ON THE GENERATORS OF THE EDGE STABILIZERS OF GROUPS ACTING ON TREES

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الخلاصـة :

لقد تم في هذا البحث البرهنة على أنه إذا أثرت الزمرة G على الشجرة X بحيث أن مجموعة رؤوس بيان الخارج X/G محدودة فإن:

(١) إذا كان المُقِر G_x لكل حرف (حَافَّة) x من X له مولدات محدودة فإن مولدات G تكون محدودة إذا كان X/G محدودة وكل مُقِر G_v لكل رأس v من X له مولدات محدودة.

(٢) إذا كان المقر G_v لكل رأس v من X له تقديم محدود فإن G لها تقديم محدود إذا وفقط إذا كان X/G محدودة وكل مقر G_x لكل حرف x من X له مولدات محدودة.

ABSTRACT

In this paper we prove that if G is a group acting on a tree X such that the set of vertices of the quotient graph X/G is finite, then: (*i*) if the stabilizer G_x of each edge x of X is finitely generated, then G is finitely generated if and only if X/G is finite and the stabilizer G_v of each vertex v of X is finitely generated; (*ii*) if the stabilizer G_v of each vertex v of X is finitely presented then G is finitely presented if and only if X/G is finite and the stabilizer G_x of each edge x of X is finitely generated.

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1. INTRODUCTION

Baumslag [1] proved that a generalized free product of two finitely presented groups is finitely presented if and only if the amalgamated subgroup is finitely generated. In this paper we generalize this result to groups acting on trees.

In [2], Bass and Serre have given a structure theorem for groups acting on trees without inversions. The case with inversions was settled by Mahmud [3]. We shall use the definitions and notation developed in [3] and [4].

For a graph X we shall write $V(X)$ for the set of vertices of X and $E(X)$ for the set of edges of X. For $x \in E(X), o(x)$, and $t(x)$ are called the ends of x and $\overline{x}$ is called the inverse of x. We note that $o(\overline{x}) = t(x)$ and $\overline{\overline{x}} = x$. The case $\overline{x} = x$ is possible. Suppose that a group G acts on a tree X in general, *i.e.* with inversions. We write X/G for the quotient graph obtained from the action G on X. If $x \in X$ (vertex or edge) we write $G(x)$ for the orbit containing x. Moreover, if x and y are in X, we write $G(x,y)$ for the set $\{g \in G : g(y) = x\}$, and $G(x,x) = G_x$, the stabilizer of x.

If T and Y are two subtrees of X such that $T \subseteq Y$ then T is called a tree of representatives for the action of G on X if T contains exactly one vertex form each G-vertex orbit and Y is called a fundamental domain for the action of G on X if each edge of Y has an end in T and Y contains exactly one edge y from each G-edge orbit such that $G(\overline{y}, y) = \emptyset$ and exactly one pair x and $\overline{x}$ from each G-edge orbit such that $G(\overline{x}, x) \neq \emptyset$. It is clear that if Y consists of exactly one vertex v then $Y = T = X = G(v) = \{v\}$ and $G_v = G$. For the existence of T and Y see Khanfar and Mahmud [5]. For the rest of this section G will be a group acting on a tree X, T a tree of representatives and Y a fundamental domain for the action of G on X such that $T \subseteq Y$ and Y contains more than one element. We have the following definition and notation:

(*i*) For each $v \in V(T)$ let v^* be the unique vertex of T such $G(v, v^*) \neq \emptyset$. It is clear that $v^* = v$, if $v \in V(T)$, and in general $(v^*)^* = v^*$. Also if $G(u,v) \neq \emptyset$, then $u^* = v^*$ for $u, v \in V(X)$.

(*ii*) For each edge y of Y define $[y]$ to be an element of $G(t(y), (t(y))^*)$, that is, $[y]((t(y))^*) = t(y)$, to be chosen as follows:

- (*a*) $[y] = 1$ if $y \in E(T)$;
- (*b*) $[y](y) = \overline{y}$ if $G(\overline{y}, y) \neq \emptyset$. We define $[\overline{y}] = [y]^{-1}$ if $G(\overline{y}, y) = \emptyset$ and $[\overline{y}] = [y]$ otherwise. It is clear that $[y][\overline{y}] = 1$ if $G(\overline{y}, y) = \emptyset$ and $[y][\overline{y}] = [y]^2 \in G_y$ otherwise.

(*iii*) For each edge y of Y let $-y = [y]^{-1}(y)$ if $o(y) \in V(T)$, otherwise $-y = y$, and let $+y = [y](-y)$. So $+y = y$ if $o(y) \in V(T)$, otherwise $+y = [y](y)$. It is clear that if $o(y) \in V(T)$ and $G(\overline{y}, y) \neq \emptyset$ then $(o(y))^* = o(y)$ and $-y = +y = \overline{y}$. Also, $t(-y) = (t(y))^*$ and $o(+y) = (o(y))^*$.

(*iv*) By a <u>word</u> of G we mean an expression w of the form $w = g_o \cdot y_1 \cdot g_1 \cdot \cdots \cdot y_n \cdot g_n, n \geq 0$, where $y_i \in E(Y)$, $i = 1, \ldots, n$, such that:

- (1) $g_o \in G_{(o(y_1))^*}$,
- (2) $g_i \in G_{(t(y_i))^*}$, for $i = 1, \ldots, n$,
- (3) $(t(y_i))^* = (o(y_{i+1}))^*$, for $i = 1, \ldots, n-1$.

We define $o(w) = (o(y_i))^*$ and $t(w) = (t(y_n))^*$. If $o(w) = t(w)$ then w is called a <u>*closed*</u> word of type $o(w)$. We define n to be the <u>*length*</u> of w, and denote it by $|w|$. The <u>*value*</u> $[w]$ of w is the element $[w] = g_o[y_1]g_1 \ldots [y_n]g_n$ of G. w is called a <u>*reduced word*</u> of G if w contains no expression of the form:

(1) $y_i g_i \overline{y}_i$ if $g_i \in G_{-y_i}$, for $i = 1, \ldots, n$

or

(2) $y_i g_i y_i$ if $g_i \in G_{y_i}$ with $G(\overline{y_i}, y_i) \neq \emptyset$, for $i = 1, \ldots, n$.

If for each i, $i = 1, \ldots, n, g_i$ is the identity element of G then w above can be written as $w = y_1 y_2 \ldots y_n$.

Lemma 1.1. G is generated by the generators of G_v, and by the elements $[y]$, where v runs over $V(T)$ and y over $E(Y)$.

Proof. See Mahmud [3].

Lemma 1.2. If $w_1 = f_o.x_1.f_1.\ldots.x_m, f_m$ and $w_2 = g_c.y_1.g_1 \ldots y_n.g_n$ are two reduced and closed words of G of the same type and same value, then $m = n$ and $x_i = y_i$ (or $x_i = \overline{y}_i$ if $G(\overline{y}_i, y_i) \neq \emptyset$) for $i = 1, \ldots, n$.

Proof. See Mahmud [4].

The following propositions are needed for the proof of the main result of this paper.

Proposition 1.3. Let x and y be two edges of Y such that x and y are not in T and $[x] = [y]$. Then $x = y$ or $x = \overline{y}$ if $G(\overline{y}, y) \neq \emptyset$.

Proof. Let v be a vertex of T. Then there exist unique reduced paths $(x_1, \ldots, x_p), (x_{p+1}, \ldots, x_m), (y_1, \ldots, y_q)$ and $(y_{q+1}, \ldots, y_n)$ in T such that $o(x_1) = v, t(x_p) = (o(x))^*, (t(x))^* = o(x_{p+1}), t(x_m) = o(y_1) = v, t(y_q) = (o(y))^*, (t(y))^* = o(y_{q+1})$ and $t(y_n) = v$. We get the words $w_1 = x_1.\ldots.x_p.x.x_{p+1}.\ldots.x_m$ and $w_2 = y_1.\ldots.y_q.y.y_{q+1}.\ldots.y_n$. It is clear that w_1 and w_2 are closed words of G of the same value and type v. Since x and y are not in T, therefore w_1 and w_2 are reduced. Therefore by Lemma 1.2, $m = n$, $p = q$ and $x = y$ or $x = \overline{y}$ if $G(\overline{y}, y) \neq \emptyset$. This completes the proof.

Proposition 1.4. If $y \in E(Y)$ and $y \notin E(T)$ then $[y] \notin G_v$, for all $v \in V(T)$.

Proof. Since $y \notin E$ (T), therefore $[y] \neq 1$. Let $v \in V(T)$. We need to show that $[y] \notin G_v$. For if $[y] \in G_v$ then we have the words:

$$w_1 = [y] \text{ and } w_2 = y_1.\ldots.y_n y.\overline{y}_n.\ldots.\overline{y}_1,$$

where $y_1, \ldots, y_n$ is the unique path in T joining v and $(o(y))^*$. Since $y \notin E(T)$, therefore $y \notin \{y_n, \overline{y}_n\}$. It is clear that $[w_1] = [w_2]$ and, w_1 and w_2 are reduced and closed words of G of type v. Therefore by Lemma 1.2, $|w_1| = |w_2|$, *i.e.* $0 = 2n + 1$. Contradiction.

Consequently $[y] \notin G_v$. This completes the proof.

Proposition 1.5. If G is finitely generated then $Y - T$ is finite.

Proof. To show that $Y - T$ is finite is equivalent to showing that the set of edges of Y that are not in T is finite. Since G is finitely generated, by Lemma 1.1, G is generated by finitely many $G_v, \in V(T)$, and the set $A = \{[y] : y \in E(Y), y \notin E(T)\}$ is finite. Since by Proposition 1.4, $A \cap G_v = \emptyset$ for all $v \in V(T)$, by Lemma 1.1, the set $\{y \in E(Y) : y \notin E(T)\}$ is finite. This completes the proof.

Convention. Throughout the next section the generators of groups acting on trees will be those of Lemma 1.1 and the presentations will be those of Corollary 5.5 of [3].

2. THE MAIN RESULT

The main result of this section is the following theorem.

Theorem 2.1. Let G be a group acting on tree X such that $V(X/G)$ is finite. Then

(i) if G_x is finitely generated for all $x \in E(X)$, then G is finitely generated if and only if X/G is finite and G_v is finitely generated for all $v \in V(X)$;

(ii) if G_v is finitely presented for all $v \in V(X)$, then G is finitely presented if and only if X/G is finite and G_x is finitely generated for all $x \in E(X)$.

Proof. Let T be a tree of representatives for the action of G on X and Y a fundamental domain such that $T \subseteq Y$. It is clear that X/G is finite if and only if Y is finite, and $V(X/G)$ is finite if and only if T is finite. By assumption this implies that T is finite. If Y consists of one vertex v then $Y = T = X = G(v) = \{v\}$, and, $G_v = G$, and the proof is clear. Assume that Y has more than one element.

(i) It is clear that if X/G is finite and G_v is finitely generated for all $v \in V(X)$, then by Lemma 1.1, G is finitely generated.

Let G and G_x be finitely generated for all $x \in E(X)$. Therefore by Proposition 1.5, Y is finite. So X/G is finite. Now we show that each vertex stabilizer of X is finitely generated. This is equivalent to showing that the stabilizer of each vertex of T is finitely generated. For each $v \in V(T)$ let H_v be the subgroup of G_v generated by the generators of G that lie in G_v and the generators of G_{-y}, where $y \in E(Y)$ is such that $(t(y))^* = v$.

Since G and G_{-y} are finitely generated and Y is finite, H_v is finitely generated. We need to show that $G_v = H_v$, *i.e.* G_v is finitely generated. For, if G_v were not finitely generated then G_v would be the union of a properly ascending chain of finitely generated subgroups

$$H_v = H_v^o < H_v^1 < H_v^2 < \dots,.$$

For each non-negative integer n let H^n be the subgroup of G generated by the generators of H_v^n, the generators of H_u and $[y]$ for all $u \in V(T)$ and all $y \in E(Y)$. From the above H^n is finitely generated and $H^n \leq H^{n+1}$.

Now we show that for every non-negative integer j we have $H^j \neq H^{j+1}$ (if G_v is not finitely generated).

For, if for some non-negative integer s we have $H^s = H^{s+1}$ then the case $H_v^s < H_v^{s+1}, H_v^{s+1}$ implies that there exists an element g of G such that $g \in H_v^{s+1}$ and $g \notin H_v^s$. As $g \in H^s, g$ can be expressed as a product $g_o[y_1]g_1 \dots [y_k]g_k$, where $g_i \in H_{u_i}$ for some vertices $u_o, u_i, \dots, u_k$ in T and edges $y_1, \dots, y_k$ in Y.

By taking the unique reduced paths in T between v and u_o, between u_o and $(o(y_i))^*$, between $(t(y_1))^*$ and $u_1, \dots, u_k$, between $(t(y_k))^*$ and u_k, and between u_k and v we may choose this product so that $w_o = g_o.y_1.g_1.\dots.y_k.g_k$ is a closed word of G of value g and type v. By performing the following operations on w_o

(1) replacing $y.g'.\overline{y}$ by $[y]g'[\overline{y}]$ if $g' \in G_{-y}$ and $y \in \{y_1, \dots, y_k\}$;

(2) replacing $y.g'.y$ by $[y]g'[y]$ if $g' \in G_y, G(\overline{y}, y) \neq \emptyset$ and $y \in \{y_1, \dots, y_k\}$, yields a reduced word $w = f_o.x_1.f_1.\dots.x_n.f_n$ of G such that $[w] = g, f_i \in H_{v_i}$ where $v_i = (t(x_i))^*$ for $i = 1, \dots, n$ and w is of type v.

This implies that the words g and w are reduced words of G of the same value and of type v. Lemma 1.2 implies that $n = 0$. Then $g \in H_v \leq H_v^s$. Contradiction. Therefore $H^o < H^1 < H^2 \dots$ is a proper ascending chain of finitely generated subgroups of G. Since $G = \bigcup_{n \geq 0} H^n$, this contradicts the assumption that G is finitely generated because a finitely generated group cannot be the union of an ascending sequence of proper finitely subgroups. So $G_v = H_v$. Hence the stabilizer of each vertex of X is finitely generated.

(*ii*) By Corollary 5.5 of [3], G has the presentation:

$$\langle G_v x, l | rel\ G_v, G_m = G_{\overline{m}}, x.[x]^{-1}G_x[x].x^{-l} = G_x, l.G_l.l^{-1} = G_l.l^2 = [l]^2\rangle$$

via $G_v \to G_v, x \to [x]$ and $l \to [l]$, where v runs over $V(T)$, m over $E(T)$, x over $E(Y)$ such that $t(x) \notin T$ and $G(\overline{x}, x) = \emptyset$, and l over $E(Y)$ such that $t(l) \notin T$ and $G(\overline{l}, l) \neq \emptyset$.

If G_v is finitely presented for all $v \in V(T), G_y$ is finitely generated for all $y \in E(Y)$ and Y is finite, then G is finitely presented.

Conversely, let G and G_v be finitely presented for all $v \in V(T)$. By (*i*) Y is finite. Now G is isomorphic to the factor group F/R of the free group F on G_v, x and l and by the smallest congruence R of F containing the relations of G. Since G_v is finitely presented for all $v \in V(T)$, Y is finite and G is finitely presented, by a well-known theorem of Neumann [6], R is finitely generated. For each $z \in E(X)$ let H_z be the subgroup of G_z generated by the generators of G_z which make G finitely related. It is clear that H_z is finitely generated. Therefore R is generated by the following relations:

(1) rel $G_v, v \in V(T)$;

(2) $H_m = H_{\overline{m}}, m \in E(T)$;

(3) $x.[x]^{-1}H_x[x].x^{-1} = H_x, x \in E(Y), t(x) \notin V(T), G(\overline{x}, x) = \emptyset$,

(4) $l.H_l.l^{-1} = H_l, l \in E(Y), t(l) \notin V(T), G(\overline{l}, l) \neq \emptyset$,

(5) $l^2 = [l]^2$, where l is as in (4).

We now map F onto F/R by mapping $G_v \to G_v$, $x \to [x]$ and $l \to [l]$. We observe that the homomorphism from F to F/R induced by the above mapping satisfies the relations of G. Also the kernel of this homomorphism contains all the elements (relators) obtained from the relations (1–5) above. Therefore this homomorphism induces an isomorphism from G onto F/R.

This gives rise to an action of G on X such that the vertex stabilizer of $v \in V(X)$ equals G_v and the edge stabilizer of $y \in E(X)$ equals H_y.

Now we show that G_y is finitely generated, or equivalently $G_y = H_y$ for all $y \in E(Y)$. For if $G_y \neq H_y$ for some $y \in E(Y)$ then there exists $g \in G_y$, $g \notin H_y$. Let $o(y) \in V(T)$. It is clear that $[y]^{-1}g[y] \in G_{-y}$ and $[y]^{-1}g[y] \notin H_{-y}$. Consider the words $w_1 = g$ and $w_2 = y.[y]^{-1}g[y].\overline{y}$. Then w_1 and w_2 are closed reduced words of G of type $o(y)$ and value g relative to the new presentation with the relations (1–5) above. Therefore by Lemma 1.2 $|w_1| = |w_2|$. This contradicts the fact that $|w_1| = 0$ and $|w_2| = 2$. If $o(y) \notin V(T)$, *i.e.* $t(y) \in V(T)$ then similarly we get a contradiction. This implies that G_y is finitely generated for all $y \in E(X)$. This completes the proof.

We have the following corollaries of Theorem 2.1.

Corollary 2.2. If every finitely subgroup of G_v is finitely presented for all $v \in V(X)$ and every subgroup of G_y is finitely generated for all $y \in E(X)$, then every finitely generated subgroup of G of finite quotient graph is finitely presented.

Proof. Let H be a finitely generated subgroup of G such X/H is finite. We need to show that H is finitely presented. Then H acts on X by restriction. It is clear that for each $x \in X$, (vertex or edge) $H_x = H \cap G_x$. If $H = H_v$ then $H \leq G_v$ and by assumption H is finitely presented. Let $y \in E(Y)$. Then $H_y \leq G_y$, and by assumption $H_y \leq G_y$ is finitely generated. Since H is finitely generated and X/H is finite, by Theorem 2.1-(*i*) H_v is finitely generated for all $v \in V(X)$. Therefore H_v is finitely presented. Hence by Theorem 2.1-(*ii*) H is finitely presented. This completes the proof.

The following corollary is a consequence of Corollary 2.2.

Corollary 2.3. If H is a finitely generated subgroup of G such that X/H is finite and every finitely generated subgroup of G_v is finitely presented for all $v \in v(X)$ and $H \cap G_y$ is finitely generated for all $y \in E(X)$, then H is finitely presented.

Corollary 2.4. Let $G = \prod_{i \in I} (A_i;\, U_{jk} = U_{kj})$ be a tree product of the groups A_i, $i \in I$ such that I is finite. Then:

(1) If U_{jk} is finitely generated for all $j,\ k \in I$ then G is finitely generated if and only if A_i is finitely generated for all $i \in I$;

(2) If A_i is finitely presented for all $I \in I$ then G is finitely presented if and only if U_{jk} is finitely generated for all $j,\ k \in I$.

Proof. There exists a tree X on which G acts and, a tree of representatives T and a fundamental domain Y for the action of G on X such that $Y = T$ and $V(T)$ is in one-to-one correspondence with I, and if $v \in V(X)$ and $y \in E(X)$ then G_v is a conjugate of A_i, for some $i \in I$ and G_y is a conjugate of U_{jk} for some $j,\ k \in I$. Then by Theorem 2.1, the proof of Corollary 2.4 follows.

Corollary 2.5. Let $G = \langle K, t_i | rel K, t_i A_i t_i^{-1} = B_i \rangle$ be an HNN group of base K and associated pairs $(A_i B_i)$ of subgroups of K, $i \in I$. Then:

(1) If A_i is finitely generated for all $i \in I$ then G is finitely generated if and only if K is finitely generated and I is finite;

(2) If K is finitely presented then G is finitely presented if and only if A_i is finitely generated for all $i \in I$ and I is finite.

Proof. There exists a tree X on which G acts and a tree of representatives T consisting of exactly one vertex and a fundamental domain Y for the action of G on X such that $T \subset Y$ and the set of all unordered edges of Y is in one-to-one correspondence with I, and if $v \in V(X)$ and $y \in E(X)$ then G_v is a conjugate of K and G_y is a conjugate of A_i, for some $i \in I$. Then by Theorem 2.1, the proof of Corollary 2.5 follows.

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