A VECTORIAL CALCULATION OF THE DIRECTION OF QIBLA

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Historically the determination of the direction of Mecca provided a great impetus for the development of mathematics among the Muslims. For a spherical Earth the problem was solved by spherical trigonometry[1] essentially developed by the Arabs[2]. It is of some interest to see how to achieve the same result with the more economical methods of vector calculus. For this purpose it will be assumed that the Earth is of unit radius. Then any location on its surface can be specified [3] by a unit radial vector.

$$\vec{u}_o = (\cos l \cos L, \sin i \cos L, \sin L) \quad (1)$$

$$\text{where} \begin{cases} L = \text{latitude of location} \\ l = \text{longitude of location} \end{cases}$$

with West longitudes positive and East longitudes negative.

At this location (l,L) the unit vectors in the North, South, East, West directions are given by

$$\vec{u}_N = (-\cos l \sin L, -\sin l \sin L, \cos L) = -\vec{u}_S \quad (2)$$

$$\vec{u}_E = (\sin l, \cos l, 0) = -\vec{u}_W$$

The infinitesimal vectorial difference between two locations is then given by

$$d\vec{u}_o = (-\sin l \cos L, \cos l \cos L, 0)\, dl + (-\cos l \sin L, -\sin l \sin L, \cos L)\, dL$$

or

$$d\vec{u}_o = \vec{u}_W \cos L\, dl + \vec{u}_W\, dL \quad (3)$$

so that

$$|d\vec{u}_o| = \sqrt{(\cos^2 L\, dl^2 + dL^2)} \quad (4)$$

and

$$\frac{d\vec{u}_o}{|d\vec{u}_o|} = \frac{\vec{u}_W \cos L\, dl + \vec{u}_N\, dL}{\sqrt{(\cos^2 L\, dl^2 + dL^2)}} \quad (5)$$

is a unit vector in the direction of $d\vec{u}_o$.

If $d\vec{u}_o$ represents a differential displacement vector along the Great Circle course from (l,L) to (l_M, L_M) the location of Mecca, then $d\vec{u}_o$ must be coplanar with the two unit radius vectors for the observer location and the location of Mecca $\vec{u}_M$.

$$\vec{u}_M = (\cos_M l \cos L_M, \sin l_M \cos L_M, \sin L_M) \quad (6)$$

Thus

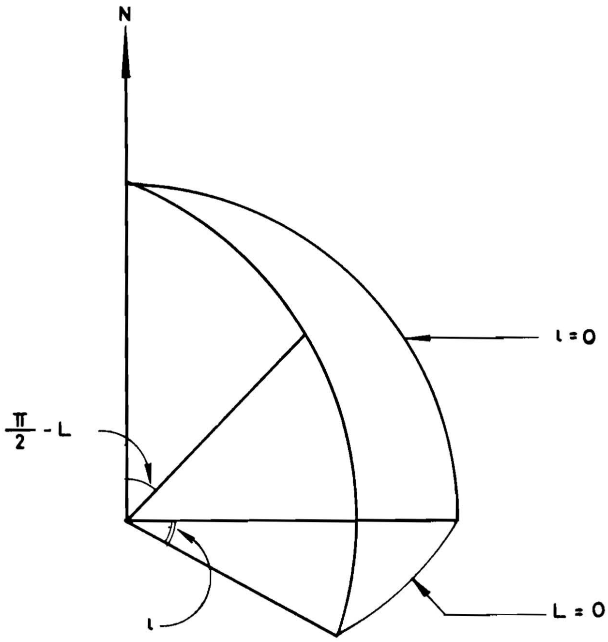
$$d\vec{u}_o \cdot \vec{u}_o \times \vec{u}_M = 0 \quad (7)$$

so that substitution of (3) yields

$$\vec{u}_o \times \vec{u}_M \cdot \vec{u}_W \cos L\, dl + \vec{u}_o \times \vec{u}_M \cdot \vec{u}_N\, dl = 0$$

or

$$\frac{-dl}{dL} \cos L = \frac{\vec{u}_o \times \vec{u} \cdot \vec{u}_N}{\vec{u}_o \times \vec{u} \cdot \vec{u}_W} \quad (8)$$



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The cosine of the bearing θ of Mecca from the observer location $\vec{u}_o$ is then given by the scalar product of (5) and $\vec{u}_N$.

$$\cos\theta = \frac{dL}{\sqrt{(\cos^2 L\, dl^2 + dL^2)}} = \frac{1}{\sqrt{\{[\cos L\,(dl/dL)]^2 + 1\}}} \tag{9}$$

or in terms of tan θ by

$$\tan\theta = -\frac{dl}{dL}\cos L = \frac{\vec{u}_o \times \vec{u}_M \cdot \vec{u}_N}{\vec{u}_o \times \vec{u}_M \cdot \vec{u}_W} \tag{10}$$

upon substituting from (8). Thus the bearing of Mecca may be expressed by the determinantal ratio

$$\tan\theta = \frac{\begin{vmatrix} \cos l\ \cos L & \sin l\ \cos L & \sin L \\ \cos l_M \cos L_M & \sin l_M\ \cos L_M & \sin\ L_M \\ -\cos l\ \ \sin L & -\sin l \sin L & \cos L \end{vmatrix}}{\begin{vmatrix} \cos l\ \ \cos L & \sin l\ \ \cos L & \sin L \\ \cos l_M \cos L_M & \sin l_M\ \cos L_M & \sin L_M \\ -\sin\ l & \cos l & 0 \end{vmatrix}} \tag{11}$$

This, of course, does not hold at Mecca nor its antipode. Incidentally any point (l′, L′) on the Great Circle course between the observer location and Mecca is seen to satisfy the following relationship [4]:

$$\begin{vmatrix} \cos l\ \ \cos L & \sin\ l\ \cos\ L & \sin L \\ \cos l_N \cos L_N & \sin l_N \cos L_N & \sin L_N \\ \cos l'\ \cos L' & \sin l'\ \cos L' & \sin L' \end{vmatrix} = 0 \tag{12}$$

For the case of a non-spherical Earth the determination of a tangent at observer location to a geodesic connecting observer location with Mecca would follow from the extremalization of $\int$ ds with

$$ds^2 = [\,(1+\varepsilon)^2 \cos^2 L + (\partial_l\ \varepsilon)^2]\, dl^2 +$$

$$2(\partial_L\, \varepsilon)(\partial_l\, \varepsilon)\ dLdl + [\,(1+\varepsilon)^2 + (\partial_L\, \varepsilon)^2\ {}^2dL$$

and

$$\varepsilon = \varepsilon\ (l,L),$$

the discrepancy from unit radius at (l,L) so that the (non-unit) radius vector R at (l,L) is

$$R = (1+\varepsilon)\vec{u}_o.$$

The details of this calculation are left to the reader.

As mentioned above the original problem was solved by spherical trigonometry. Because Arab knowledge retained individual steps a didactic reasoning is presumed. In any case it is not inconceivable that the corresponding problem for a non-spherical Earth could once again stimulate the development of Muslim mathematics.

REFERENCES

[1] Carra de Vaux, *Legacy of Islam.* London: Oxford University Press, 1931, p. 396.

[2] Nasir Al-Din Tusi, *Treatise on the Quadrilateral* (translated by Caratheodory Pasha). Constantinople: 1891.

[3] A. Kyrala, *Theoretical Physics.* Philadelphia; Saunders, 1967, p. 13, Ex. 4.

[4] *Ibid.*, p. 14, Ex. 6.