FOUR-WEEK EMPLOYEE SCHEDULING USING THE (7/3, 7/3, 6/2) DAYS-OFF SCHEDULE

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الخلاصـة :

هذا البحث يُقدم طريقة تقريبية فعّالة لحل مسألة من واقع الحياة العملية لجدولة أيام العطل خلال دورة عمل تتكون من أربعة أسابيع . وباستخدام ترتيب (أيام العمل/أيام العطل) فإن الجدول المعني يمكن وصفه بالترميز (٣/٧ – ٣/٧ – ٢/٦) . والمفترض أن هناك مستويين من العمالة المطلوبة ، أحدهما لأيام العمل المعتادة و الآخر لأيام العطلة الأسبوعية . إنَّ الهدف الأول هو تقليل العدد الكلي للعمال ، والهدف الثاني هو تقليل عدد الورديات النشطة (المستخدمة) . وطريقة الحل المقترحة لا تشمل البرمجة الخطية أو برمجة الأعداد الصحيحة، ولكنها تستغل نتيجة حلِّ النموذج الثنائي لإيجاد العدد الكلي الأقل للعمال وتحديد وردية أيام العطل المطلوبة لكل عامل . ونظراً لاستخدامها حسابات بسيطة فإن هذه الطريقة الجديدة أكثر كفاءة من برمجة الأعداد الصحيحة في حل هذه المسألة .

ABSTRACT

An efficient heuristic technique is developed for a real-life labor days-off scheduling problem with a four-week cycle. Using a workdays/off-days notation, this days-off arrangement is referred to as the (7/3, 7/3, 6/2) schedule. Given two different levels of labor demands, D for weekdays and E for weekends, the primary objective is to minimize the workforce size. The secondary objective is to reduce the number of active days-off patterns required. The solution technique does not include linear or integer programming, but it utilizes the dual solution to determine the workforce size and feasible days-off assignments. Requiring only simple calculations, the new technique offers an efficient alternative to integer programming.

Keywords: labor scheduling, heuristics, staffing, integer programming.

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INTRODUCTION

Workforce scheduling is a complex problem that involves several conflicting objectives, such as labor cost, customer demands, and employee availability, skills, and preferences. Efficient scheduling minimizes labor cost, which is a large proportion of the total cost for most organizations. Days-off scheduling is a problem that concerns organizations that operate seven days a week. Since employees cannot work continuously, they are assigned to different days off patterns, in order to give each employee a break without interrupting the work flow. The aim is to decide the number of employees assigned to each days-off pattern, in order to satisfy daily labor demands with the minimum number or cost of employees.

This paper is concerned with the (7/3, 7/3, 6/2) days-off scheduling problem, in which each employee is assigned three work stretches of 7, 7, and 6 consecutive workdays, separated by three breaks of 3, 3, and 2 consecutive days off. This is real-life work schedule used by a major company to schedule employees in remote locations. This and similar remote-area schedules are widely applied in such fields as oil exploration, mining, and road and railway construction. The main advantage of these schedules is the reduced cost of transportation to remote work locations. Under the (7/3, 7/3, 6/2) schedule, for example, each employee gets only three off periods instead of the usual four weekends during the four-week cycle. Although this schedule has practical as well as theoretical significance, it has not been addressed in the literature.

The primary objective of the (7/3, 7/3, 6/2) problem is to minimize the workforce size, *i.e.*, total number of workers. In order to reduce transportation costs further, a secondary objective is added: to minimize the number of *active* days-off patterns (*i.e.*, patterns to which some employees are actually assigned). It is assumed that all employees assigned to the same days off pattern can be transported together (*e.g.*, by aircraft flights). Therefore, total transportation cost will be proportional to the number of active days-off patterns.

To represent real-world situations, weekend labor demands are not assumed to be equal to those of regular workdays. The size and complexity of the pure integer programming model of the problem makes the optimum solution impractical. Therefore, a heuristic method is presented to produce efficient, near optimum solutions. The heuristic method seeks to obtain the minimum workforce size with the least number of active days-off patterns. To find an efficient solution, the heuristic method utilizes the dual solution and primal-dual relations.

Workforce scheduling problems are classified into three types: (1) shift scheduling; (2) days-off scheduling; and (3) tour scheduling, which combines shift and days-off scheduling. Nanda and Browne [1] provide a comprehensive survey of literature on all these types. Our focus in this paper is limited to the days-off scheduling problem. A lot of attention has been focused on the (5/2) days-off problem, in which two consecutive days off are given per week. Morris and Showalter [2] describe an iterative, cutting plane procedure to minimize the workforce size. Bechtold and Showalter [3] develop another iterative, manual procedure utilizing three simple rules.

Bartholdi III *et al.* [4] present a technique to obtain the optimum integer solution from the solution of the continuous linear programming (LP) relaxation. Vohra [5] develops an expression for the minimum workforce size for the (5/2) problem. Alfares and Bailey [6] develop a lower bound on the workforce size. Alfares [7] shows that this lower bound is equal to the minimum workforce size determined by Vohra, and combines the bound with Bartholdi III *et al.*'s LP relaxation method.

Several methods have been developed to minimize the workforce size under two assumptions: (1) D workers are required on weekdays and E workers on weekends, where $D \geq E$, and (2) each worker must have A out of B weekends off. Bechtold [8] develops methods for scheduling full- and part-time employees working α to β days per week in multiple locations. Giving each worker 2 days off per week, Burns and Carter [9] and Burns and Koop [10] consider a single type of workers, while Emmons and Burns [11] and Narasimhan [12] consider multiple worker types.

Billionnet [13] uses integer programming to schedule a hierarchical workforce to meet varying labor demands over the week, allowing each worker n off-days per week. Hung and Emmons [14] introduce multiple-shift models for 3-day and 4-day workweeks. Alfares [15] describes a single-shift optimum algorithm for 3-day workweeks. Burns *et al.* [16] present a set-processing algorithm for 3-day and 4-day workweeks with work stretch constraints. Yura [17] uses linear goal programming to minimize overtime needed to satisfy workers' days-off preferences under due-date constraints.

The remainder of this paper is organized as follows. First, the integer programming models of the (7/3, 7/3, 6/2) problem and its dual are presented. Subsequently, the procedures for determining the lower bound on workforce size and assigning workers to days-off patterns are developed. Next, the heuristic's performance in terms of the workforce size is analyzed, and a step-by-step description of the algorithm is given. Finally, an example is solved and conclusions are given.

INTEGER PROGRAMMING MODELS

The (7/3, 7/3, 6/2) days-off scheduling problem is represented by the integer linear programming model shown below. Objective function (1) includes two prioritized goals: first, to minimize the total number of workers; second, to minimize the number of active days-off patterns. The lesser weight of the secondary objective is reflected by the small value of its coefficient ε. Constraints (2) and (3) ensure that labor demands are satisfied for each day during the four-week cycle, where (2) represents *weekday constraints*, and (3) represents *weekend constraints*. Finally, (4) are logical constraints to ensure that v_j is equal to 1 if x_j is positive, and equal to 0 if x_j is equal to 0.

$$\text{Minimize } Z = \sum_{j=1}^{28} x_j + \varepsilon \sum_{j=1}^{28} v_j \tag{1}$$

subject to

$$\sum_{j=1}^{28} a_{ij} x_j \geq D, \qquad i \in WD \tag{2}$$

$$\sum_{j=1}^{28} a_{ij} x_j \geq E, \qquad i \in WE \tag{3}$$

$$L\, v_j \geq x_j, \qquad j = 1, 2, \ldots, 28 \tag{4}$$

$$x_j \geq 0 \text{ and integer}, \qquad j = 1, 2, \ldots, 28 \tag{5}$$

$$v_j = 0 \text{ or } 1, \qquad j = 1, 2, \ldots, 28 \tag{6}$$

where

- x_j = number of workers assigned to weekly days-off pattern j, *i.e.*, number of workers whose first 3 days off start on day j
- ε = small constant ($\varepsilon \ll 1$)
- v_j = binary variable used to indicate the presence of the jth days-off pattern in a solution
- a_{ij} = 1 if i is a workday for days-off pattern j, 0 otherwise. Matrix $A = \{a_{ij}\}$ is shown in Table 1
- D = number of workers required on each weekday
- E = number of workers required on each weekend day
- WD = workdays = $\{n + 7k, n = 1, 2, \ldots, 5, k = 0, 1, 2, 3\} = \{1, 2, \ldots, 28\} - WE$
- WE = weekends = $\{n + 7k, n = 6, 7, k = 0, 1, 2, 3\} = \{6, 7, 13, 14, 20, 21, 27, 28\}$
- L = large constant ($L > \lceil \max(D, E)/3 \rceil$, which is the maximum possible value of x_j, as will be shown in the Appendix)
- $\lceil a \rceil$ = smallest integer greater than or equal to a.

The above formulation represents a formidable pure integer programming (IP) problem, involving 56 constraints and 56 integer variables, half of which (x_1, ..., x_{28}) are general integer, while the other half (v_1, ..., v_{28}) are binary. Because of the size and pure-integer nature of this model, optimum solution by integer programming is not feasible. Using both Hyper Lindo® and Excel Solver® on a 450-MH Pentium II with 96MB of memory, computational experiments have been performed on a small number of initial test problems with different characteristics. Optimum integer programming solutions could not be obtained in several hours. Thus, an efficient heuristic procedure will be presented next to solve this scheduling problem.

In order to develop the heuristic, the first step taken is to simplify the above IP model. All the variables (v_1, ..., v_{28}) and constraints (4 and 6) relevant to the secondary objective of minimizing the number of active days-off patterns are removed. Rules pertaining to the secondary objective will be included at a later stage in the development of the heuristic. In the simplified model, only constraints (2), (3), and (5) remain, which are imposed on the new objective function:

$$\text{Minimize } W = \sum_{j=1}^{28} x_j \ , \tag{7}$$

where

W = workforce size, *i.e.*, number of workers assigned to all 28 days-off patterns.

Table 1. Days-off Matrix $A = \{a_{ij}\}$ for the (7/3, 7/3, 6/2) Schedule.

j / i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0
2	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0
3	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1
4	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
5	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
6	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1
7	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1
8	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1
9	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1
10	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0
11	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0
12	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0
13	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1
14	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1	1
15	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1	1
16	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1	1
17	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1	1
18	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1	1
19	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0	1
20	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0	0
21	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	0
22	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1
23	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1
24	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1
25	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1
26	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1	1
27	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0	1
28	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	0	0	0

A formulation developed by Bartholdi III and Ratliff [18] for the (5/2) problem is used to obtain a sparser matrix in order to simplify the derivation of the solution. Since $\sum_{j=1}^{28} x_j$ is equal to W, (2) and (3) can be written as:

$$\sum_{j=1}^{28} a_{ij}{}^c x_j \leq d, \qquad i \in WD \tag{8}$$

$$\sum_{j=1}^{28} a_{ij}{}^c x_j \leq e, \qquad i \in WE, \tag{9}$$

where

$$a_{ij}{}^c = 1 - a_{ij}, \qquad i = 1, 2, \ldots, 28, \qquad j = 1, 2, \ldots, 28 \tag{10}$$

$$d = W - D, \tag{11}$$

$$e = W - E. \tag{12}$$

As in the case of (2) and (3), (8) and (9) respectively represent *weekday constraints* and *weekend constraints*. To facilitate discussion, the 28 variables representing days-off patterns ($x_1, \ldots, x_{28}$) are divided into two sets: *weekday patterns* corresponding to $x_j, j \in WD$, and *weekend patterns* corresponding to $x_j, j \in WE$. The dual of the LP relaxation of the simplified days-off scheduling model, defined by (2), (3), and (7), with dual variables $y_1, \ldots, y_{28}$, is:

$$\text{Maximize } W = D \sum_{m \in WD} y_m + E \sum_{n \in WE} y_n \tag{13}$$

subject to

$$\sum_{i=1}^{28} a_{ij} y_i \leq 1, \qquad j = 1, 2, \ldots, 28 \tag{14}$$

$$y_i \geq 0, \qquad i = 1, 2, \ldots, 28. \tag{15}$$

CALCULATING THE LOWER BOUND ON WORKFORCE SIZE

Let us first ignore the secondary objective in (1) and the integrality constraints in (5). Given workday and weekend labor demands D and E, the minimum workforce size W_l corresponding to the simplified model defined by (2), (3), and (7) can be easily obtained using the dual model shown above, without integer programming. To solve the dual problem we allocate the unit resource — right hand side of (14) — among the 28 dual variables in order to maximize the dual objective W. Based on a complete enumeration of all dual solutions, there are three possible optimum solutions.

1. It is possible to allocate the unit right hand side of (14) only to the 20 weekday dual variables, *i.e.*, all y_i where $i \in WD$. Matrix A shown in Table 1 indicates that there is a maximum of 15 of these variables active or present ($a_{ij} = 1$) in each constraint, or $\sum_{i \in WD} a_{ij} \leq 15$. Therefore, we can assign a value of 1/15 to each of the 20 weekday variables. In this case, $W_l = 20D/15$.
2. A similar solution is to allocate the unit right hand side of (14) only among the eight weekend dual variables, *i.e.*, all y_i where $i \in WE$. There is a maximum of six such variables active in each constraint ($\sum_{i \in WE} a_{ij} \leq 6$). Thus, we can assign a value of 1/6 to each of the eight weekend variables. In this case, $W_l = 8E/6$.
3. Since each constraint j in (14) contains only 20 active dual variables ($\sum_{i=1,\ldots,28} a_{ij} = 20$), it is possible to divide the unit right-hand side of each constraint among those 20 variables. Thus we can assign a value of 1/20 to all 28 dual variables. In this case, $W_l = (20D + 8E)/20$.

To determine the workforce size corresponding to (2), (3), and (7), we choose the maximum value of W_l obtained from the three above cases. Thus:

$$W_l = \max\ \{\ 4D/3, 4E/3, D + 0.4E\ \}. \tag{16}$$

To determine the integer lower bound on workforce size W, we must round up W_l in case it is not an integer. Therefore, we obtain the following expression for the lower bound:

$$W \geq \lceil \max\ \{\ 4D/3, 4E/3, D + 0.4E\ \} \rceil. \tag{17}$$

Expression (17) can be written as:

$$W \geq \lceil \max\ \{4/3, 4/3(E/D), 1 + 0.4E/D\} \times D \rceil. \tag{18}$$

Ignoring D, the 3 arguments of (18) are linear functions of E/D. Using simple algebra, we can determine the range in which each argument is maximum. The results obtained are summarized in Table 2.

DETERMINING DAYS-OFF ASSIGNMENTS

Using the three dual solutions presented in Table 2, basic primal-dual relationships will be used for obtaining the solution of the primal (original) days-off scheduling problem. The principle of complementary slackness dictates that a basic (non-zero) dual variable corresponds to a primal equation, and a dual equation corresponds to a basic primal variable. Days-off assignments $x_1, ..., x_{28}$ will depend on the value of E/D; thus there are three possible cases.

Case 1. $E/D \leq 5/6$

In this case, $W_l = 4D/3$. In the optimum dual solution, the 20 weekday variables are basic, and the eight weekend constraints are equations. Therefore, in the corresponding primal solution, the 20 weekday constraints (8) are equations, and the eight weekend variables are basic. Each weekday constraint (8) is an equation containing two weekend variables, since $\sum_{j \in WE} a_{ij}^c = 2$ for $i \in WD$. On the other hand, each weekend constraint (9) is an inequality containing three weekend variables, since $\sum_{j \in WE} a_{ij}^c = 3$ for $i \in WE$. The two types of constraints typically look like:

$$x_b + x_c = d, \qquad b, c \in WE \tag{19}$$

$$x_f + x_g + x_h \leq e, \qquad f, g, h \in WE. \tag{20}$$

Ignoring integrality restrictions, $W_l = 4D/3$, and the primal solution is given as:

$$x_i = \begin{cases} d/2 = D/6 = W_l/8, & \text{if } i \in WE \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Equation (21) obviously satisfies (19). To show that (20) is also satisfied, we substitute the values of (21), making the left hand side of (20) equal to: $3d/2 = D/2$. By definition, the right hand side of (20) is: $e = W_l - E$. But for Case 1, $E \leq 5D/6$, thus: $e \geq W_l - 5D/6$. Since $W_l = 4D/3$, $e \geq (4D/3 - 5D/6 = D/2)$.

Table 2. Values of *W* and Corresponding Dual Solutions for Each Value of *E/D*.

Case	Range of E/D	W	Dual solution
1	$E/D \leq 5/6$	$\lceil 4D/3 \rceil$	$y_i = 1/15$ if $i \in WD$, $y_i = 0$ if $i \in WD$
2	$E/D \geq 15/14$	$\lceil 4E/3 \rceil$	$y_i = 0$ if $i \in WD$, $y_i = 1/6$ if $i \in WD$
3	$5/6 \leq E/D \leq 15/14$	$\lceil D + 0.4E \rceil$	$y_i = 1/20$, $i = 1, 2, \ldots, 28$

When $E/D \le 5/6$, Equation (21) is optimum in terms of the minimum-workforce objective (7). However, the original objective function (1) also includes a secondary objective, which is the minimization of the number of active days-off patterns. The solution specified by Equation (21) may include up to eight active patterns (*i.e.*, all weekend variables). However, if $E/D \le 2/3$, a better solution in terms of the original objective (1) can be obtained, with a maximum of four active patterns. If only four weekend variables (x_j, j = 7, 14, 21, 28) are active, constraints 7, 14, 21, and 28 will be inequalities with two active variables, while all other constraints will be equations with one active variable. In this case, the solution is given by:

$$x_i = \begin{cases} d = D/3 = W_l/4, & \text{if } i = 7, 14, 21, 28 \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

The solutions specified by (21) and (22) are not necessarily integer, but feasibility of the relaxed problem is obtained by dividing W_l equally among all active days-off patterns. To obtain integer results while maintaining feasibility, we allocate workforce W among active patterns as evenly as possible using a simple rounding procedure. The idea is to divide the *remaining* workforce by the *remaining* number of active patterns, rounding down the result. First $W = \lceil 4D/3 \rceil$ is computed, then the basic *weekend* variables are calculated by Equations (23) and (24). Equation (24) below must be applied in the order of j, *i.e.*, all variables x_{j+7k} corresponding to the current value of j must be calculated before going to the next value of j.

If $E/D \le 2/3$

$$x_{7+7k} = \lfloor (W - \Sigma x_p)/(4 - n_p) \rfloor, \qquad k = 0, 1, 2, 3 \tag{23}$$

If $2/3 \le E/D \le 5/6$

$$x_{j+7k} = \lfloor (W - \Sigma x_p)/(8 - n_p) \rfloor, \qquad j = 6, 7, k = 0, 1, 2, 3 \tag{24}$$

where

$\lfloor a \rfloor$ = largest integer less than or equal to a

Σx_p = sum of all variables calculated prior to the current variable (please see the example at the end of the paper)

n_p = number of all previously calculated variables.

Equations (23) and (24) always produce feasible solutions for their respective ranges of E/D. This means that the minimum workforce size for Case 1 is equal to the lower bound specified by (17).

Case 2. $E/D \ge 15/14$

In this case, $W_l = 4E/3$. In the optimum dual solution, the eight weekend variables are basic, and the 20 weekday constraints are equations. Therefore, in the corresponding primal solution, the eight weekend constraints (9) are equations, and the 20 weekday variables are basic. Each weekday constraint (8) is an inequality containing six weekend variables, while each weekend constraint (9) is an equation containing five weekday variables. We proceed in a similar fashion to Case 1, aiming to reduce the number of active days-off patterns.

The number of weekday variables present in each constraint, and consequently the number of active days-off patterns, varies with increasing values of E/D. For each range of E/D, first $W = \lceil 4E/3 \rceil$ is computed, then the basic *weekday* variables are calculated by the equations below, adhering to the order of j in computing the values of x_{j+7k}. As in Case 1, these equations produce feasible solutions to the days-off scheduling problem.

If $15/14 \le E/D \le 12/11$

$$x_{j+7k} = \lfloor (W - \Sigma x_p)/(20 - n_p) \rfloor, j = 1, 2, ..., 5, \quad k = 0, 1, 2, 3 \tag{25}$$

If $12/11 \le E/D \le 9/8$

$$x_{j+7k} = \lfloor (W - \Sigma x_p)/(16 - n_p) \rfloor, j = 1, 2, 3, 4, \quad k = 0, 1, 2, 3 \tag{26}$$

If $9/8 \le E/D \le 6/5$

$$x_{j+7k} = \lfloor (W - \Sigma x_p)/(12 - n_p) \rfloor, j = 2, 3, 5, \quad k = 0, 1, 2, 3 \tag{27}$$

If $6/5 \le E/D \le 3/2$

$$x_{j+7k} = \lfloor (W - \Sigma x_p)/(8 - n_p) \rfloor, j = 2, 3, \quad k = 0, 1, 2, 3 \tag{28}$$

If $E/D \ge 3/2$

$$x_{2+7k} = \lfloor (W - \Sigma x_p)/(4 - n_p) \rfloor, \quad k = 0, 1, 2, 3 \tag{29}$$

Case 3. $5/6 \le E/D \le 15/14$

In this case, $W_l = D + 0.4E$, all dual variables are basic ($y_1 = \cdots = y_{28} = 1/20$), and all dual constraints are equations. Therefore all primal variables are also basic and all primal constraints are equations. Inequality constraints (2) and (3) are transformed into equations. The solution of the 28×28 linear system of equations is given by:

$$x_i = -0.25D + 0.3E, \quad i \in WD \tag{30}$$

$$x_i = 0.75D - 0.7E, \quad i \in WE. \tag{31}$$

The above equations cannot always be used to obtain the solution, since they may produce noninteger, and even negative, values. To guarantee feasibility, we have two options:

Option (a)

According to matrix A, if we assign a M employees to each of the 20 weekday days-off patterns, then $14M$ workers are assigned to each weekday, and $15M$ workers are assigned to each weekend day. The remaining labor demands for weekdays and weekends are denoted by D' and E', which are defined by:

$$D' = D - 14M \tag{32}$$

$$E' = E - 15M. \tag{33}$$

In order to make Case 1 applicable to the remaining demands, we must have $E'/D' \le 5/6$, or

$$(E - 15M)/(D - 14M) \le 5/6$$

thus

$$M \ge -0.25D + 0.3E. \tag{34}$$

Notice that the above lower limit on the value of M is equal to the value of weekday variables given by (30), which corresponds to the solution as a system of equations. Moreover, there must be at least $14M$ workers required on each weekday ($D \ge 14M$), and at least $15M$ workers required on each weekend day ($E \ge 15M$), thus:

$$M \le \min(D/14, E/15). \tag{35}$$

Since we are considering the case where $E/D \le 15/14$, then $E/15 \le D/14$. Combining (34) and (35), and restricting M to integer values, we obtain:

$$M = \min(\lceil -0.25D + 0.3E \rceil, \lfloor E/15 \rfloor). \tag{36}$$

The two arguments of (36) are respectively denoted by M_{I} and M_{II}. Assigning M workers to each of the 20 weekday patterns, the remaining demands D' and E' will be satisfied according to whether M_{I} or M_{II} is the minimum. Thus, there are two possibilities:

(a).I. $M_{\mathrm{I}} \le M_{\mathrm{II}}$

In this case, $M = M_{\mathrm{I}} = \lceil -0.25D + 0.3E \rceil$, and $E'/D' \le 5/6$. Thus, *weekday* days-off assignments are given by: $x_i = M,\ i \in WD$, while *weekend* assignments are obtained by applying the appropriate Case 1 equation (depending on the ratio E'/D') to the remaining demands D' and E'. The remaining workforce size W' is calculated as $\lceil 4D'/3 \rceil$, and the total workforce size is given by:

$$\begin{aligned} W &= 20M + \lceil 4D'/3 \rceil = 20M + \lceil 4(D - 14M)/3 \rceil \\ &= \lceil (4(D + M)/3 \rceil . \end{aligned} \tag{37}$$

(a).II. $M_{\mathrm{II}} \le M_{\mathrm{I}}$

In this case, $M = M_{\mathrm{II}} = \lfloor E/15 \rfloor$ and $E'/D' \ge 5/6$. Thus, we cannot apply Case 1 equations to D' and E'. To satisfy the remaining demands D' and E', a feasible solution is given by:

$$x_1' = x_7' = \lceil \max(D', E')/2 \rceil, \quad x_4' = \lfloor \max(D', E')/2 \rfloor . \tag{38}$$

Thus, the remaining workforce size W' is equal to $\lceil 3\max(D', E')/2 \rceil$. Days-off assignments and total workforce size W can be calculated by:

$$\begin{aligned} &x_i = M, \qquad \text{for all } i \in WD, \text{ except } x_1 \text{ and } x_4 \\ &x_1 = M + x_1', \quad x_4 = M + x_4', x_7 = x_7' \end{aligned} \tag{39}$$

$$\begin{aligned} W &= 20M + \lceil 3\max(D', E')/2 \rceil \\ &= \max\left(\lceil 20M + 3(D - 14M)/2 \rceil, \lceil 20M + 3(E - 15M)/2 \rceil \right) \\ &= \max\left(\lceil (3D - 2M)/2 \rceil, \lceil (3E - 5M)/2 \rceil \right). \end{aligned} \tag{40}$$

Option (b)

As indicated by matrix A, if we assign N employees to each of the eight weekend variables, then $6N$ workers are assigned to each weekday, and $5M$ workers are assigned to each weekend day. The remaining labor demands for weekdays and weekends are denoted by D'' and E'', which are defined by:

$$D'' = D - 6N \tag{41}$$

$$E'' = E - 5N. \tag{42}$$

In order to make Case 2 applicable to the remaining demands, we must have $E''/D'' \ge 15/14$, or:

$$(E - 5N)/(D - 6N) \ge 15/14$$

thus

$$N \ge 0.75D - 0.7E. \tag{43}$$

Again, we notice that this lower limit on the value of N is equal to the value of weekend variables given by (31), which corresponds to the solution as a system of equations. Moreover, there must be at least $6N$ workers required on each weekday ($D \ge 6N$), and at least $5N$ workers required on each weekend day ($E \ge 5N$), thus

$$N \le \min(D/6, E/5). \tag{44}$$

Since we are considering the case in which $E/D \geq 5/6$, then $D/6 \leq E/5$. Combining (43) and (44), and restricting N to integer values, we obtain:

$$N = \min(\lceil 0.75D - 0.7E \rceil, \lfloor D/6 \rfloor). \tag{45}$$

The two arguments of (45) are respectively denoted by N_{I} and N_{II}. Assigning N workers to each of the eight weekend patterns, the remaining demands D'' and E'' will be satisfied according to whether N_{I} or N_{II} is the minimum. Thus, there are two possibilities:

(b).I. $N_{\mathrm{I}} \leq N_{\mathrm{II}}$

In this case, $N = N_{\mathrm{I}} = \lceil 0.75D - 0.7E \rceil$ and $E''/D'' \geq 15/14$. Thus, *weekend* days-off assignments are given by: $x_i = N$, $i \in WE$, while *weekday* assignments are obtained by applying the appropriate Case 2 equation (according to ratio E''/D'') to remaining demands D'' and E''. First, the remaining workforce size W'' is calculated as $\lceil 4E''/3 \rceil$, then the total workforce size is given by:

$$\begin{aligned} W &= 8N + \lceil 4E''/3 \rceil = 8N + \lceil 4(E - 5N)/3 \rceil \\ &= \lceil (4(E + N)/3 \rceil. \end{aligned} \tag{46}$$

(b).II. $N_{\mathrm{II}} \leq N_{\mathrm{I}}$

In this case, $N = N_{\mathrm{II}} = \lfloor D/6 \rfloor$ is minimum and $E''/D'' \leq 15/14$. Thus, we can apply system (38) to the remaining demands D'' and E''. Denoting the remaining workforce size by $W'' = \lceil 3\max(D'', E'')/2 \rceil$, days-off assignments and the total workforce size W are calculated by:

$$\begin{aligned} x_1 &= \lceil \max(D'', E'')/2 \rceil, \qquad x_4 = \lfloor \max(D'', E'')/2 \rfloor \\ x_i &= N, \text{ for all } i \in WE, \text{ except } x_7 = N + \lceil \max(D'', E'')/2 \rceil \end{aligned} \tag{47}$$

$$\begin{aligned} W &= 8N + \lceil 3\max(D'', E'')/2 \rceil \\ &= \max\left(\lceil 8N + 3(D - 6N)/2 \rceil, \lceil 8N + 3(E - 5N)/2 \rceil\right) \\ &= \max\left(\lceil (3D - 2N)/2 \rceil, \lceil (3E + N)/2 \rceil\right). \end{aligned} \tag{48}$$

The four possibilities resulting from the two options may produce different values of the workforce size W. Since the primary objective is to minimize workforce size, we choose the feasible option with the least value of W. Once the minimum W is determined, the corresponding system of equations is applied to calculate days-off assignments $x_1, \ldots, x_{28}$. If there is a tie for the minimum W, we choose the option with the least number of active days-off patterns. Table 3 summarizes the number of active days-off patterns for each option, possibility, and ratio of remaining demands (E'/D' or E''/D'').

From Table 3, we can develop explicit rules for choosing the solution with the minimum number of active days-off patterns. Specifically, if there are ties between the two possibilities within the same option, we choose possibility II, which corresponds to system (38). If there is a tie between the two options, we choose Option (b) unless $N_{\mathrm{I}} < N_{\mathrm{II}}$ and we have one of these two conditions:

$$M_{\mathrm{I}} \leq M_{\mathrm{II}},\ E'/D' \leq 2/3, \text{ and } E''/D'' \leq 12/11 \tag{49}$$

$$M_{\mathrm{II}} \leq M_{\mathrm{I}}, \text{ and } E''/D'' \leq 9/8\,. \tag{50}$$

HEURISTIC PERFORMANCE IN TERMS OF THE PRIMARY OBJECTIVE

In both Case 1 and Case 2, the lower bounds on workforce size W defined by (17) are always tight, *i.e.*, feasible solutions are obtained by Equations (23)-(29). This means that the heuristic solution is optimum in terms of the primary objective for cases 1 and 2. For Case 3, satisfying the demands with the lower bound W is not guaranteed. However, the lower bounds on workforce sizes obtained by (37), (40), (46), and (48) are equal to the optimum values defined by (17). For example, if Option (a) is chosen with $M_{\text{I}} \leq M_{\text{II}}$, then $M = M_{\text{I}} = \lceil -0.25D + 0.3E \rceil$, and the workforce size specified by (37) can be written as:

$$\begin{aligned} W = \lceil (4(D + \lceil -0.25D + 0.3E \rceil)/3 \rceil &\geq \lceil (4(D - 0.25D + 0.3E)/3 \rceil \\ &= \lceil (4(0.75D + 0.3E)/3 \rceil \\ &= \lceil D + 0.4E \rceil . \end{aligned} \tag{51}$$

This heuristic lower bound on W is equal to the theoretical lower bound for Case 3 as defined by (17). Using a similar approach for the remaining possibilities, it can be shown that the corresponding heuristic lower bounds are given as:

$$\text{Option (a): } \boldsymbol{M_{\text{II}} \leq M_{\text{I}}}, \qquad W \geq \lceil (4E/3 \rceil \tag{52}$$

$$\text{Option (b): } \boldsymbol{N_{\text{I}} \leq N_{\text{II}}}, \qquad W \geq \lceil D + 0.4E \rceil \tag{53}$$

$$\text{Option (b): } \boldsymbol{N_{\text{II}} \leq N_{\text{I}}}, \qquad W \geq \lceil 4D/3 \rceil . \tag{54}$$

The three lower bounds above are equal to the theoretical lower bound values for the three cases defined by (17). Moreover, it must be remembered that the minimum value of W among all feasible possibilities is always chosen.

DESCRIPTION OF THE ALGORITHM

To sum up the above discussion, a step-by-step description of the algorithm, streamlining the required calculations, is provided as follows:

0. Initialization

   Given D and E, calculate E/D.

1. Case 1: if $E/D \leq 5/6$

   Calculate $W = \lceil 4D/3 \rceil$

   If $E/D \leq 2/3$, use Equation (23).

   If $2/3 \leq E/D \leq 5/6$, use Equation (24).

Table 3. Number of Active Days-off Patterns Obtained With Different Possible Solutions.

Possibility		I			II		
Option	Ratio of remaining demands	x_i, $i \in WD$	x_i, $i \in WE$	Total	x_i, $i \in WD$	x_i, $i \in WE$	Total
(a)	$E'/D' \leq 2/3$	20	4	24	20	1(x_7)	21
	$E'/D' \geq 2/3$	20	8	28			
(b)	$E''/D'' \leq 12/11$	20	8	28	2 (x_1, x_4)	8	10
	$12/11 \leq E''/D'' \leq 9/8$	16	8	24			
	$9/8 \leq E''/D'' \leq 6/5$	12	8	20			
	$6/5 \leq E''/D'' \leq 3/2$	8	8	16			
	$3/2 \leq E''/D''$	4	8	12			

2. Case 2: if $E/D \geq 15/14$

   Calculate $W = \lceil 4E/3 \rceil$

   If $15/14 \leq E/D \leq 12/11$, use Equation (25).

   If $12/11 \leq E/D \leq 9/8$, use Equation (26).

   If $9/8 \leq E/D \leq 6/5$, use Equation (27).

   If $6/5 \leq E/D \leq 3/2$, use Equation (28).

   If $E/D \geq 3/2$, use Equation (29).

3. Case 3: if $5/6 \leq E/D \leq 15/14$

   (*a*). Calculate $M_{\mathrm{I}} = \lceil -0.25D + 0.3E \rceil$, $M_{\mathrm{II}} = \lfloor E/15 \rfloor$, and $M = \min(M_{\mathrm{I}}, M_{\mathrm{II}})$.

   I. If $M_{\mathrm{I}} \leq M_{\mathrm{II}}$, calculate $W_{a\mathrm{I}}$ by (37), otherwise $W_{a\mathrm{I}} = \infty$

   II. If $M_{\mathrm{II}} \leq M_{\mathrm{I}}$, calculate $W_{a\mathrm{II}}$ by (40), otherwise $W_{a\mathrm{II}} = \infty$

   Let $W_a = \min(W_{a\mathrm{I}}, W_{a\mathrm{II}})$. If there is a tie choose $W_{a\mathrm{II}}$.

   (*b*) Calculate $N_{\mathrm{I}} = \lceil 0.75D - 0.7E \rceil$, $N_{\mathrm{II}} = \lfloor D/6 \rfloor$, and $N = \min(N_{\mathrm{I}}, N_{\mathrm{II}})$.

   I. If $N_{\mathrm{I}} \leq N_{\mathrm{II}}$, calculate $W_{b\mathrm{I}}$ by (46), otherwise $W_{b\mathrm{I}} = \infty$

   II. If $N_{\mathrm{II}} \leq N_{\mathrm{I}}$, calculate $W_{b\mathrm{II}}$ by (48), otherwise $W_{b\mathrm{II}} = \infty$

   Let $W_b = \min(W_{b\mathrm{I}}, W_{b\mathrm{II}})$. If there is a tie choose $W_{b\mathrm{II}}$.

   Let $W = \min(W_a, W_b)$. If tied, choose W_b unless $N_{\mathrm{I}} < N_{\mathrm{II}}$ and either (49) or (50) is valid.

   If $W = W_{a\mathrm{I}}$,

   Let $x_i = M$, $i \in WD$, $W' = W - 20M$, use (23) or (24) to determine x_i, $i \in WE$, depending on ratio E'/D'.

   If $W = W_{a\mathrm{II}}$,

   Determine x_i, $i \in 1, 2, ..., 28$, by (39).

   If $W = W_{b\mathrm{I}}$,

   Let $x_i = N$, $i \in WE$, $W'' = W - 8N$, use (25, 26, …, or 29) to determine x_i, $i \in WD$, depending on ratio E''/D''.

   If $W = W_{b\mathrm{II}}$,

   Determine x_i, $i \in 1, 2, ..., 28$, by (47).

A SOLVED EXAMPLE

Given: $D = 19$ and $E = 20$,

Since $5/6 \leq (E/D = 20/19) \leq 15/14$,

Case 3 is applicable.

(*a*) $M_{\mathrm{I}} = \lceil -0.25(19) + 0.3(20) \rceil = 2$, $M_{\mathrm{II}} = \lfloor 20/15 \rfloor = 1$, $M = \min(2, 1) = 1$.

I. Since $M_{\mathrm{I}} > M_{\mathrm{II}}$, $W_{a\mathrm{I}} = \infty$

II. Since $M_{\mathrm{II}} < M_{\mathrm{I}}$, $W_{a\mathrm{II}} = \max(\lceil (3(19)-2)/2 \rceil, \lceil (3(20)-5)/2 \rceil) = 28$

$W_a = \min(\infty, 28) = 28$.

(b) $N_{\mathrm{I}} = \lceil 0.75(19) - 0.7(20) \rceil = 1$, $N_{\mathrm{II}} = \lfloor 19/6 \rfloor = 3$, $N = \min(1, 3) = 1$.

I. Since $N_{\mathrm{I}} < N_{\mathrm{II}}$, $W_{b\mathrm{I}} = \lceil (4(20 + 1)/3 \rceil = 28$

II. Since $N_{\mathrm{II}} > N_{\mathrm{I}}$, $W_{b\mathrm{II}} = \infty$

$W_b = \min(28, \infty) = 28$.

$W = \min(28, 28) = 28$.

Since there is a tie between W_a and W_b, with $N_{\mathrm{I}} \le N_{\mathrm{II}}$ and $M_{\mathrm{II}} \le M_{\mathrm{I}}$, we must check the conditions specified in (50):

$$E''/D'' = (20 - 5)/(19 - 6) = 15/13 > 9/8$$

Since the conditions specified in (50) are not valid, we choose W_b (corresponding to $W_{b\mathrm{I}}$). Therefore:

$$x_i = 1,\ i \in WE = \{6, 7, 13, 14, 20, 21, 27, 28\}$$

For remaining demands:

$$W'' = 28 - 8 = 20$$

Since:

$$9/8 \le (E''/D'' = 15/13) \le 6/5,$$

we use Equation (27) with initial values $\sum x_p = 0$, $n_p = 0$:

$j = 2$

$x_2 = \lfloor (20 - 0)/(12 - 0) \rfloor = 1$, $\sum x_p = 1$, $n_p = 1$

$x_9 = \lfloor (20 - 1)/(12 - 1) \rfloor = 1$, $\sum x_p = 2$, $n_p = 2$

$x_{16} = \lfloor (20 - 2)/(12 - 2) \rfloor = 1$, $\sum x_p = 3$, $n_p = 3$

$x_{23} = \lfloor (20 - 3)/(12 - 3) \rfloor = 1$, $\sum x_p = 4$, $n_p = 4$

$j = 3$

$x_3 = \lfloor (20 - 4)/(12 - 4) \rfloor = 2$, $\sum x_p = 6$, $n_p = 5$

$x_{10} = \lfloor (20 - 6)/(12 - 5) \rfloor = 2$, $\sum x_p = 8$, $n_p = 6$

$x_{17} = \lfloor (20 - 8)/(12 - 6) \rfloor = 2$, $\sum x_p = 10$, $n_p = 7$

$x_{24} = \lfloor (20 - 10)/(12 - 7) \rfloor = 2$, $\sum x_p = 12$, $n_p = 8$

$j = 5$

$x_5 = \lfloor (20 - 12)/(12 - 8) \rfloor = 2$, $\sum x_p = 14$, $n_p = 9$

$x_{12} = \lfloor (20 - 14)/(12 - 9) \rfloor = 2$, $\sum x_p = 16$, $n_p = 10$

$x_{19} = \lfloor (20 - 16)/(12 - 10) \rfloor = 2$, $\sum x_p = 18$, $n_p = 11$

$x_{26} = \lfloor (20 - 18)/(12 - 11) \rfloor = 2$.

CONCLUSIONS

An efficient heuristic method for a real-life days-off scheduling problem has been developed. This method can be applied to scheduling employees on a workday/off-day sequence represented by (7/3, 7/3, 6/2) within a four-week cycle. This schedule is mainly used for scheduling employees in remote areas, where transportation cost is high. It gives each employee three days-off breaks instead of the usual four weekends during the four-week cycle. Daily labor demand is assumed to have two different levels: D for regular weekdays, and E for weekends, which is representative of real-life staffing applications.

The primary objective of the solution technique is to minimize the total number of workers assigned. The secondary objective is to minimize transportation cost by minimizing the number of active days-off patterns. Since optimum solution by integer programming is not feasible, a near-optimum, computationally-efficient heuristic solution algorithm has been presented. The solution utilizes primal–dual LP relations, but does not involve linear or integer programming. The simplicity of the algorithm makes it easy to implement manually.

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REFERENCES

[1] R. Nanda and J. Browne, *Introduction to Employee Scheduling*. New York: Van Nostrand Reinhold, 1992.

[2] G.M. Morris and M.J. Showalter, "Simple Approaches to Shift, Days-off and Tour Scheduling Problems", *Management Science*, **29** (1983), pp. 942–950.

[3] S.E. Bechtold and M.J. Showalter, "Simple Manpower Scheduling Methods for Managers", *Production and Inventory Management*, **26** (1985), pp. 116–133.

[4] J.J. Bartholdi, J.B. Orlin, and H.D. Ratliff, "Cyclic Scheduling via Integer Programs with Circular Ones", *Operations Research*, **28** (1980), pp. 1074–1085.

[5] R.V. Vohra, "The Cost of Consecutivity in the (5, 7) Cyclic Staffing Problem", *IIE Transactions*, **29** (1987), pp. 942–950.

[6] H.K. Alfares and J.E. Bailey, "Integrated Project Task and Manpower Scheduling", *IIE Transactions*, **29** (1997), pp. 711–718.

[7] H.K. Alfares, "An Efficient Two-Phase Algorithm for Cyclic Days-off Scheduling", *Computers & Operations Research*, **25** (1998), pp. 913–923.

[8] S.E. Bechtold, "Implicit Optimal and Heuristic Labor Staffing in Multiobjective, Multilocation Environment", *Decision Science*, **19** (1988), pp. 353–372.

[9] R.N. Burns and M.W. Carter, "Workforce Size and Single Shift Schedules with Variable Demands", *Management Science*, **31** (1985), pp. 599–607.

[10] R.N. Burns and G.J. Koop, "A Modular Approach to Optimal Multiple-Shift Manpower Scheduling", *Operations Research*, **35** (1987), pp. 100–110.

[11] H. Emmons and R.N. Burns, "Off-day Scheduling with Hierarchical Worker Categories", *Operations Research*, **39** (1991), pp. 484–495.

[12] N. Narasimhan, "An Algorithm for Single Shift Scheduling of Hierarchical Workforce", *European Journal of Operational Research*, **96** (1996), pp. 113–121.

[13] Billionnet, "Integer Programming to Schedule a Hierarchical Workforce with Variable Demands", *European Journal of Operational Research*, **114** (1999), pp. 105–114.

[14] R. Hung and H. Emmons, "Multiple-Shift Workforce Scheduling Under the 3-4 Compressed Workweek with a Hierarchical Workforce", *IIE Transactions*, **25** (1993), pp. 82–89.

[15] H.K. Alfares, "Dual-Based Optimization of Cyclic Three-Day Workweek Scheduling", *Asia–Pacific Journal of Operational Research*, **17** (2000), pp. 137–148.

[16] R.N. Burns, R. Narasimhan, and L.D. Smith, "A Set-Processing Algorithm for Scheduling Staff on 4-Day or 3-Day Work Weeks", *Naval research Logistics*, **45** (1998), pp. 839–853.

[17] K. Yura, "Production Scheduling to Satisfy Worker's Preferences for Days Off and Overtime Under Due-Date Constraints", *International Journal of Production Economics*, **33** (1994), pp. 265–270.

[18] J.J. Bartholdi, and H.D. Ratliff, "Unnetworks, with Applications to Idle Time Scheduling", *Management Science*, **24** (1978), pp. 850–858.

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APPENDIX: CALCULATING THE MINIMUM VALUE OF *L*

The minimum value of L in constraints (4) must be greater than the maximum possible value of x_j, which is obtained in either Case 1 or 2 if we have only four active days-off patterns. Using equation (23) with Case 1, $W = \lceil 4D/3 \rceil$ and $x_j \le \lceil D/3 \rceil$. Using equation (29) with Case 2, $W = \lceil 4E/3 \rceil$ and $x_j \le \lceil E/3 \rceil$. Therefore, $L > \lceil \max(D, E)/3 \rceil$.