ON THE ZEROS OF SOLUTIONS OF QUASILINEAR DIFFERENTIAL EQUATIONS OF THE FIFTH ORDER

Abdullah Al-Rashed*

and

Salem Sahab*

R.I.I. Abdel Karim considered the differential equations of the fifth order of the form

(1) $(p(x)y'')''' + q(x)y = 0$

and

(2) $(p(x)z''')'' - q(x)z = 0,$

where $p(x)>0$ and $p(x) \geqslant 0$ are continuous functions ot $x \in (-\infty,\infty)$, and $q(x)=0$ does not hold in any interval. He has shown in [1] that under certain conditions, the existence of a nontrivial solution of (1), such that $y, y', py'', (py'')'$, and $(py'')''$ do not vanish in $(-\infty,a)$ and $\text{sgn}(y) = -\text{sgn}(y') = \text{sgn}(y'') = -\text{sgn}(py'')' = \text{sgn}(py'')''$.

He has also shown in [1] that under certain conditions, the existence of a nontrivial solution of (2), such that $z, z', pz'', (pz'')'$, and $(pz'')''$ do not vanish in (a,∞) where $\text{sgn}(z) = \text{sgn}(z') = \text{sgn}(z'') = \text{sgn}(z''') = \text{sgn}(pz''')'$. In this paper we generalize his result.

Consider the fundamental set of solutions of (1) given by y_1, y_2, y_3, y_4, y_5, where

$$w(y_1,y_2,y_3,y_4,y_5) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y'_1 & y'_2 & y'_3 & y'_4 & y'_5 \\ y''_1 & y''_2 & y''_3 & y''_4 & y''_5 \\ (py''_1)' & (py''_2)' & (py''_3)' & (py''_4)' & (py''_5)' \\ (py''_1)'' & (py''_2)'' & (py''_3)'' & (py''_4)'' & (py''_5)'' \end{vmatrix} = \begin{vmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$$

and the fundamental set of solutions of (2) given by z_1, z_2, z_3, z_4, z_5, where

$$w(z_1,z_2,z_3,z_4,z_5) = \begin{vmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ z'_1 & z'_2 & z'_3 & z'_4 & z'_5 \\ z''_1 & z''_2 & z''_3 & z''_4 & z''_5 \\ (pz''_1)' & (pz''_2)' & (pz''_3)' & (pz''_4)' & (pz''_5)' \\ (pz''_1)'' & (pz''_2)'' & (pz''_3)'' & (pz''_4)'' & (pz''_5)'' \end{vmatrix} = \begin{vmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$$

Theorem 1:

Consider the differential equation

$$(p(x)y'')''' + q(x)y = 0$$

where $p(x)>0$ and $q(z)>0$ are continuous functions of $x \in (-\infty,\infty)$, and $q(x)=0$ does not hold in any interval.

If y is a nontrivial solution, such that $y(a) \leqslant 0$, $y'(a) \geqslant 0$. $y''(a) \leqslant 0$, $(py'')'(a) \geqslant 0$, $(py'')''(a) < 0$, then $y, y', y'', (py'')$, and $(py'')''$ do not vanish for $x < a$.

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* *Student, University of Petroleum and Minerals, Dhahran, Saudi Arabia*

Proof

First, we observe that $(py'')''<0$ in $(a-\epsilon,a)$ for some $\epsilon>0$.

If $(py'')''(a)<0$, then $(py'')''(x)<0$ for all x in $(a-\epsilon,a)$ for some $\epsilon>0$, since $(p''y)''$ is continuous.

If $(py'')''(a)=0$, then at least one of the other four inequalities is different from zero, since y is a nontrivial solution. So assume $(py'')'(a)>0$, then $(py'')'(x)>0$ for all x in $(a-\epsilon,a)$ for some $\epsilon>0$. Therefore py'' is increasing in $(a-\epsilon,a)$, but $(py'')(a)\leqslant 0$ implies that $(py'')(x)<0$ for $a-\epsilon<x<a$. Hence y' is decreasing in $(a-\epsilon,a)$, but $y'(a)\geqslant 0$ implies that $y'(x)>0$ for $a-\epsilon<x<a$. But since $(py'')'''=-qy$, it follows that $(py'')'''\geqslant 0$ in $(a-\epsilon,a)$.

Hence $(py'')''$ is increasing in $(a-\epsilon,a)$. Hence $(py'')''(x)<0$ for $a-\epsilon<x<a$. A similar result is reached if $y''(a)<0$, $y'(a)>0$, or $y(a)<0$.
Now we are ready to show the theorem.

Suppose x_1 is the first zero of $(py'')''(x)$ to the left of a, and consider the interval $(x_1,a)=I$
We know that $(py'')''(x)<0$, for $x\in I$ from the above observation.

Hence $(py'')'$ is increasing in I, but $(py'')'(a)\leqslant 0$ implies that $(py'')'>0$ in I. Hence py'' is increasing in I, but $(py'')(a)\leqslant 0$ implies that $py''<0$ in I. Since $p>0$, it follows that $y''<0$ in I.

Therefore y' is decreasing in I, but $y'(a)\geqslant 0$, implies that $y'>0$ in I.

Hence y is increasing in I, but $y(a)\leqslant 0$, implies that $y<0$ in I.

From the differential equation we have $(py'')'''=-qy$. Hence $(py'')'''\geqslant 0$.

Therefore $(py'')''$ is increasing in I, but $(py'')''(a)\leqslant 0$ and $(py'')''(x_1)=0$, and this is a contradiction.

Therefore $(py'')''(x)<0$ for all $x<a$. Hence $(py'')'$ is decreasing for all $x<a$, but $(py'')'(a)\geqslant 0$ implies that $(py'')'>0$ for all $x<a$. It follows that py'' is increasing for all $x<a$, but $(py'')(a)\leqslant 0$, implies that $(py'')(x)<0$ for all $x<a$. Since $p>0$, it follows that $y''<0$ for all $x<a$. Hence y' is decreasing for all $x<a$, but $y'(a)\geqslant 0$ implies that $y'>0$ for all $x<a$. Hence y is increasing for all $x<a$, but $y(a)\leqslant 0$ implies that $y<0$ for all $x<a$. This completes the proof of the theorem.

Theorem 2:

Consider the following fifth order differential equation:

$$(p(x)z''')''-q(x)z=0$$

where $p(x)>0$ and $q(x)\geqslant 0$ are continuous functions of x on the interval $(-\infty,\infty)$, and $q(x)=0$ does not hold in any interval. Let z be a nontrivial solution such that $z(a)\geqslant 0$, $z''(a)\geqslant 0$ $z'''(a)\geqslant 0$ and $(pz''')'(a)\geqslant 0$. Then, the functions z,z',z'',z''', and $(pz''')'$ do not vanish for $x>a$.

Proof:

We will first show that $(pz''')'(x)>0$ for all x in $(a, a+\epsilon)$, where ϵ is some positive number. If $(pz''')'(a)>0$, then $(pz''')'(x)>0$ for all x in $(a, a+\epsilon)$ [by the continuity of $(pz''')'(x)$]. If $(pz''')'(a)=0$, then at least one of the other given initial conditions is different from zero since the solution z is nontrivial. So assume $z'''(a)>0$.

Then $z'''(x)>0$ for all x in $(a, a+\epsilon)$ for some $\epsilon>0$, since $z'''(x)$ is continuous. This in turn implies that $z''(x)$ is monotone increasing in $(a,a+\epsilon)$. And since $z''(a)\geqslant 0$, it follows that $z''(x)>0$ for all x in $(a,a+\epsilon)$.

Therefore $z'(x)$ is monotone increasing in $(a,a+\epsilon)$. But $z'(a) \geqslant 0$ implies that $z'(x) > 0$ for all x in $(a,a+\epsilon)$. This implies that $z(x)$ is monotone increasing in $(a,a+\epsilon)$. Hence $z(x)>0$ in $(a,a+\epsilon)$ since $z(a)>0$. From the differential equation, we have $(p(x)z''')'' = q(x)z$. Therefore $(pz''')''(x) \geqslant 0$, for all x in the interval $(a,a+\epsilon)$. Hence $(pz''')'(x)$ is monotone increasing in $(a,a+\epsilon)$. Therefore $(pz''')'(x)>0$ in $(a, a+\epsilon)$ since $(pz''')'(a) = 0$.

A similar result is reached if $z''(a)>0$, $z'(a)>0$, or $z(a)>0$.

Suppose x_1 is the first zero of $(pz''')'(x)$ to the right of a. From the observation above, it follows that $(pz''')'(x)>0$ for all x in (a,x_1).

Hence $z''(x)>0$ in (a,x_1). Similarly $z'(x)$ and $z(x)$ are positive in (a,x).

From the differential equation we have $(pz''')'' = qz$. Therefore $(pz''')(x)'' \geqslant 0$ in (a,x_1). Hence $(pz''')'(x)$ is increasing in (a,x_1), but this is a contradiction, since $(pz''')'(a) \geqslant 0$ and $(pz''')'(x_1) = 0$. Therefore $(pz''')'(x)>0$ for all x in (a,∞). Hence $(pz''')(x)$ is monotone increasing. But $(pz''')(a)>0$ implies that $(pz''')(x)>0$ for all $x>a$. Since $p(x)>0$, it follows that $z'''(x)>0$ for all $x>a$. Hence $z''(x)$ is monotone increasing for all $x>a$. But $z''(a) \geqslant 0$ implies that $z''(x)>0$ for all $x>a$. Therefore $z'(x)$ is monotone increasing for all $x>a$. Since $z'(a) \geqslant 0$, it follows that $z'(x)>0$ for all $x>a$, which implies that $z(x)$ is monotone increasing for all $x>a$. But $z(a) \geqslant 0$ implies that $z(x)$ is greater than zero for all $x>a$, and the proof of the theorem is completed.

Remark:

The conclusions of theorem 1 and 2 remain valid, if the inequalities stated in the initial conditions are replaced by their opposite signs.

REFERENCE:

(1) R.I.I. Abdel Karim, "Existence of Solutions Without Zeros for Quasilinear Differential Equations of the Fifth Order," *Arabian Journal for Science and Engineering*, Vol. 1. No.1 (1975), 30-42.