SUFFICIENT CONDITIONS FOR AN OPERATOR T TO BELONG TO THE SECOND COMMUTANT OF $\{T^2, T^3\}$

Abdullah H. Al-Moajil*

الخلاصة

في هذا البحث نقدم شروط كافية لجعل المؤثر T المعرف على مجال هلبرت تابعاً للمبادل الثاني $\{T^2,T^3\}$ وتتلخص النتيجة في العبارة الآتية :

« ان المؤثر T يكون تابعاً للمبادل الثاني $\{T^n,T^m\}$ لأي أعداد صحيحة موجبة خاضعة للشرط $(n,m)=1$ ".

ABSTRACT

Let T be an operator on a Hilbert space. Some sufficient conditions on the range and kernel of T are given that ensure T to belong to the second commutant of (T^2, T^3) The conclusion is equivalent to the statement:"T belongs to the.secondcommutatn of (T^n, T^m) for any two relatively prime positive integers n and m

INTRODUCTION

Given a ring $\mathcal{R}$ we let $A(\mathcal{R}) = \{ x\in\mathcal{R} : xy = yx$ whenever $x^n y = yx^n$ and $x^{n+1} y = yx^{n+}$ for some $n\}$. It is easy to show that if $x \in A(\mathcal{R})$, n and m are relatively prime positive integers, $x^n y = yx^n$ and $x^m y = yx^m$ then $xy = yx$ [1]. The author proved that in a prime ring $\mathcal{R}$, $A(\mathcal{R}) = \mathcal{R}$ if and only if $\mathcal{R}$ contains no nilpotent elements [1]. This leads to an investigation of the set $A(\mathcal{R})$ for some special types of rings.

An involution on $\mathcal{R}$ is a map $_*$from $\mathcal{R}$ onto $\mathcal{R}$ such that for all $x,y \in \mathcal{R}$ $x^{**} = x$, $(x + y)^* = x^*+y^*$ and $(xy)^* = y^*x^*$. If, moreover, $\mathcal{R}$ is a complex algebra then $(a x)^* = ax^*$ for all complexes a and all $x \in \mathcal{R}$. If there is an involution defined on $\mathcal{R}$, we say $\mathcal{R}$ is a $*$ - ring.

An involution on $\mathcal{R}$ is said to be proper if $x=0$ whenever $x^*x=0$. In this case we say $\mathcal{R}$ is a proper $*$ - ring. We say $\mathcal{R}$ is prime if $IJ = 0$, where I and J are ideals in $\mathcal{R}$; this implies $I = 0$ or $J = 0$. Given $x \in \mathcal{R}$ we let $AR(x)$ $(AL(x))$ denote the right (left) annihilator of x in $\mathcal{R}$, i.e.,
$AR(x) = y \{\in \mathcal{R} : xy=0\}$ and
$AL(x) = \{ y \in \mathcal{R} : yx = 0 \}$.

The author proved in [1] that in a proper $*$ - ring $\mathcal{R}$ which is prime, an element $x \in \mathcal{R}$ will belong to $A(\mathcal{R})$ if x commutes with everything that commutes with x^n and x^{n+} for some integer n greater than one. So, to check whether an element x belongs to $A(\mathcal{R})$, it suffices to check whether it commutes with everything that commutes with x^2 and x^3.

It was also shown that in a prime proper $*$ - ring $\mathcal{R}$, if $x \in A(\mathcal{R})$ then one of the following conditions must hold : (i) $AR(x) = 0$, (ii) $AL(x) = 0$, or (iii) $AR(x) = AR(x^2)$ and $AL(x) = AL(x^2)$. It is easy to check that each of the conditions (i) and (ii) imply that $x\in A(\mathcal{R})$. Whether condition (iii) implies that $x\in A(\mathcal{R})$ is not known. The purpose of this paper is to find conditions that are closely related to condition (iii), which will insure that a Hilbett space operator T will belong to $A(\mathcal{R})$ where $\mathcal{R}$ is the ring of all bounded operators on the Hilbert space.

OPERATORS ON HILBERT SPACE

Let H be a Hilbert space and let B(H) denote the algebra of bounded operators on H. We will use the notation A(H) to mean A(B(H)). It is well known that B(H) is a prime proper $*$ - ring [2].

If $T \in B(H)$ we let ker T denote the null space of $T \{ (x \in H\colon Tx = 0 \})$ and $\mathcal{R}(T)$ denote the range of T. We also let T^* stand for the adjoint of T [3]. If K is a subset of H we let K denote the closure of K in H. The conditions (i), (ii), and (iii) become in

* *Department of Mathematics, University of Petroleum and Minerals, Dhahran, Saudi Arabia.*

this case, (i) ker $T = 0$, (ii) ker $T^* = 0$ and (iii) ker $T =$ ker T^2 and ker $T^* =$ ker T^{*2}. Since the orthogonal complement of ker T^* is $\overline{\mathcal{R}(T)}$ [3], the condition ker $T^* =$ ker T^{*2} is equivalent to $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^2)}$ which is equivalent to saying $\mathcal{R}(T)$ is a subset of $\overline{\mathcal{R}(T^2)}$.

The author proved that if T is an algebraic operator, then the condition $T \in A(H)$ is equivalent to any of the following conditions: (a) ker $T =$ ker T^2, (b) $\mathcal{R}(T) = \mathcal{R}(T^2)$. (c) $\mathcal{R}(T) \cap$ ker $T = (0)$ and (d) $H = \mathcal{R}(T) +$ ker T [1]. In the same reference it was also shown that if ker $T =$ ker T^2 and $T(\overline{\mathcal{R}(T)}) = \overline{\mathcal{R}(T)}$ then $T \in A(H)$. Thus, if T has a closed range and T satisfies condition (iii) then $T \in A(H)$.

CONDITIONS ON T TO BE IN A(H)

In this section we show that certain conditions on ker T and $\mathcal{R}(T)$ will imply $T \in A(H)$.

Lemma:

Suppose that ker $T =$ kerT^2. If S commutes with T^2 and T^3, then $T^2S = TST = ST^2$.

Proof:

We have $T^2(ST - TS) = T^2ST - T^3S = ST^3 - T^3S = 0$. Hence $T(ST - TS) = 0$, i.e. $TST = T^2S = ST^2$.

Theorem:

Let ker $T \subset$ ker $T^* =$ ker T^{*2}, then $T \in A(H)$.

Proof:

First note that if $T^2x = 0$, then $T^*Tx = 0$, hence $\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = 0$. Therefore ker $T =$ ker T^2. Let S commute with T^2 and T^3 and let $C = TS - ST$. Then by the lemma $TC = 0$. Hence $T^*C = 0$. We also have $T^{*2}S^*C = S^*T^{*2}C = 0$, hence $T^*S^*C = 0$. Therefore, $C^*C = S^*T^*C - T^*S^*C = 0$ and thus $C=0$.

Theorem:

Suppose that ker $T^* =$ ker T^{*2} and ker $T \cap \overline{\mathcal{R}(T)} = (0)$. Then $T \in A(H)$.

Proof:

Let S commute with T^2 and T^3. Then S^* commutes with T^{*2} and T^{*3}. Hence, by the lemma, and taking adjoints we get $T^2S = TST = ST^2$.

Let $C = ST - TS$. Since $TC = 0$, we have $\mathcal{R}(C) \subset$ ker T.

Taking orthogonal complements of ker T^* and ker T^{*2} we get $\overline{\mathcal{R}(T)} = \overline{\mathcal{R}(T^2)}$. Hence, if $x \in H$ then $STx = \lim_n ST^2x_n = \lim_n T^2Sx_n$ for some sequence $(x_n) \subset H$. Hence $\mathcal{R}(ST) \subset \overline{\mathcal{R}(T)}$. Therefore $\mathcal{R}(C) \subset \overline{\mathcal{R}(T)}$. But ker $T \cap \overline{\mathcal{R}(T)} = 0$, which implies $\mathcal{R}(C) = (0)$, or $C=0$.

Theorem:

If ker $T =$ ker T^2 and $H=$ ker $T + \mathcal{R}(T)$, then $T \in A(H)$.

Proof

Let S and C be as in the previous theorem. If $Tx = 0$, then $T^2Sx = ST^2x = 0$. Hence $TSx = 0$. Thus $Cx = STx - TSx = 0$. This says that $C(\ker T) = 0$. We also have $C(\mathcal{R}(T)) = 0$, hence $C(\overline{\mathcal{R}(T)}) = 0$. Thus $C(H) = 0$, i.e., $C = 0$.

For the next theorem we remind the reader that the condition $\mathcal{R}(T) \subset \overline{\mathcal{R}(T^2)}$ is equivalent to ker $T^* =$ ker T^{*2} (by taking orthogonal complements).

Theorem:

Suppose that ker $T =$ ker T^2 and $\mathcal{R}(T) \subset \overline{\mathcal{R}(T^2)}$ such that given $x \in H$ there exists a bounded sequence $(x_n) \subset H$ with $Tx = \lim_{n \to \infty} T^2x_n$. Then $T \in A(H)$.

Proof:

Taking orthogonal complements of ker T and ker T^2 we get $\mathcal{R}(T^*) \subset \overline{\mathcal{R}(T^{*2})}$.

Let S and C be as before, and choose $x, y \in H$ with $\|x\| \leq 1$. Choose $(x_m) \subset H$ with $\|x_m\| \leq t$ for all m and $Tx = \lim_m T^2x_m$. Choose (y_n) and $(z_n) \subset H$ such that $T^*S^*y = \lim_n T^{*2}y_n$ and $T^*y = \lim_n T^{*2}z_n$. For each n and m let $a_{nm} = (T^{*3}y_n - T^*S^*T^{*2}z_n, x_m)$. Then $|a_{nm}| \leq \|t\,T^{*3}y_n - T^*S^*T^{*2}z_n\|$ for all n and m. Hence, $\lim_m |a_{nm}| \leq t\,\|T^{*3}y_n - T^*S^*T^{*2}z_n\|$ for all n and m. Hence, $\lim_m |a_{nm}| \leq t\,\|T^{*3}y_n - T^*S^*T^{*2}z_n\|$ for all n, and thus $\lim_n \lim_m |a_{nm}| \leq t\,\|T^{*2}S^*y - T^*S^*T^*y\| = 0$. Hence $(y, Cx) = (C^*y, x) = (T^*S^*y - S^*T^*y, x) = \lim_n (T^{*2}y_n - S^*T^{*2}z_n, x) = \lim_n (T^*y_n - S^*T^*z_n, Tx) = \lim_n \lim_m (T^*y_n - S^*T^*z_n, T^2x_m) = \lim_n \lim_m (T^{*3}y_n - T^{*2}S^*T^*z_n, x_m) = \lim_n \lim_m a_{nm} = 0$. Since x and y were arbitrary, we get $C=0$.

REFERFNCES

[1] A.H. Al-Moajil, "The commutants of relatively prime powers in operator algebras", *Proceedings of the American Mathematical Society*, to appear.

[2] I., Kaplansky, *Rings of Operators*, New York: W.A. Benjamin, 1968.

[3] F. Riesz and B.SZ-Nagy, *Functional Anàlysis*, New York: Frederick Ungar Publishing, 1965.

Reference Code for ASJE Information Retrieval : QA 1276 MO 1. Paper Received June 30, 1975.