DISTINGUISHED SEQUENCES

Waleed M. Deeb

الخلاصة

لنفرض أن D مجال من نوع Δ ، $H^\infty(\Delta)$ هي جبريات بناخ للدوال التحليلية المحدودة المعرفة على D ، M(D) هي فراغ المثاليات العظمى لـ $H^\infty(D)$.

اذا اعتبرنا المتتابعة $\{z_n\}$ حيث $z_n \in D$ و $z_n \to \infty$ فاننا في هذا البحث نعطي شروط كافية لـ $\{z_n\}$ ان تكون محتواه في نفس جزء جليسون المحتوى على D . كذلك نعطي شروط كافية لجعل $\overline{\{z_n\}}\setminus\{z_n\}$ منفصلة عن جزء جليسون المحتوى على D .

ABSTRACT

Let D be a domain obtained from the open unit disk by deleting the origin and a sequence of disjoin closed disks $\Delta_n = \overline{\Delta(c_n,r_n)} = \{z:\Delta|z-c_n| \leqq r_n\}$ with $c_n \to 0$ and $\sum_{n=1}^{\infty} \frac{r_n}{|c_n|} < \infty$. Let $H^\infty(D)$ denote as usual the Banach Algebra of bounded analytic functions on D and M(D) its maximal ideal space.

Let $\{z_n\}$ be a sequence in D such that $z_n \to 0$.

In this paper we give conditions sufficient for the closure $\overline{\{z_n\}}$ of $\{z_n\}$ in M(D) to be contained in the same Gleason part as D. We also give conditions sufficient for $\overline{\{z_n\}} \setminus \{z_n\}$ to be disjoint from the Gleason part containing D.

INTRODUCTION

Let D be a bounded domain in the complex plane. Let $H^\infty(D)$ denote the Benach algebra of bounded analytic functions on D and M(D) its maximal ideal space. The Corona Conjecture asserts that D is weak* dense in M(D). In [2], Carleson proved that the open unit disk Δ is dense in $M(\Delta)$. In [7], Stout extended Carleson's result to finitely connected domains. In [3], Gamelin showed that the Corona problem is local.

By a Δ-domain we mean a domain obtained from the open unit disk Δ by deleting the origin and a sequence of disjoint closed disks $\Delta_n = \Delta\ \overline{(c_n,r_n)}$ with $c_n \to 0$ and $\sum_{n=1}^{\infty} \frac{r_n}{|c_n|} < \infty$. In [1] Behrens showed that if the Corona fails for any domain, it fails for a Δ-domain. This result of Behrens focused the attention on Δ- domains, together with Gamelin's localization of the Corona; [3] reduced the Corona problem to asking if $M_o(D) \supset \bar{D}$, where $M_o(D)$ is the fiber over the origin and $\bar{D}$ is the closure of D in M(D). See [4] for more details. In [8] Zalcman showed that the condition $\sum_{n=1}^{\infty} \frac{r_n}{|c_n|} < \infty$ implies that $\phi_o(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z} dz$ is a homomorphism of $H^\infty(D)$ which lies in $M_o(D)$. He called it the distinguished homomorphism.

Let A be a function algebra and M(A) its maximal ideal space. Two homomorphisms ψ and ϕ in M(D) are in the same Gleason part if $\| \psi - \phi \| < 2$. The pseudo-hyperbolic distance between two points ψ

* *Mathematics Department, University of Petroleum and Minerals, Dhahran, Saudi Arabia.*

and ϕ in M(D) is defined by
$$\rho(\psi,\phi)=\text{Sup } \{|\psi(f)| : f \in A, \|f\| \leq 1 \text{ and } \phi(f)=0\}.$$

The relation $\rho(\psi,\phi)<1$ is an equivalence relation and the equivalence classes are the Gleason parts of M(A), see [6].

Gamelin and Garnett showed in [5] that ϕ_o is in the same Gleason part of M (D) as D. They also showed in the same paper that if $\{z_n\}$ is a sequence in D such that $z_n \to 0$ and $d(z_n,\partial D) \geqslant c\,|z_n|$ for some $c>0$ then $z_n \to 0$ is the norm in M(D), where D is any domain such that $0 \in \partial D$ and d denotes the distance.

NOTATIONS AND DEFINITIONS

Let $$(P_n f)(z)= \frac{1}{2\pi i} \int_{\partial\Delta_n} \frac{f(w)}{w-z}\,dw, \text{ for } z \in \Delta_n^c$$
and n=1,2,...

Let $P(\phi_o)$ denote the Gleason part of M(D) which contains ϕ_o. A_o is the restriction of $H^\infty(D)$ to $M_o(D)$, it is a closed subalgebra of $C(M_o(D))$. For a sequence $\{z_n\}$ in D, $\overline{\{z_n\}}$ will denote its closure in M(D), and

$$\overline{\{z_n\}}^{*} = \overline{\{z_n\}} \setminus \{z_n\}$$

Let D be a Δ-domain; in section 2 we prove the following:

Theorem 1:

Let $\{z_n\}$ be a sequence in D which converges to 0 and satisfies

1. $\exists A>0$ such that for each k, $d(z_k, \Delta_n) \geqq A\,|z_n|$ for all $n \neq k$

2. $\overline{\lim_k}\,|(P_k f)(z_k) - (P_k f)(0)|<2$ for all $f \in H^\infty(D)$ with $\|f\| \leq 1$. Then $\{z_n\} \subset P(\phi_o)$.

Theorem 2:

Let $\{z_n\}$ be a sequence in D which converges to 0 and satisfies

1. $\exists A>0$ such that for each k, $d(z_k, \Delta_n) > A\,|z_k|$ for all $n \neq k$.

2. Given $\varepsilon > 0$, $\exists f \in H^\infty(D)$ such that $\underline{\lim}\,|(P_k f)(z_n) -(P_k f)(0)|>2-\varepsilon$. Then $\overline{\{z_k\}}^{*} \cap P(\phi_o) =\phi$.

SECTION 1: PRELIMINARIES

Theorem 1.1:

Let D be a bounded domain, $\lambda \in \partial D$ and let U be an open neighborhood of λ. For $\phi \in M_\lambda(D \cap U)$, define $\bar{\phi}$ on $H^\infty(D)$. by $\bar{\phi}(f) = \phi(f|_{D \cap U})$ where $f \in H^\infty(D)$. Then $\bar{\phi} \in M_\lambda(D)$ and the map

$$\Phi: M_\lambda(D \cap U) \to M_\lambda(D)$$

defined by $\Phi(\phi)=\bar{\phi}$, is a homeomorphism.

Also for $f \in A_\lambda(D)$, let f^* be in $H^\infty(D)$ such that $f^*|_{M_\lambda} = f$ and let $F^*=f^*|_{D \cap U}$, $F=F^*|_{M_\lambda(D \cap U)}$.

Then the map

$$\psi: A_\lambda(D) \to A_\lambda(D \cap U) \text{ defined by } \psi(f) = F$$

is an isomorphism.

Proof: See [3].

Corollary 1.1: The map ψ is an isometry.

Proof: For $f \in H^\infty(D)$.

$$\|f\|_{M_\lambda(D)} = \sup\{|f(\bar{\phi})| : \bar{\phi} \in M_\lambda(D)\}$$
$$= \sup\{|f(\Phi(\phi))| : \phi \in M_\lambda(D \cap U)\}$$
$$= \sup\{|\phi(f^*|_{D \cap U})| : \phi \in M_\lambda(D \cap U)\}$$
$$= \sup\{|\phi(\psi(f))| : \phi \in M_\lambda(D \cap U)\}$$
$$= \|\psi(f)\|.$$

Lemma 1.1: Let $\phi, \phi' \in M_\lambda(D)$ then

$$\|\phi-\phi'\|_{A_\lambda(D)} = \|\phi-\phi'\|_{H^\infty(D)}$$

Proof: Let $A = \|\phi-\phi'\|_{A_\lambda(D)}$

$$= \sup\{|f(\phi)-f(\phi')| : f \in A_\lambda(D), 0< \|f\| \leq 1\}.$$

and let

$B = \|\phi - \phi'\|_{H^\infty(D)}$

$= \sup \{|f(\phi) - f(\phi')| : f \in H^\infty(D), 0 < \|f\| \leq 1\}.$

Clearly $B \leq A$

Let $\varepsilon > 0$ be given. $\exists f \in A_\lambda$ such that

$0 < \|f\| \leq 1$ and $A \leq |f(\phi) - f(\phi')| + \varepsilon$.

Also [5] $\exists h \in H^\infty(D)$ such that

$h|_{M_\lambda(D)} = f$ and $\|h\| \leq \|f\| + \varepsilon$.

Let $g = \dfrac{h}{\|f\| + \varepsilon}$; then $g \in H^\infty(D), 0 < \|g\| \leq 1$,

we have

$$B \geqslant |g(\phi) - g(\phi')| = \left| \frac{\phi(h) - \phi'(h)}{\|f\| + \varepsilon} \right| \leq \frac{|\phi(h) - \phi'(h)|}{1 + \varepsilon}$$

$$= \frac{|\phi(f) - \phi'(f)|}{1 + \varepsilon}.$$

So $B + \varepsilon B \geq |\phi(f) - \phi'(f)| \leqq A - \varepsilon$. Since ε was arbitrary, we get $B \geqslant A$.

Lemma 1.2: If $\phi, \phi' \in M_\lambda(D \cap U)$ then

$$\|\phi - \phi'\|_{A_\lambda(D \cap U)} = \|\bar{\phi} - \bar{\phi}'\|_{A_\lambda(D)}$$

Proof: Immediate from the above corollary.

Lemma 1.3: For $\phi, \phi' \in M_\lambda(D \cap U)$

$$\|\phi - \phi'\|_{H^\infty(D \cap U)} = \|\phi - \phi'\|_{H^\infty(D)}.$$

Proof: $\|\phi - \phi'\|_{H^\infty(D \cap U)} = \|\phi - \phi'\|_{A_\lambda(D \cap U)}$

$$= \|\phi - \phi'\|_{A_\lambda(D)} = \|\phi - \phi'\|_{H^\infty(D)}$$

Let D be a domain obtained from the open unit disc by deleting a sequence of disjoint closed discs $\Delta_n = \overline{\Delta(c_n, r_n)}$ converging to 0 and satisfying

$$\sum_{n=1}^{\infty} \frac{r_n}{|c_n|} < \infty$$

For purposes of integration, we orient the circle $|z| = 1$ in the counterclockwise direction and $\partial\Delta_n$ $(n \geqslant 1)$ in the clockwise sense.

As mentioned in the introduction ϕ_o, which is defined on $H^\infty(D)$ by $\phi_o(f) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial D} \frac{f(\zeta)\, d\zeta}{\zeta}$, is a homomorphism in $M_o(D)$ called the distinguished homomorphism.

Lemma 1.4: Let D be a domain of the type described

D_k let $= D \cup \bigcup_{n=1}^{k-1} \Delta_n$ and denote by

ϕ_{ok} the distinguished homomorphism of $H^\infty(D_k)$, then $\phi_{ok} = \phi_o|_{H^\infty(D_k)}$.

Proof: Let $f \in H^\infty(D_k)$.

$$\phi_o(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta}\, d\zeta =$$

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)}{\zeta}\, d\zeta + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{f(\zeta)}{\zeta}\, d\zeta + \frac{1}{2\pi i} \sum_{n=k}^{\infty} \int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta}\, d\zeta, \text{ and}$$

because $\dfrac{f(\zeta)}{\zeta}$ is analytic in Δ_n for $1 \leq n \leq k-1$,

so $\phi_o(f) = \phi_{ok}(f)$.

SECTION 2: DISTINGUISHED SEQUENCES

Throughout this section D will denote a Δ-domain.

Recall that P_n is defined on $H^\infty(D)$ by

$$P_n(f)(z) = \frac{1}{2\pi i} \int_{\partial\Delta_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } n = 1, 2, \ldots$$

for $z \in \Delta_n^c = \mathbb{C} \setminus \Delta_n$, and $P(\phi_o)$ denotes the Gleason part of $M(D)$ which contains ϕ_o.

Proof of Theorem 1: Let $f \in H^\infty(D)$ with $\|f\| \leq 1$, fix an integer k.

$$|f(z_k)-\phi_o(f)| =$$

$$\left|\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta-z_k}\,d\zeta-\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta}\,d\zeta\right|$$

$$\leq\left|\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\partial\Delta_n}\frac{f(\zeta)}{\zeta-z_k}\,d\zeta-\right.$$

$$\left.\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{\partial\Delta_n}\frac{f(\zeta)}{\zeta}\,d\zeta\right|$$

$$+\frac{1}{2\pi}\left|\int_{\partial\Delta}\frac{z_k f(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$\leq\frac{1}{2\pi}\left|\sum_{n=1}^{k-1}\int_{\partial\Delta_n}\frac{z_k f(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$+|(P_kf)(z_k)-(P_kf)(0)|$$

$$+\frac{1}{2\pi}\left|\sum_{n=k+1}^{\infty}\int_{\partial\Delta_n}\frac{z_k f(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$+\frac{1}{2\pi}\left|\int_{\partial\Delta}\frac{z_k f(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

Let $0<\varepsilon<1/2$ be given. Choose k large so that

$$\frac{1}{A}\sum_{n=k+1}^{\infty}\frac{r_n}{|c_n|-r_n}<\varepsilon \text{ and } |z_k|<\varepsilon \quad \forall n\geq k.$$

Let $D_\varepsilon = D\cup\bigcup_{n=1}^{k-1}\Lambda_n$, let $\phi_{o\varepsilon}=\phi_o\Big|_{H^\infty(D)}$.

If $f\in H^\infty(D_\varepsilon)$, $\varepsilon\,\|f\|\leq 1$ then $f\in H^\infty(D)$, $\|f\|\leq 1$ and

$$\int_{\partial\Delta_n}\frac{f(\zeta)}{\zeta-z_k}\,d\zeta = 0 \text{ for } n<k.$$

Now for $n\geq k$ we have

$$|f(z_n)-\phi_{o\varepsilon}(f)|\leq\frac{1}{2\pi}\left|\sum_{m=k+1}^{\infty}\int_{\partial\Delta_m}\frac{z_n f(\zeta)}{\zeta(\zeta-z_n)}\,d\zeta\right|$$

$$+|(P_nf)(z_n)-(P_nf)(0)|+\frac{1}{2\pi}\left|\int_{\partial\Delta}\frac{z_n f(\zeta)}{\zeta(\zeta-z_n)}\,d\zeta\right|$$

$$\leq\sum_{\substack{m=k+1\\ m\neq n}}^{\infty}\frac{|z_n|}{d(z_n,\Delta_m)}\;\frac{\|f\|\;r_m}{|c_m|-r_m}$$

$$+|(P_nf)(z_n)-(P_nf)(0)|+\frac{|z_n|}{1-\varepsilon}$$

$$\leq\frac{1}{A}\sum_{m=k+1}^{\infty}\frac{r_m}{|c_m|-r_m}+\frac{\varepsilon}{1-\varepsilon}+|(P_nf)(z_n)-(P_nf)(0)|$$

$$\leqq 3\varepsilon+|(P_nf)(z_n)-(P_nf)(0)|,$$

so $\overline{\lim_n}|f(z_n)-\phi_{o\varepsilon}(f)|\leq\overline{\lim_n}|(P_nf)(z_n)-(P_nf)(0)|+3\varepsilon$

Let $\phi\in\overline{\{z_n\}}\cap M_o(D)$, then [3] $\phi_\varepsilon=\phi\Big|H^\infty(D_\varepsilon)$ is in $\overline{\{z_n\}}\cap M_o(D_\varepsilon)$ and $\|\phi-\phi_o\|=\|\phi_\varepsilon-\phi_{o\varepsilon}\|$.

For any $f\in H^\infty(D_\varepsilon)$ with $\|f\|\leq 1$, $\exists$ a sub-sequence $\{z_{k_j}\}$ of $\{z_k\}$ such that

$$\lim_{j\to\infty}f(z_{n_j})=\phi_\varepsilon(f),$$

$$|\phi_\varepsilon(f)-\phi_{o\varepsilon}(f)|=\lim_{j\to\infty}|f(z_{n_j})-\phi_{o\varepsilon}(f)|$$

$$\leq\overline{\lim_k}|f(z_k)-\phi_{o\varepsilon}(f)|$$

$$\leq\overline{\lim_k}|(P_kf)(z_k)-(P_kf)(0)|+3\varepsilon$$

If ε is small enough, there exists $a>0$ such that

$|\phi_\varepsilon(f)-\phi_{o\varepsilon}(f)|\leq 2-a$;

thus, $\|\phi_\varepsilon-\phi_{o\varepsilon}\|<2\Longrightarrow\|\phi-\phi_o\|<2\Longrightarrow$

$\phi\in P(\phi_o)$.

Corollary 2.1: If $\{z_n\}$ is a sequence in D which satisfies

1. $\exists\ A>0$ such that for each k, $d(z_k, \Delta_n) \geqq A|z_k|$, for all $n \neq k$,
2. $d(z_k, \Delta_k) \geqslant Br_k$, $B>1$, for all k,
3. $|z_k| < C|c_k|$, $C<2$, for all k,

then $\overline{\{z_k\}} \subset P(\phi_o)$

Proof: Let $f \in H^\infty(D)$, $\|f\| \leq 1$, for any k,

$$|(P_kf)(z_k)-(P_kf)(0)| = \left|\frac{1}{2\pi i}\int_{\partial\Delta_k}\left(\frac{f(\zeta)}{\zeta - z_k} - \frac{f(\zeta)}{\zeta}\right)d\zeta\right|$$

$$\leq \frac{1}{2\pi}\left|\int_{\partial\Delta_k}\frac{z_kf(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$\leq \frac{1}{2\pi}\ \frac{|z_k|}{d(z_k, \Delta_k)}\ \frac{2\pi r_k}{|c_k|-r_k}\ \|f\|$$

$$\leq \frac{C}{B}\ \frac{|c_k|}{|c_k|-r_k}.$$

So $\overline{\lim_k}|(P_kf)(z_k)-(P_kf)(0)| \leq \frac{C}{B} < 2.$

Applying the theorem now we get the required result.

Proof of theorem 2: Let $0 < \varepsilon < 1$ be given, choose N so that

$$\sum_{m>N}\frac{r_m}{|c_m|-r_m} < \varepsilon \text{ and } |z_k| < \varepsilon \text{ for } k \geqq N$$

$\exists f \in H^\infty(D)$, $\|f\| \leq 1$, such that

$\lim_k |(P_kf)(z_k)-(P_kf)(0)| > 2-\varepsilon$. Fix $k>N$,

$$|f(z_k)-\phi_o(f) + (P_kf)(z_k)-(P_kf)(0)|$$

$$\leqq \left|\frac{1}{2\pi i}\sum_{\substack{n=1\\ n\neq k}}^{\infty}\int_{\partial\Delta_n}\frac{f(\zeta)}{\zeta-z_k} - \frac{f(\zeta)}{\zeta}\,d\zeta\right|$$

$$+\left|\frac{1}{2\pi i}\int_{\partial\Delta}\frac{z_kf(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$\leq \left|\sum_{n=1}^{N}(P_nf)(z_k)-(P_nf)(0)\right|$$

$$+\left|\frac{1}{2\pi i}\sum_{\substack{n=N+1\\ n\neq k}}^{\infty}\int_{\partial\Delta_n}\frac{z_kf(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$+\left|\frac{1}{2\pi i}\int_{\partial\Delta}\frac{z_kf(\zeta)}{\zeta(\zeta-z_k)}\,d\zeta\right|$$

$$\leq \left|\sum_{n=1}^{N}(P_nf)(z_n)-(P_nf)(0)\right|$$

$$+\frac{1}{2\pi}\sum_{\substack{n=N+1\\ n\neq 1}}^{N}\frac{|z_k|\,2\pi}{d(z_k, \Delta_n)}\ \frac{r_n}{|c_n|-r_n}$$

$$+\frac{|z_k|}{1-\varepsilon}$$

or $|f(z_k)-\phi_o(f)+(P_kf)(z_k)-(P_kf)(0)|$

$$\leqq \left|\sum_{n=1}^{N}(P_nf)(z_k)-(P_nf)(0)\right|+3\varepsilon\ \cdot$$

Since $\sum_{n=1}^{N} P_nf - (P_nf)(0)$ is continuous at 0 and vanishes there, $\exists\delta$ such that if $|z| < \delta$ then

$$\left|\sum_{n=1}^{N}(P_nf)(z)-(P_nf)(0)\right| < \varepsilon\ .$$

Choosing $K > N$ so that if $k \geqq K$ then $|z_k| < \delta$, we get for any such k

$|f(z_k)-\phi_o(f)+(P_kf)(z_k)-(P_kf)(0)| \leq 4\varepsilon$

or $|f(z_k)-\phi_o(f)| + 4\varepsilon \geqq |(P_kf)(z_k)-(P_kf)(0)|$,

so $\lim_k |f(z_k)-\phi_o(f)| + 4\varepsilon \geqq \lim|(P_kf)(z_k)-(P_kf)(0)|$

$> 2-\varepsilon$.

If $\phi \in \overline{\{z_n\}}^*$ then $\exists$ a subsequence $\{z_{n_j}\}$ such that

$$\phi(f) = \lim_{j\to\infty} f(z_{n_j}),$$

so $|\phi(f)-\phi_o(f)| = \lim_{j\to\infty}|f(z_{n_j})-\phi(_of)| \geqq \lim_n |f(z_n)-\phi_o(f)|$

$$\geqq 2-5\varepsilon$$

Hence $\|\phi-\phi_o\| = 2$ so $\phi \notin P(\phi_o)$.

Corollary 2.2: if $\{z_n\}$ is a sequence in D which converges to 0 and satisfies

1. for each k, $d(z_k, \Delta_n) \geqslant A \mid z_k \mid$ for some $A>0$ and all $n \neq k$

2. given $\varepsilon > 0 \ \exists$ f such that $f(\phi_o) = 0$ and

$$\lim_k |(P_k f)(z_k) - (P_k f)(0)| > 1 - \varepsilon \text{ then}$$

$$\overline{\{z_k\}}^* \bigcap P(\phi_o) = \phi.$$

Proof: Let $\varepsilon > 0$ be given, as in the proof of theorem 2.2 we get

$$| f(z_k) | \geqslant | (P_k f)(z_k) - (P_k f)(0) | - 4\varepsilon ,$$

so $\lim_k | f(z_k) | \geqq 1 - 5\varepsilon$

If $\phi \in \overline{\{z_k\}}^*$, then $\exists$ a subsequence $\{z_{n_j}\}$ such that

$$\phi(f) = \lim_{j\to\infty} f(z_{n_j}).$$

Now $|\phi(f)| = \lim_{j\to\infty} | f(z_{nj}) | \geqslant \lim_n |f(z_n)| > 1 - \varepsilon$,

so $\rho(\phi, \phi_o) = 1$ and hence $\phi \notin P(\phi_o)$.

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