MEASURES ORTHOGONAL

TO $H^{\infty}(D)$

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الخلاصة :

لنفرض أن D مجال محدود في المستوى المركب. ولنفرض أن $H^{\infty}(D)$ تمثل جبر يات بناخ للدوال التحليلية المحدودة المعرفة على D . ولنفرض أن M (D) تمثل فراغ المثاليات العظمى .

فاذا كانت μ مقياس بورلي معرفة على M (D) فاننا نعرف المقياس μ على $\overline{D}$ كما يلي $\mu(E)=\mu(Z^{-1}(E))$ حيث $E \subset \overline{D}$.

في هذا البحث نبرهن على أنه اذا كانت μ عمودية على $H^{\infty}(D)$ و $M(D) \setminus \hat{Z}^{-1}(D)$ تحتوي على دعم μ المغلق وكانت μ تامة الانفراد فإن $\mu = 0$.

ABSTRACT

Let D be abounded domain in the complex plane. Let $H^{\infty}(D)$ be the Banach algebra of bounded analytic functions on D. Let μ be a regular Borel measure on the maximal ideal space M(D) of $H^{\infty}(D)$.

Define $\bar{\mu}$ on $\overline{D}$ by $\mu(E) = \mu(\hat{Z}^{-1}(E))$ for $E \subset \overline{D}$, where Z is the coordinate function on D and $\hat{Z}$ is its Gelfand transform.

In this paper we prove that if μ is orthogonal to $H^{\infty}(D)$, $\mu \neq 0$, the closed support of μ is contained in $M(D) \setminus \hat{Z}^{-1}(D)$ and μ is completely singular then $\mu = 0$'.

INTRODUCTION

Let D be a bounded domain in the complex plane $\mathbb{C}$, let $H^{\infty}(D)$ be the Banach algebra of bounded analytic function on D. M(D) will denote the maximal ideal space of $H^{\infty}(D)$. $\hat{Z}$ is the Gelfand transform of the function Z defined by $Z(\lambda) = \lambda$, for all λ in D. Also $\hat{f}$ denotes the Gelfand transform of f for $f \in H^{\infty}(D)$.

It was shown in [3] that $\hat{Z}(M(D)) = \overline{D}$ and $\hat{Z}^{-1}(D)$ is homeomorphic to D. $M_{\lambda}(D) = \hat{Z}^{-1}\{\lambda\}$ is the fiber over λ, for $\lambda \in D$. See [3] for detailed description of this algebra.

For a compact set $K \subset \mathbb{C}$, R(K) denotes the algebra of all functions in C(K) which can be approximated uniformly on K by rational functions with poles off K.

All measures considered in this paper are regular Borel measures.

For a measure μ on M(D) define $\bar{\mu}$ on $\overline{D}$ by $\bar{\mu}(E) = \mu(\hat{Z}^{-1}(E))$ so $\int_D f d\bar{\mu} = \int_{M(D)} f o \hat{Z} d\mu$ for all continuous functions on D.

Lemma 1:

If μ is a non-zero measure on M(D) and $\mu \perp H^{\infty}(D)$ i.e. $\int f d\mu = 0$ for every $f \in H^{\infty}(D)$, then $\bar{\mu} \perp R(\overline{D})$.

Proof:

Claim $f o \hat{Z} = f$, for every $f \in R(\overline{D})$, let $\phi \in M(D)$ and assume $\phi \in M_{\lambda}$ then $(f o \hat{Z})(\phi) = f(\hat{Z}(\phi)) = f(\lambda)$ but f is analytic at λ so [1] $f(\lambda) = f(\phi)$ so $\int_{\overline{D}} f d\bar{\mu} = \int_{M(D)} f o \hat{Z} d\mu = \int_{M(D)} \hat{f} d\mu = 0$ for all $f \in R(\overline{D})$, so $\bar{\mu} \perp R(\overline{D})$.

Definition 1 :

Let μ be a finite measure on $\mathbb{C}$ with compact

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support, the Cauchy transform of μ is defined by $\mu^*(\omega) = \int \frac{d\mu(z)}{z-\omega}$ Clearly μ^* is an analytic function off the closed support of μ.

Notations: $\overline{\text{supp }\mu}$ denotes the closed support, of μ, and $\partial D = \overline{D} \setminus D$.

Theorem A:

Let μ be a measure on a compact subset K of C then μ^* vanishes off K if $\mu \perp R(K)$.

Proof:

The proof of this theorem is given in reference [2], page 46.

Lemma 2:

Let μ be a non-zero measure on $\overline{D}$, $\mu \perp R(\overline{D})$ and $\overline{\text{supp }\mu} \subset \partial D$ then $\exists\, z_0 \in D$ such that $\mu^*(z_0) \neq 0$.

Proof:

Assume $\mu^*(z)=0$, for all $z \in D$; since $\mu \perp R(\overline{D})$, then by theorem A, $\mu^*(z) = 0$ for all $z \in C \setminus \overline{D}$ so $\mu^*(z)=0$ for all $z \notin \partial D$, again using theorem A, this implies $\mu \perp R(\partial D)$ but [2], $R(\partial D) = C(\partial D)$ hence $\mu = 0$, a contradiction.

Corollary 1:

For μ as in Lemma 1, $\mu^*(z) \neq 0$ for all $z \in D$, except at a discrete set in D.

Proof:

This is clear from the fact that μ^* is a nalytic in D.

Definition 2:

Let A be a function algebra on X, let ϕ be a maximal ideal in A, (or a non-zero complex-valued algebra homomorphism on A). A representing measure for ϕ is a positive measure μ on X, such that $\phi(f) = \int f\, d\mu$ for all $f \in A$.

Definition 3:

If μ_1, μ_2 are two measures on X, we say μ_1 is absolutely continuous with respect to μ_2 if $\mu_1(A) = 0$ for each set A for which $|\mu_2|(A) = 0$, we write $\mu_1 \ll \mu_2$ where $|\mu_2|$ is the total variation of μ_2.

Theorem B:

Let A be a function algebra on X, let ϕ be a maximal ideal in A, let μ be a complex representing measure for ϕ, then there exists a representing measure ν for ϕ such that $\nu \ll \mu$.

Proof:

The proof of this theorem is given in Reference [2], page, 33.

Corollary 2:

Let μ be a non-zero measure on $\overline{D}$. $\overline{\text{supp }\mu} \subset \partial D$ and $\mu \perp R(D)$. If $z \in D$ with $\mu^*(z) \neq 0$ then $\exists$ a representing measure μ_z for z such that $\mu_z \ll \mu$.

Proof:

Let $z_0 \in \overline{D}$, $\mu^*(z_0) \neq 0$, then $|\mu^*(z_0)| < \infty$ because $\overline{\text{supp }\mu} < \partial D$. Let f be a rational function with poles off D, then $\frac{f(z)-f(z_0)}{z-z_0}$ is also a rational function with poles off D. So

$$\int \frac{f(z)-f(z_0)}{z-z_0}\, d\mu(z) = 0, \text{ hence}$$

$$f(z_0) = \frac{1}{\mu^*(z_0)} \int \frac{f(z)}{z-z_0} d\mu(z) \text{ for all rational functions}$$

f with poles off D. By taking limits we get

$$f(z_0) = \frac{1}{\mu^*(z_0)} \int \frac{f(z)}{z-z_0}\, d\mu(z) \text{ for all } f \in R(D).$$

So $\frac{1}{\mu^*(z_0)} \frac{d\mu(z)}{z-z_0}$ is a complex representing measure for z_0 which is absolutely continuous with respect to μ. Apply theorem B to get the required result.

Definition 4:

If μ_1, μ_2 are two measures on X, we say μ_1 and μ_2 are mutually singular if there exist two sets A and B in X such that $X = A \cup B$ and $|\mu_1|(A) = |\mu_2|(B) = 0$, we write $\mu_1 \perp \mu_2$.

Definition 5:

Let A be a function algebra on X. For $\phi \in MA$ = the maximal ideal space of A, define M_ϕ to be the set of all representing measures for ϕ. A measure μ on X is said to be completely singular if $\mu \perp \nu$, for every $\nu \in M_{\phi}$, for every $\phi \in MA$.

Theorem:

Let μ be a non-zero measure on $M(D)$, $\mu \perp H^{\infty}(D)$ and $\overline{\text{supp } \mu} \subset M(D) \setminus \hat{Z}^{-1}(D)$. If μ is a completely singular measure, then $\mu = 0$.

Proof:

Assume $\mu \neq 0$, by lemma 1, $\overline{\mu} \perp R(\overline{D})$ and $\overline{\text{supp } \overline{\mu}} \subset \partial D$. By lemma 2, $\exists\, z_o \in D$ such that $\overline{\mu}^*(z_o) \neq 0$ and $|\overline{\mu}^*(z_o)| < \infty$. If $f \in H^{\infty}(D)$, then $\frac{f(z) - f(z_o)}{z - z_o}$ is also in $H^{\infty}(D)$, considering its Gelfand transform,

$$\int_{M(D)\setminus D^*} \frac{f(\phi) - f(\phi_o)}{\hat{Z}(\phi) - z_o}\, d\mu = 0, \quad \text{where } \phi_o = \hat{Z}^{-1}(z_o)$$

and $D^* = Z^{-1}(D)$.

$$\text{So} \int_{M(D)\setminus D^*} \frac{f(\phi)\, d\mu}{\hat{Z}(\phi) - z_o} = f(\phi_o) \int_{M(D)\setminus D^*} \frac{d\mu}{\hat{Z}(\phi) - z_o} =$$

$$f(\phi_o) \int_{\partial D} \frac{d\mu}{\omega - z_o} = f(\phi_o)\, \overline{\mu}^*(z_o)$$

$$\text{So } \hat{f}(\phi_o) = \frac{1}{\overline{\mu}^*(z_o)} \int_{M(D)\setminus D} \frac{f(\phi)}{\hat{Z}(\phi) - z_o}\, d\mu \quad \text{for all } f \in H^{\infty}(D)$$

So $\nu = \frac{1}{\overline{\mu}^*(z_o)} \frac{d\mu}{\hat{Z} - z_o}$ is complex representing measure for ϕ_o in $H(D)$ and $\nu \ll \mu$. Using theorem B, $\exists\, \nu_o$ representing measure for ϕ_o such that $\nu_o \ll \nu << \mu$, so μ is not completely singular.

Corollary 3:

If μ is as in the theorem, then $\Gamma(D) \subset \overline{\text{supp } \mu}$, where $\Gamma(D)$ is the Shilov boundary for $H^{\infty}(D)$.

Proof:

From the theorem $|f(z)| \leq c(z) \|f\|_{\overline{\text{supp } \mu}}$, for every $f \in H^{\infty}(D)$ and all $z \in D$ such that $\mu^*(z) \neq 0$, where $c(z)$ is a constant, which depends on z, so $|f^n(z)| = |f(z)|^n \leq c(z) \|f\|^n_{\overline{\text{supp } \mu}}$. Hence $f(z) \leq [(z)]^n \|f\|_{\overline{\text{supp } \mu}}$.
If $c(z) < 1$ then $f(z) = 0$, if $c(z) \geqslant 1$ then $|f(z)| \leq \|f\|_{\overline{\text{supp } \mu}}$, for every $z \in D$ with $\mu^*(z) \neq 0$. By Corollary 1, we have $|f(z)| \leq \|f\|_{\overline{\text{supp } \mu}}$ for every $z \in D$, so $\Gamma(D) \subset \overline{\text{supp } \mu}$.

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