TEACHING INEQUALITIES TO PRE-CALCULUS STUDENTS

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If some recent pre-calculus books are any indication, the subject of inequalities has suffered backwards progress in the last two hundred years. The dullest method presented in some of these texts would, for example solve $(x-1)(x-2)(x-3) > 0$ by considering four cases: one case where all factors are positive, and three other cases where one factor is positive and the other two factors are negative. Hardly a more tedious and narrow exercise in logic could be used to stifle the minds of students. In view of the large number of mathematics instructors who might be seeking alternate methods of presenting inequalities, the following exposition seems to be in order, Presented here is one method of solving inequalities which might be suitable for pre-calculus students. This article follows the author's current classroom practice. In my classes, I begin by first describing the advertised method as a theorem with but little generality. A second step presents further examples which extend the method beyond the theorem of the first day.

Theorem:

Let $f : Re \rightarrow Re$ have at most a finite number of discontinuities and zeros. (This finite set $\{b_1, \ldots, b_n\}$ of discontinuities and zeros will be called the *solution boundary*. Then, if x_i is any point inbetween two adjacent boundary points b_i, b_{i+1}, (x_i will be called a *test* point), then f(x) has the same sign as $f(x_i)$ for all $x \in (b_i\ ,\ b_{i+1})$.

The proof is an easy exercise for almost every instructor. It may be argued that since a pre-calculus student does not know a definition of continuity, hence cannot prove the theorem, hence he must not use the theorem. But this misguided paedagogical argument is invalid on at least three points.

First: If the argument is paedagogically valid, then since the typical pre-calculus student cannot adequately define the Reals, he must not use any real number. But we do require the use of real numbers.

Second: The argument is unfair to our students for the mathematical geniuses of past centuries did not naturally carry out the program of the argument. In fact it is now historically clear that the use of a concept before its precise definition was essential to know what it was that needed to be defined.

Third: As shown in the following, most students can appreciate the essentials of a proof; essentials they can understand at their level.

A one-dimensional example (illustrating the above theorem):

If the solutions of $x+1 \geqslant \dfrac{x+6}{x-2}$ are required, then

$f(x) = (x+1) - \dfrac{x+6}{x-2} = \dfrac{(x-4)(x+2)}{x-2}$ has the

solution boundary $\{-2, ②, 4\}$. The student now picks his favorite numbers, x_i in each of the four parts of the line separated by these three solution boundary points. The work may be conveniently tabulated:

x	−4	0	3	5
f (x)	−	+	−	+
solution intervals	☒	[−2, 2)	☒	[4,∞]

Hence the required solution set is: $[-2, 2) \cup [4,\infty]$

There are several things the instructor must point out to the students. Note that the value of $f(x_i)$ need not be calculated, rather the sign of $f(x_i)$ need only be determined. This is usually easier than a calculation.

Note the circle around 2 in the display of the solution boundary. My students get so involved in solving $f(x) = 0$, that they tend to forget discontinuities. These circles also remind the worker about the endpoints of the solution intervals.

It usually must be pointed out to the students the relationship between "$\geqslant$" in the original inequality

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and determination of the endpoints of solution intervals. This is best explained by sketching f.

In fact, a sketch of f quickly points out the geometric essentials of a proof of the above theorem.

A two dimensional example (given during the second lecture):

If the graph of the solutions of $\frac{x^2 + y^2 - 4}{x - y} > 0$ is required, then the union of the circle $x^2 + y^2 = 4$ and the line $x = y$ gives our solution boundary. A student might determine $f(-3, 0) < 0$, $f(-1, 0) > 0$, $f(1, 0) < 0$ and $f(3, 0) > 0$; hence the unbounded upper left region and the bounded lower right region comprise the graph.

The poorer students must be warned against using a boundary point such as (0,0) as one of the test points (x_i, y_i). Most students need to be reminded that they need only pick convenient test points. There is no need to test with $(-2, 5.1)$ when $(-3,0)$ will do.

Please note that algebraically this method amounts to replacing an inequality $f(x) \leqslant 0$ with the equality $f(x) = 0$ and the determination of some signs. The following example is not usually presented to sophomores, but could be explained.

Example (Cauchy - Bunyakovski - Schwarz inequality):

If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are vectors in the plane R^2, then: $x \cdot y \leqq |x|\,|y|$, where $x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2$ is the usual dot product.

Proof:

We obtain the solution boundary by replacing the above inequality with its corresponding equation. Note $x \cdot y = |x|\,|y| \cos\theta$, where θ is the angle between x and y, then substitute and simplify to obtain $\cos\theta = 1$. Hence there is a scalar a such that $x = a \cdot y$. Either we recognize our solution boundary $x = a \cdot y$ immediately as a two-dimensional plane in the four-dimensional space

$R^4 = \{(x_1, x_2, y_1, y_2)\}$, or we reason as follows

that $x = a \cdot y$ has another form in terms of coordinates:

$$x_1 = a \cdot y_1$$

$$x_2 = a \cdot y_2$$

Each of these two scalar equations describes a different three-dimensional hyperplane in 4-space. Thus $x = a \cdot y$ describes their intersection, at most a two-dimensional plane embedded in 4-space. Therefore, $x = a \cdot y$ does not separate 4-space, hence we need only one test point, say $(x, y) = ((1,0), (0,1))$.

The same argument could be applied to the higher dimensional forms of the CBS - inequality. Since there are only two vectors f and g, the concept of the angle θ between them is a two-dimensional one, hence easily defined; similarly for $\cos\theta$. The concept of co-dimension will need to be defined in the infinite dimensional case, and $f.g = |f|\,|g| \cos\theta$ will need to be proved.

In conclusion, observe that this method, as illustrated in the first two examples above, reinforces the lessons the pre-calculus student has already received on sketching, and further, the examples of this method can introduce three of the most common types of discontinuities that the pre-calculus student is likely to meet: removable, jump, and infinite. Therefore our method immediately relates the solution of inequalities to other parts of mathematics, surely a desirable objective. The last example above indicates some of the applicability and liabilities of the method to the serious worker in inequalities.