INFEASIBILITY ANALYSIS FOR LINEAR SYSTEMS, A SURVEY

Katta G. Murty*
Department of Industrial and Operations Engineering
University of Michigan, Ann Arbor, MI-48109-2117, USA

Santosh N. Kabadi
Faculty of Administration, University of New Brunswick
Fredericton, NB, E3B 5A3, Canada
Phone: 506-453-4869; e-mail: kabadi@unb.ca

and

R. Chandrasekaran
University of Texas at Dallas, MsJ047
Richardson, TX 75083-0688, USA
Phone: 214-883-2032; e-mail: chandra@utdallas.edu

الخلاصـة :

تُعنى هذه الورقة بدراسة التغيرات التي نحتاج لعملها لتحويل نظام مكّون من قيود خطية ليس لها حل إلى نظام له حلّ.

ABSTRACT

We discuss infeasibility analysis (study of changes needed to make an infeasible system feasible) for systems of linear constraints.

Keywords: Infeasibility analysis.

*To whom correspondence should be addressed.
Phone: 734-763-3513
e-mail: katta_murty@umich.edu
Web page: http://www-personal.engin.umich.edu/~murty/

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1 INFEASIBILITY ANALYSIS FOR SYSTEMS OF LINEAR EQUATIONS

Mathematical models for real world problems often involve systems of linear equations of the form "$Ax = b$". Some of these models may contain more equations than variables.

Consider such a model in which the coefficient matrix A is of order $m \times n$. Then the vector of right hand side constants in the model (RHS constants vector) $b \in R^m$.

In such models, the coefficients in the A matrix come from things like properties of materials which are combined, *etc.*, which are very hard to change. The RHS constants vector usually comes from requirements that are to be met, or targets to be achieved, *etc.*, which are easier to modify if a need arises.

Suppose it turns out that the model is inconsistent, *i.e.*, it has no solution. Mathematically there is nothing more that can be done on the current model. But the real world problem does not go away, it has to be solved somehow. In this situation, we are forced to investigate what practically feasible changes can be carried out on the model, to modify it into a consistent or feasible system. Infeasibility analysis is a study of such changes (see [1–6]).

Since it is very hard to change the coefficient vectors of the variables, changes in them are rarely considered in applications. In most cases, it is the RHS constants which are changed, this is what we consider in this paper.

Historically, before the advent of linear programming [7], if the system "$Ax = b$" is infeasible, people used to find an approximate solution for it using the method of least squares, which is reported to have been developed by the 19th century mathematician Carl Friedrich Gauss while studying linear equations for approximating the orbit of the asteroid Ceres. In this method, the approximate solution is taken as an optimum solution of the unconstrained minimization problem in the variables x:

$$\text{Minimize} \quad ||Ax - b||^2.$$

Let $\bar{x}$ be an optimum solution of this problem. $\bar{x}$ is known as a least squares solution of the inconsistent system "$Ax = b$". Accepting $\bar{x}$ as an approximate solution of the system, is equivalent to changing the RHS constants vector b to $\bar{b} = A\bar{x}$, to make the system feasible. This $\bar{b}$ is unique, it is the point in the linear hull of the columns of A that is nearest by Euclidean distance to b.

The disadvantage of this method of least squares is that the user has no control on which RHS constants b_i are changed to make the system feasible. Normally, there are costs associated with changing the values of b_i, and these are different for different i. The least squares method does not take this information into account, to find a least costly modification of the b-vector to make the system feasible.

1.1 How is Infeasibility Detected?

Our original system is:

$$Ax = b, \tag{1}$$

where A is an $m \times n$ matrix, $b \in R^m$, and $x = (x_1, \ldots, x_n)^T$ is the column vector of decision variables. The alternate system for (1) based on the same data as in (1) is:

$$\pi A = 0, \quad \pi b = 1 \tag{2}$$

where $\pi = (\pi_1, \ldots, \pi_m)$ is the vector of variables in the alternate system. The classical theorem of alternatives states that (1) has no solution x iff (2) has a solution π.

Both these systems can be processed simultaneously by the Gauss-Jordan (GJ) method applied on system (1) (see [8]). For this, it is convenient to record (1) in the form of a detached coefficient tableau. The GJ method tries to carry out a GJ pivot step in each row of this tableau, with the aim of creating a unit submatrix of order m on its left hand side. In this process, at each stage, each row in the current tableau will always be a linear combination of rows in (1). For each row vector in the current tableau, the coefficients in this linear combination will be denoted by a row vector $\mu = (\mu_1, \dots, \mu_m)$. These μ-vectors for the various rows in the current tableau are stored and updated under a "memory matrix". These μ-vectors for the various rows in the original tableau are the unit vectors in R^m. These are recorded in the original tableau before beginning the application of the GJ method.

Memory matrix*				Original tableau				
μ_1	μ_2	$\dots$	μ_m	x_1	x_2	$\dots$	x_n	
1	0	$\dots$	0	a_{11}	a_{12}	$\dots$	a_{1n}	b_1
0	1	$\dots$	0	a_{21}	a_{22}	$\dots$	a_{2n}	b_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	0	$\dots$	1	a_{m1}	a_{m2}	$\dots$	a_{mn}	b_m

* Coeff. vector for expressing the row on the right of this memory matrix as a linear combination of rows in the original tableau.

Carrying out all the computations involved in the pivot steps, also on the columns of the memory matrix, updates it. Here is a summary of the method.

1. Select the order in which rows 1 to m in the tableau are to be chosen as pivot rows.
2. **General Step:** Suppose row r is the pivot row for the pivot step in the present tableau. Let $\bar{a}_{r1}, \dots, \bar{a}_{rn}, \bar{b}_r$ be the coefficients of the variables and the updated RHS constant in row r in the present tableau.

   2.1. If $(\bar{a}_{r1}, \dots, \bar{a}_{rn}) \neq 0$ select a variable x_j for a j such that $\bar{a}_{rj} \neq 0$ as the basic variable in row r, and the column of x_j in the present tableau as the pivot column, and perform the GJ pivot step. If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far, or to 4 otherwise. If row r is not the last pivot row in the selected order, with the resulting tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

   2.2. If $(\bar{a}_{r1}, \dots, \bar{a}_{rn}) = 0$ and $\bar{b}_r = 0$, this row is called the "0 = 0" equation. This indicates that the constraint in the original system (1) corresponding to this row is a redundant constraint and can be eliminated without changing the set of solutions.

   If row r is the last pivot row in the selected order, go to 3 if 2.3 given below has never occurred so far, or to 4 otherwise. If row r is not the last pivot row in the selected order, with the present tableau go back to 2 to perform a pivot step with the next pivot row in the selected order.

   2.3. If $(\bar{a}_{r1}, \dots, \bar{a}_{rn}) = 0$ and $\bar{b}_r \neq 0$, this row is called the "$0 = \alpha$" equation for $\alpha = \bar{b}_r \neq 0$, or an inconsistent or infeasible equation. In this case the original system (1) has no solution.

   If $(\bar{\mu}_1, \dots, \bar{\mu}_m)$ is the row in the memory matrix in row r in the present tableau, then $\bar{\pi} = (1/\bar{b}_r)(\bar{\mu}_1, \dots, \bar{\mu}_m)$ is a solution of the alternate system (2).

   If it is only required to either find a solution to (1) or determine that it is inconsistent, the method can terminate here.

   But to carry out infeasibility analysis, the method moves to 4 if row r is the last pivot row in the selected order, or to 2 with the present tableau to perform a pivot step with the next pivot row in the selected order.

3. Make all the nonbasic variables in the final tableau equal to 0, and the basic variable in each row equal to the updated RHS constant in that row in the final tableau. This is a basic solution to (1), terminate.

4. **Infeasibility Analysis:** In this case the original system (1) is infeasible.

   One possible way to make the system (1) feasible is:

   > for each $i = 1$ to m such that the ith equation in the final tableau is "$0 = \bar{b}_i$" for some $\bar{b}_i \neq 0$, change b_i in the original system (1) to $b_i - \bar{b}_i = b'_i$

   and leave the other b_i in the original system (1) unchanged. This change of b to b' in (1) converts all inconsistent equations "$0 = \bar{b}_t$" for $\bar{b}_t \neq 0$ in the final tableau into redundant equations "$0 = 0$".

   A basic solution of the modified system $Ax = b'$ is obtained by making all the nonbasic variables in the final tableau equal to 0, and the basic variable in each row equal to the updated RHS constant in that row in the final tableau. Terminate.

As a numerical example, we consider the following system.

Original system								
x_1	x_2	x_3	x_4	x_5	x_6	x_7	b	b'
1	0	1	−1	1	1	0	−7	−7
0	−1	2	1	−1	0	1	8	8
1	−2	5	1	−1	1	0	9	9
1	1	0	2	1	0	0	10	10
3	0	5	5	1	1	2	35	29
0	0	1	3	1	0	0	15	15
3	0	7	11	3	1	2	55	59

b' is the modified RHS vector found under the method to make original system feasible.

The method is carried out by choosing rows 1 to 7 in natural order as pivot rows. The following final tableau is obtained.

Memory matrix							Final tableau								
μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
−3	−4	0	4	0	−5	0	x_1	1	0	0	0	−8	−3	4	18
5	4	0	−5	0	7	0	x_2	0	1	0	0	13	5	−6	−38
−1	−2	1	0	0	0	0		0	0	0	0	0	0	0	0
3	3	0	−3	0	4	0	x_3	0	0	1	0	7	3	−3	−15
−1	2	0	−2	1	0	0		0	0	0	0	0	0	0	6
−1	−1	0	1	0	−1	0	x_4	0	0	0	1	−2	−1	1	10
−1	2	0	−2	0	−2	1		0	0	0	0	0	0	0	−4

"BV" is basic variable selected in row

The first inconsistent equation to be obtained in the method is the one in row 5 of the final tableau, "$0 = 6$", it leads to the solution:

$$\pi = (-1, 2, 0, -2, 1, 0, 0)/6$$

of the alternate system. The second inconsistent equation to be obtained is "0 = −4" in the last row of the final tableau; it leads to the solution:

$$\pi = (1, -2, 0, 2, 0, 2, -1)/4$$

of the alternate system.

One way of making the original system feasible that is revealed by the information in the final tableau is to change b to b' (*i.e.*, change b_5 from 35 to 29, and b_7 from 55 to 59). This leads to a modified system for which a basic solution is $x = (18, -38, -15, 10, 0, 0, 0)^T$.

Here we changed the values of two RHS constants to make the system feasible, we decreased b_5 from 35 to 29, and increased b_7 from 55 to 59. This is only one possible modification of the original b vector to make the system feasible, not necessarily the best. Models for determining the best possible change in the RHS constants vector to make the original infeasible system into a feasible one are discussed next.

1.2 Models for Finding Optimal Changes in RHS Constants Vector to Make System Feasible

In the example discussed above, we processed the rows in the order from top to bottom as pivot rows, and ended up with two inconsistent equations "$0 = \bar{b}_t$" for some $\bar{b}_t \neq 0$, and one redundant equation "0 = 0" in the final tableau. Does the number of inconsistent equations in the final tableau depend on the order in which the rows of the tableau are selected as pivot rows? It does. Consider the following example.

Original system				
x_1	x_2	x_3	b	Eq. no.
1	0	0	1	1
0	1	0	1	2
0	0	1	1	3
1	1	1	2	4
1	1	1	3	5
1	1	1	3	6

Here are the final tableaus under different orders of selection of rows in the tableau as pivot rows.

Final tableau when eqs. used as pivot rows in order 1 to 6				
x_1	x_2	x_3	RHS	Eq. no.
1	0	0	1	1
0	1	0	1	2
0	0	1	1	3
0	0	0	−1	4
0	0	0	0	5
0	0	0	0	6
No. inconsistent equations found = 1				

Final tableau when eqs. used as pivot rows in order 4, 1, 2, 3, 5, 6				
x_1	x_2	x_3	RHS	Eq. no.
1	0	0	1	4
0	1	0	1	1
0	0	1	0	2
0	0	0	1	3
0	0	0	1	5
0	0	0	1	6
No. inconsistent equations found = 3				

So, if we selected the equations in this system in the order 1 to 6 as pivot rows in the GJ method, we have to decrease just one RHS constant (the 4th in the original system) by one to make the system feasible by the above procedure. Changing the pivot row order to 4, 1, 2, 3, 5, 6, requires decreasing three RHS constants (the last three), each by one, to make the system feasible by the same procedure.

What is the Maximum Number of Inconsistent Equations That Can Be Discovered?

When system (1) is solved by the GJ method, and system (1) is infeasible, the method will find at least one inconsistent equation of the form "$0 = \bar{b}_t$" for some $\bar{b}_t \neq 0$ before it terminates. Also, we have seen that the total number of such inconsistent equations found under the method may depend on the order in which rows of the system are selected as pivot rows. What is the maximum possible number of such inconsistent equations?

In the GJ method, each time a new inconsistent equation of the form "$0 = \bar{b}_t$" for some $\bar{b}_t \neq 0$ is encountered, the method generates a new solution for the alternate system (2) from this row. If this equation is from row r of the current tableau, then the variable π_r in (2), which was 0 in all the solutions of (2) generated in the method in the past, has a nonzero value in the solution generated from this row. This implies that the set of all solutions of (2) generated in the method, which is the same as the number of inconsistent equations discovered in the method, is $\leq 1 +$ the dimension of the set of solutions of (2) $= 1 + m -$ rank$(A \vdots b)$ $= m -$ rank(A).

Models for Optimum Modification of the b-Vector

In practical applications, each equation in the model represents a constraint that is expected to be satisfied; it usually corresponds to a contractual obligation agreed upon. In American business, contractual obligations can only be broken at the expense of paying a certain penalty. The amount of this penalty can vary from a small amount to an enormous sum depending on the importance of the contractual obligation.

As an example, in 1999, a cable TV company, Mediaone, signed a contract to merge with another company, Comcast. After the signing of this contract, Mediaone received another merger proposal from AT&T on better terms, but in order to break the agreed-upon merger contract with Comcast, it had to pay Comcast a penalty of more than a billion dollars. In American business culture, such penalties for breaking all types of contracts are already an established business practice, and this practice is becoming widely adopted all over the world.

We consider models for changing the inconsistent system $Ax = b$ where A is of order $m \times n$, by changing the b-vector.

The Smallest Changes Model: This model for modifying the inconsistent system "$Ax = b$", seeks a modification of the RHS constants vector $b = (b_i)$ to $b' = (b'_i)$ to make the system feasible with the smallest number of changes, *i.e.*, to minimize the number of i for which $b_i \neq b'_i$.

The Smallest Penalty Model: If the cost of changing b_i is the penalty (or fixed cost) f_i, which is known for all i, this model seeks to find the set of b_i to change, to minimize the associated sum of penalties for making the system feasible. Actually the smallest changes model is a special case of this model obtained by taking $f_i = 1$ for all i. These two models are suitable to use if the penalties are fixed costs that only depend on the number of changes and not on the amount of each change.

The Smallest Variable Cost Model: In some applications, the cost of change may be a variable cost depending on the amount of each change, but not on the number of changes. In this case let:

$0 \le c_i^+$ = cost per unit increase in the value of b_i

$0 \le c_i^-$ = cost per unit decrease in the value of b_i.

This model seeks to change b_is to make the system feasible so as to minimize the total variable cost of all the changes. It leads to the linear program (LP)

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m} (c_i^- u_i^+ + c_i^+ u_i^-) \\ \text{subject to} \quad & Ax + u^+ I - u^- I = b \\ & u^+, u^- \ge 0 \end{aligned}$$

where $u^+ = (u_1^+, \dots, u_m^+)^T$, $u^- = (u_1^-, \dots, u_m^-)^T$, and I is the unit matrix of order m.

This model was introduced under the name *elastic programming* by G. Brown and G. Graves in a talk they gave at an ORSA-TIMS Conference in 1977, and discussed more fully by Chinnek and Dravineks [2]. They call the variables u_i^+, u_i^- *elastic variables* since they allow the constraints to "stretch" to make the feasible region nonempty. This LP has an optimum solution. If $(\bar{x}, \bar{u}^+, \bar{u}^-)$ is an optimum solution of this LP, then $b' = b - \bar{u}^+ + \bar{u}^-$ is the optimum modification of b under this model; and $\bar{x}$ is a feasible solution of the modified model.

So, this variable cost model can be solved very efficiently by linear programming techniques only.

The Smallest Variable Cost Model with Bounds: This is the same as the above model, except that bounds are imposed on the changes. For $i = 1$ to m, let

$p_i \ge 0$ denote the maximum possible increase allowed in the value of the RHS constant b_i

$q_i \ge 0$ denote the maximum possible decrease allowed in the value of the RHS constant b_i.

If some of the RHS constants b_i cannot be increased (decreased) from their present values, then we set $p_i = 0$ ($q_i = 0$), and if the value of b_i cannot be changed at all, we set both p_i, q_i equal to zero, for those i. This model seeks to change b_is subject to the bounds given above, to make the system feasible so as to minimize the total variable cost of all the changes. It leads to the linear program (LP)

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m} (c_i^- u_i^+ + c_i^+ u_i^-) \\ \text{subject to} \quad & Ax + u^+ I - u^- I = b \\ & 0 \le u_i^+ \le q_i, \quad 0 \le u_i^- \le p_i, \quad \text{for } i = 1 \text{ to } m \end{aligned}$$

where I is the unit matrix of order m.

If this LP is infeasible, it means that the bounds specified for the changes are too tight to make the original system of equations consistent.

On the other hand, if $(\bar{x}, \bar{u}^+, \bar{u}^-)$ is an optimum solution of this LP, then $b' = b - \bar{u}^+ + \bar{u}^-$ is the optimum modification of b under this model; and $\bar{x}$ is a feasible solution of the modified model.

1.3 Results on the Smallest Penalty Models

In order to find the change of b to b' that makes the inconsistent system $Ax = b$ feasible with the smallest number of changes, we need to determine the order in which the rows in the system have to be chosen as pivot rows in the GJ method, to get the smallest number of inconsistent equations in the final tableau. We will prove that this problem is NP-hard.

Theorem 1: Consider the inconsistent system of linear equations, $Ax = b$. Determining the smallest number of changes in the b-vector that will make this system feasible is NP-hard.

Proof. We review some definitions first. Let D be a matrix of order $m \times n$ and rank r such that the system

$$Dx = d \qquad (3)$$

is feasible. Let $D_{.j}$ denote the jth column vector of the matrix D. A solution $\bar{x} = (\bar{x}_j)$ of this system is said to be a basic solution if $\{D_{.j} : j \text{ such that } \bar{x}_j \neq 0\}$ is linearly independent. So, the number of basic solutions of (3) is $\leq \begin{pmatrix} n \\ r \end{pmatrix}$.

A basic solution of (3) is said to be a

nondegenerate basic solution	if the number of nonzero variables in it is r
degenerate basic solution	if this number is $\leq r - 1$.

The solutions of (3) with the smallest number of nonzero variables are always basic solutions of (3).

Let $M \geq 2, N \geq 2$ be positive integers. Let $\{a_1, \ldots, a_M\}$, $\{b_1, \ldots, b_N\}$ be two sets of positive integers satisfying the balance condition $a_1 + \ldots + a_M = b_1 + \ldots + b_N$. Consider the following system of $M + N - 1 + MN$ constraints in MN double subscripted variables x_{ij}, $i = 1$ to M, $j = 1$ to N.

$$\left.\begin{array}{ll} \sum_{j=1}^{N} x_{ij} = a_i, & i = 1, \ldots, M \\ \sum_{i=1}^{M} x_{ij} = b_j, & j = 1, \ldots, N-1 \end{array}\right\} \qquad (4)$$

$$x_{ij} = 0, \quad i = 1, \ldots, M; \;\; j = 1, \ldots, N \;\; \} \qquad (5)$$

(4) is the system of equality constraints in a balanced transportation problem, it is of full row rank. When (4) is solved by the GJ method, the final tableau will have a basic variable selected in each row, and the updated RHS constant in it becomes the value of that basic variable in the corresponding basic solution for (4).

Since (5) requires all the variables x_{ij} to be zero, the combined system (4), (5) is inconsistent. Also, it is clear that (4), (5) can be made consistent by changing $M + N - 2$ or less RHS constants iff (4) has a degenerate basic solution. However, it has been shown in (CKM [9]) that checking whether (4) has a degenerate basic solution is NP-hard. So checking whether (4), (5) can be made consistent by changing the values of $M + N - 2$ or less RHS constants is NP-hard. This implies that the problem of finding the smallest number of RHS constants in a general inconsistent system of linear equations, to make it consistent, is NP-hard. •

Even though the smallest changes model is NP-hard, quite often, optimum solutions of the smallest variable cost model turn out to be also optimal to the smallest changes model. So, a reasonable heuristic approach to solve the smallest penalty model is to take as an approximate solution for it the optimum solution of the smallest variable cost model with both c_i^+, c_i^- equal to f_i/s_i, where s_i is an estimate of the range of change of b_i to achieve feasibility.

2 INFEASIBILITY ANALYSIS FOR SYSTEMS OF LINEAR CONSTRAINTS INCLUDING INEQUALITIES

For the same reasons as mentioned in Section 1, it is hard to make changes in the coefficient matrix of the constraints in systems including inequalities. Hence, here also, we will only consider changes in the RHS constants that will modify an infeasible system into a feasible one.

Models of systems of linear constraints including linear inequalities, usually involve nonnegativity constraints on variables. Nonnegativity constraints appear naturally in models involving economic activities, since these can only occur at nonnegative levels. These nonnegativity constraints of the form: $x \geq 0$, have the important property that it is impossible to decrease the RHS constants in them. Hence, in trying to modify an infeasible model involving linear inequalities into a feasible one by changing some RHS constants, the following features may be specified: some RHS constants cannot be decreased, some others cannot be increased, while some others cannot be changed at all. Also, any possible change may be limited by a practical bound.

Let $A_{i.}, b_i$ denote the row vector of the coefficients of the variables, and the RHS constant in the ith constraint. We consider the system in the following general form (clearly, all the inequality constraints can be expressed in the $\geq$ form).

$$A_{i.}x \begin{cases} = & b_i, \quad i = 1, \ldots, m \\ \geq & b_i, \quad i = m+1, \ldots, m+p. \end{cases} \tag{6}$$

Any nonnegativity constraints on individual variables are included among the p inequalities in the above model. Let $b = (b_i)$.

First we consider the simple case when all changes in the value of any b_i are possible.

The alternate system for (6) is

$$\begin{aligned} \textstyle\sum_{i=1}^{m+p} \pi_i A_{i.} &= 0 \\ \textstyle\sum_{i=1}^{m+p} \pi_i b_i &> 0 \\ \pi_i &\geq 0 \quad i = m+1, \ldots, m+p \end{aligned} \tag{7}$$

where $\pi = (\pi_1, \ldots, \pi_{m+p})$ is the vector of variables in the alternate system. The theorem of alternatives for (6) states that (6) has no feasible solution x iff (7) has a feasible solution π.

To find a feasible solution for (6), we solve a Phase I problem that has additional variables $u^+ = (u_1^+, \ldots, u_m^+)^T$, $u^- = (u_1^-, \ldots, u_m^-)^T$, $t = (t_{m+1}, \ldots, t_{m+p})^T$ called artificial variables; which is a linear program.

$$\text{Minimize} \quad w = \sum_{i=1}^{m}(u_i^+ + u_i^-) + \sum_{i=m+1}^{m+p} t_i$$

$$\begin{aligned} \text{subject to} \quad A_{i.}x + u_i^+ - u_i^- &= b_i, \quad i = 1, \dots, m \\ A_{i.}x + t_i &\geq b_i, \quad i = m+1, \dots, m+p \\ u_i^+, u_i^-, t_i &\geq 0 \quad \text{for all } i. \end{aligned}$$

If $(\bar{x}, \bar{t}, \bar{u}^+, \bar{u}^-)$ is an optimum solution, and $\bar{w}$ is the optimum objective value in the Phase I problem, then $\bar{x}$ is a feasible solution for (6) if $\bar{w} = 0$.

If $\bar{w} > 0$, then (6) is infeasible. In this case one possible way to make the system (6) feasible is

for each $i = 1, \dots, m$, define $b_i' = b_i - (\bar{u}_i^+ - \bar{u}_i^-)$
for each $i = m+1, \dots, m+p$, define $b_i' = b_i - \bar{t}_i$

and let $b' = (b_i')$. Changing the original RHS constants vector b to b' converts (6) into a feasible system, and $\bar{x}$ is a feasible solution of the modified system. This is only one possible modification of the original b-vector to make the system feasible, not necessarily the best.

If $\bar{w} > 0$, let $\bar{\pi} = (\bar{\pi}_1, \dots, \bar{\pi}_{m+p})$ be an optimum dual solution for the Phase I problem. Then, $\bar{\pi}b > 0$, and therefore $\bar{\pi}$ is a feasible solution of the alternate system (7).

2.1 Finding All (Minimal) Infeasible Subsystems

When (6) is infeasible, there is often a mathematical interest in identifying a subset of constraints in (6) which by itself is infeasible. Such an infeasible subset can be found from any feasible solution $\pi = (\pi_1, \dots, \pi_{m+p})$ for the Phase I dual satisfying $\pi b > 0$, Murty [5]. For any such dual feasible $\pi = (\pi_i)$ satisfying $\pi b > 0$, the set of constraints with indices in the set $\{i : 1 \leq i \leq m+p$ is such that $\pi_i \neq 0\}$ is infeasible. Using this and the Phase I dual, one can generate all subsets of constraint indices that are infeasible.

Also, when (6) is infeasible, the subset of indices of nonzero variables in an extreme point π of the Phase I dual satisfying $\pi b > 0$, is a minimal infeasible (or irreducibly inconsistent) subset of constraints of (6) (a set of constraints constitutes a minimal infeasible or irreducibly inconsistent system if it is itself infeasible, but every proper subsystem of it is feasible). Hence from the Phase I dual, we can also derive all minimal infeasible subsets of constraints in (6)

When (6) is infeasible, the problem of finding the smallest cardinality subset of constraints in (6) which is infeasible is also of mathematical interest. By the above, this is equivalent to the problem of finding the most degenerate basic solution of the Phase I dual satisfying $\pi b > 0$, which is NP-hard from the results in [9, 10].

2.2 Optimum Modification of the b Vector to Make An Infeasible System Feasible

Every linear equation can be expressed as a pair of linear inequalities. Using this and the result in Theorem 1, we can conclude that when (6) is infeasible, the problem of making the smallest number of changes in (b_i) that will modify (6) into a feasible system is NP-hard.

Suppose (6) is infeasible. We will now consider the most practically useful model for modifying the b-vector in it optimally to make the system feasible. It is the smallest variable cost model with bounds, that leads to a type of Phase I problem with bounds.

For $i = 1$ to m, let

- $p_i \geq 0$ denote the maximum possible increase allowed in the value of the RHS constant b_i
- $q_i \geq 0$ denote the maximum possible decrease allowed in the value of the RHS constant b_i
- $0 \leq c_i^+$ denote the cost per unit increase in the value of b_i
- $0 \leq c_i^-$ denote the cost per unit decrease in the value of b_i.

If some of the RHS constants b_i for $1 \leq i \leq m$ cannot be decreased (increased) from its present values we set $q_i = 0 (p_i = 0)$, and we set both $q_i = p_i = 0$ if the value of b_i cannot be changed at all.

For $i = m + 1$ to $m + p$, notice that the ith constraint in (6) becomes tighter as b_i is increased, so the modification in b_i that is needed to make the system feasible is to reduce it as defined by Chinnek and Dravineks [2]. So, for these i we only consider decreasing these b_i. Hence, for $m + 1 \leq i \leq m + p$, let

- $q_i \geq 0$ denote the maximum possible decrease allowed in the value of the RHS constant b_i (q_i is set at 0 if b_i cannot be decreased)
- $0 \leq c_i^-$ denote the cost per unit decrease in the value of b_i.

This model leads to the linear program:

$$\begin{aligned}
\text{Minimize} \quad w = \sum_{i=1}^{m}(c_i^+ u_i^- + c_i^- u_i^+) + \sum_{i=m+1}^{m+p} c_i^- t_i \\
\text{subject to} \quad A_{i.}x + u_i^+ - u_i^- &= b_i, \quad i = 1, \ldots, m \\
A_{i.}x + t_i &\geq b_i, \quad i = m+1, \ldots, m+p \\
0 \leq u_i^+ \leq q_i, \ 0 \leq u_i^- \leq p_i, \ 0 \leq t_i \leq q_i & \quad \text{for all } i.
\end{aligned}$$

If this LP is infeasible, it means that the bounds specified for the changes are too tight to make the original system feasible.

On the other hand, if $(\bar{x}, \bar{u}^+, \bar{u}^-, \bar{t})$ is an optimum solution of this LP, then $b' = (b_i')$ where:

$$b_i' = \begin{cases} b_i - \bar{u}_i^+ + \bar{u}_i^-, & \text{for } i = 1 \text{ to } m \\ b_i - \bar{t}_i, & \text{for } i = m+1 \text{ to } m+p \end{cases}$$

is an optimum modification of b under this model; and $\bar{x}$ is a feasible solution of the modified model.

3 OTHER MATHEMATICAL RESULTS

3.1. Consider the smallest changes model for making the infeasible system "$Ax = b$" into a feasible one. Let A be of order $m \times n$ and rank r. Clearly $r < m$. When the GJ method discussed in Section 1.1 is applied to the system "$Ax = b$", exactly r pivot steps can be carried out, by the end of which all the remaining $m - r$ row vectors in A would become 0-vectors. So, the smallest number of changes in the b-vector needed to make the system "$Ax = b$" feasible is $\leq m - r$.

For $i = 1$ to m, let t_i denote the change in b_i to make the system feasible. Let $t = (t_1, \ldots, t_m)$. Then x, a feasible solution of the modified system, and t, together satisfy:

$$Ax - It = b, \tag{8}$$

where I is the unit matrix of order m. The smallest changes model is equivalent to finding a feasible solution $(\bar{x}, \bar{t})$ with $\bar{t}$ having the smallest number of nonzero components.

Consider the case in which b is nondegenerate in (8). Then any sequence of pivot steps performed on (8) will always keep every component of the updated RHS vector nonzero. Hence, in every basic solution of (8) corresponding to a basic vector with the maximum possible number, r, of x_j variables as basic variables, exactly $m - r$ variables from the vector t will be nonzero. This implies that in every solution of (8), at least $m - r$ variables from t will be nonzero. These facts imply the following in this case:

(i) When b is nondegenerate in (8), the number of inconsistent equations of the form $0 = \alpha$ for some $\alpha \neq 0$ discovered in the GJ method applied on "$Ax = b$" is always $m - r$, independent of the order in which the rows are selected as pivot rows.

(ii) In this case the smallest number of changes to be made in the b-vector to make the system "$Ax = b$" feasible is $m - r$.

(iii) The infeasibility analysis step in the GJ method discussed in Section 1.1 always leads to a modification of the RHS vector in "$Ax = b$" to make the system feasible with the smallest number of changes, independent of the order in which rows are selected as pivot rows in the method.

It is well known that in a probabilistic sense, most of the column vectors $b \in R^m$ will be nondegenerate in system (8). Thus even though in the worst case the smallest number of changes model is hard, for most of the systems the solution found by the GJ method of Section 1.1 will be optimum for it. Unfortunately, practical use of this argument is made difficult because checking whether a given b-vector is nondegenerate in (8) is possibly a hard problem itself.

3.2. When a system of linear constraints is infeasible, a problem of mathematical interest is to find a smallest cardinality subset of constraints whose deletion from the system will make the remaining system feasible. Chakravarti [1] shows that this problem is NP-hard even when all the constraints in the system are equations (actually our Theorem 1 in Section 1.3 also follows from the elegant proof of this result of Chakravarti, through $0-1$ integer programming). In [3] it has been shown that this problem can be solved in polynomial time if the number of variables in the system, n, is fixed; however the complexity of this algorithm grows exponentially with n.

3.3. Consider the system of linear constraints (6) containing $m + p$ constraints numbered $1, \ldots, m + p$. Let $M = \{1, \ldots, m + p\}$, the index set of all the constraints in the system. Let $b \in R^{m+p}$ be the RHS constants vector in the system. Suppose system (6) is infeasible. Let

$IS(b) =$ Class of all subsets of M which form infeasible subsystems,

$FS(b) =$ Class of all subsets of M which form feasible subsystems.

In Section 2 we have seen how the class $IS(b)$ can be completely determined from the Phase I dual. The class $IS(b)$ is closed under the operation of taking supersets (*i.e.*, if $D \in IS(b)$, then any E satisfying $D \subset E \subset M$ is also in $IS(b)$). Similarly, the class $FS(b)$ is closed under the operation of taking subsets.

The classes $IS(b), FS(b)$ are related. For example, if D is a minimal set in the class $IS(b)$, then all proper subsets of D are in the class $FS(b)$; and if E is a maximal set in the class $FS(b)$, then all strict supersets of E (*i.e.*, sets $G \neq E$ satisfying $E \subset G \subset M$) are in $IS(b)$. Using these properties, one can derive the class $FS(b)$ from the class $IS(b)$ by the following: it is the union of the classes of all proper subsets of minimal sets in the class $IS(b)$. Therefore, if

$$\begin{aligned} \alpha &= \text{smallest cardinality of a subset in the class } IS(b) \\ \beta &= \text{largest cardinality of a subset in the class } FS(b) \end{aligned}$$

then we have $\beta \geq \alpha - 1$, and in fact

$$\beta = -1 + \text{maximum cardinality among minimal sets in the class } IS(b).$$

It is interesting to study how the class $IS(b)$ varies as b varies over the nonconvex set K = set of all RHS constants vectors in (6) for which (6) is infeasible, while the conffficient matrix in (6) remains unchanged.

3.4. When a system of linear constraints is infeasible, another problem of mathematical interest is to find a partition of the constraints in it into the smallest number of subsystems such that each subsystem is feasible. Let us consider two special cases of this problem, one dealing with equality constraints only, and the other dealing with inequalities only.

Problem 1: Given an infeasible system of constraints

$$A_{i.}x = b_i, \quad i = 1, \ldots, m \tag{9}$$

find a partition $M_1, \ldots, M_r$ of $M = \{1, \ldots, m\}$ into the smallest number r of subsets such that

$$A_{i.}x = b_i, \quad i \in M_k$$

is feasible for all $k = 1$ to r.

Problem 2: Given an infeasible system of constraints

$$D_{i.}x \geq d_i, \quad i = 1, \ldots, p \tag{10}$$

find a partition $P_1, \ldots, P_s$ of $P = \{1, \ldots, p\}$ into the smallest number s of subsets such that

$$D_{i.}x \geq d_i, \quad i \in P_\ell$$

is feasible for all $\ell = 1$ to s.

A heuristic approach for solving these problems is the following greedy scheme.

The greedy scheme: Find a maximum cardinality (or at least a maximal) feasible subset of the constraints. Make this one of the subsets in the partition. Peel it off and repeat the same process with the remaining system of constraints.

Even if a maximum cardinality feasible subset of constraints is identified in each stage of this greedy scheme, we cannot guarantee that the partition generated is optimal, as the following example involving 8 equality constraints in two variables x_1, x_2 illustrates. The original system is:

Constraint	Constraint number	Constraint	constraint number
$x_1 = 1$	1	$x_1 + x_2 = 10$	5
$x_1 = 2$	2	$x_1 + 2x_2 = 15$	6
$x_1 = 3$	3	$2x_1 + x_2 = 15$	7
$x_1 = 4$	4	$x_1 + 3x_2 = 20$	8

The maximum cardinality feasible subset of constraints in the original system is $\{5, 6, 7, 8\}$, and the greedy scheme generates the partition $\{5, 6, 7, 8\}, \{1\}, \{2\}, \{3\}, \{4\}$ consisting of 5 feasible subsystems. However, the optimal partition $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$ consists of only 4 subsystems.

We have the following result.

Lemma: Problem 1 is NP-hard.

Proof. Let $a_1, \ldots, a_n, \beta$ be positive integers satisfying $\beta \neq (a_1 + \ldots + a_n)/2$. Consider SSP (subset sum problem) of finding a solution to

$$\begin{aligned} \sum_{j=1}^{n} a_j x_j &= \beta \\ x_j &\in \{0, 1\} \quad \text{for all } j \end{aligned}$$

which is a well known NP-hard problem. Let $\alpha = \sum_{j=1}^{n} a_j - \beta$. Now consider the system of $2n + 2$ equations:

$$\begin{aligned} \sum_{j=1}^{n} a_j x_j &= \beta \\ \sum_{j=1}^{n} a_j x_j &= \alpha \\ x_j &= 0 \quad j = 1, \ldots, n \\ x_j &= 1 \quad j = 1, \ldots, n. \end{aligned}$$

This system can be partitioned into exactly two subsystems both of which are feasible iff SSP has a solution. Since SSP is NP-hard, this shows that Problem 1 is NP-hard too. •

It is well known that the system of equality constraints (9) is equivalent to the following system of inequality constraints (11):

$$\left.\begin{array}{rcll} A_{i.}x & \geq & b_i & i=1,\ldots,m \\ -A_{i.}x & \geq & -b_i & i=1,\ldots,m. \end{array}\right\}. \tag{11}$$

However, Problem 1 is not equivalent to Problem 2 applied to the inequality system (11). To see this, let $a \neq 0, a \in R^n$ be a row vector, and let $b_1 < b_2 < \ldots < b_k$ be scalars. Consider the following systems of constraints:

$$\left.\begin{array}{c} ax = b_1 \\ ax = b_2 \\ \vdots \\ ax = b_k \end{array}\right\} \tag{12}$$

$$\left.\begin{array}{c} ax \geq b_1 \\ ax \geq b_2 \\ \vdots \\ ax \geq b_k \\ -ax \geq -b_1 \\ -ax \geq -b_2 \\ \vdots \\ -ax \geq -b_k \end{array}\right\}. \tag{13}$$

Systems (12) and (13) are equivalent. Problem 1 for the infeasible system of equations (12) leads to an optimum partition with k subsystems each one containing exactly one constraint from (12). However, Problem 2 for the infeasible system of inequalities (13) leads to an optimum partition of (13) into exactly two feasible subsystems.

It is not known whether Problem 1 can be posed as a special case of Problem 2. Also not known is whether Problem 2 is NP-hard.

It would be interesting to study whether Problems 1, 2 can be solved efficiently for systems involving two variables only.

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