_{[a\alpha]}$ because $I_{[\alpha]} \subset I_{[a\alpha]}$, by Lemma 2. Therefore $J_{[a\alpha]} \not\subset p$. This means that $a\alpha$ is a super-primitive element over R. □

Theorem 11. *Let R be a Noetherian domain and $\overline{R}$ the integral closure of R in its quotient field. Let α be a ultra-primitive element of degree d over R. Assume that $\overline{R}$ is a finite R-module and $d \geq 3$. Then there exists a one-to-one correspondence between $\{B_a; a \in R - \{0\}\}$ and $R - \{0\}/R^*$.*

Proof. Since α is a ultra-primitive element over R, we get $R' = R - \{0\}$. Then the assertion is clear from Theorem 5 (2). □

Remark 12. Let R''_α be the set of non-zero elements a of R such that α/a is a super-primitive element over R. Since the following statements hold, the same results as B_a and R' hold for $B_{a^{-1}} = R[\alpha/a] \cap R[a/\alpha]$ and R''_α:

(1) $\varphi_{\alpha^{-1}}(X) = X^d + \eta'_1 X^{d-1} + \cdots + \eta'_d$, where $\eta'_k = \eta_d^{-1}\eta_{d-k}$, for $1, \ldots, d$, where $\eta_0 = 1$.

(2) $J_{[\alpha]} = J_{[\alpha^{-1}]}$.

(3) α is super-primitive if and only if α^{-1} is super-primitive.

(4) $R''_\alpha = R'_{\alpha^{-1}}$, where $R'_\alpha = R'$.

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