THE MÖBIUS FUNCTION AND FINITE NETWORKS

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الخلاصـــة

ان ايضاح وتعريف دالة موبيس الخاصة بالنظرية العددية التقليدية قد امتدت الآن إلى مجموعات محددة ذات ترتيب جزئي .

ومن خلال هذا البحث يمتد تعريف دالة موبيس إلى أبعد من ذلك حيث يتناول الشبكات المحددة والمتناهية محلياً . ومن خلال هذا الامتداد والسياق تتواجد مجموعة من علوم الجبر المعقدة تتكافىء وتتساوى مع هذه الشبكة .

ABSTRACT

The definition of the Möbius function of classical number theory has been extended to locally finite partially ordered sets. In this paper, the definition of the Möbius function is extended further to locally finite networks. In the course of this extension, a family of convolution algebras is found which is equivalent to the network.

The definition of the classical Möbius function has been extended to partially ordered sets. See for example the important 1964 paper by Gian-Carlo Rota (1), or the 1969 expository paper by Robin Wilson (2).

The purpose of this paper is to extend further the definition of the Möbius function to locally finite networks. In the course of this extension we define a family of algebras equivalent to a network and make an analogy corresponding integration to a type of union of certain digraphs. This paper is restricted to locally finite networks in order to emphasize the graph theoretic detail and minimize the analytic difficulties. Later work will attack the analytic problems.

Section 1: Definitions and Examples

In this paper, a *graph* $\langle V, E\rangle$ consists of a non-empty set V, called the *vertices*, together with a set E of unordered pairs of vertices (x, y), called *edges*. Thus loops will be permitted, but never multiple or parallel edges, A *digraph* (directed graph) $\langle V, A\rangle$ consists of a set $V \neq \phi$, together with a set A of ordered pairs (x,y) of vertices, called *arcs*.

A *network* $N = \langle V, E, p\rangle$ is a graph $\langle V, E\rangle$ together with a realvalued function p defined on V. Call p, the *potential* function of N. The theory presented here requires only trivial modifications, if a network is defined to be a graph together with $F{:}E \rightarrow Re$, a *flow*.

We now associate a family of partially ordered sets with the network $N = \langle V, E, p\rangle$. For $v \geqslant 0$, the *v-poset of* N will be defined as the digraph v-po(N) = $\langle V, \leqq_v\rangle$ where $x \leqq_v y$ iff $x = y$, or if x and y are adjacent $\{x, y\} \in E\times E$) and $v + p(x) < p(y)$. Easily, if $v \geqslant 0$, then v-po(N) is indeed a poset

The *isopotential graph of* N will be defined as the graph isop (N) = $\langle V, I\rangle$ where $\{x, y\} \in I$ iff x and y are adjacent in N and $p(x) = p(y)$.

Example 1.

As an imprecise, but intuitive example of the two definitions above, consider the surface of the earth as a network with the potential $p(x)$ equal to the height of point x above sea level. The isopotential graph consists of all curves of constant elevation, a topographic map of the earth. The v-posets contain all curves of directional derivative greater than v.

A reconstruction of the Earth's surface is possible from the usual topographic map because the curves of constant elevation are labelled with their elevation. But the isopotential graph has no such potential indicated upon it; it consists solely of curves. However, we shall see that the network edges and potential are both recoverable (up to an additive constant) from the v-posets and the isopotential graph.

In order to concentrate on the graph theoretic detail, rather than analytic detail, we begin the defi-

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nition of, then restriction to, locally finite networks.

If $\langle V, E\rangle$ is a graph, then define a *chain* of $\langle V,E\rangle$ to be an alternating sequence $(\ldots, e_{i-1}, x_i, e_i, x_{i+1},\ldots)$ such that each vertex x_i is adjacent to x_{i+1} by the edge e_i, and such that no vertex nor edge is repeated in the sequence. The *length* of a chain is defined to be the number of its edges.

If $\langle V, A\rangle$ is a digraph, then define a *chain from* x_1 *to* x_n to be an alternating sequence $(x_1, a_1, x_2,\ldots, a_{1-n}, x_n)$ such that each arc a_i is oriented from its end vertex x_i to x_{i+1}, and such that no vertex nor arc is repeated in the sequence. If no confusion results, mention of the edges (or arcs) of a chain will be frequently supressed.

If $N=\langle V, E, p\rangle$ is a network, then given vertices $x, z \in V$ and $v \geqq 0$, the *closed* v-*interval* $[x, z]_v$ is defined by $[x, z]_v = \{y \in V \mid x \leqq_v y \leqq_v z\}$. Since there may be more than one chain from x to z, $[x, z]_v$ need not be a chain.

The network N will be called *locally finite* if for all $x, z \in V$, and all $v>0$, the three sets ; $[x, z]_v$, the set of all vertices adjacent to x, and the edges I of the isopotential graph $\langle V, I\rangle$ are all finite. Therefore, if N is a locally finite network, then all of the v-posets of N are locally finite posets.

Henceforth, all networks considered in this paper will be locally finite and the term "network" will be redefined to mean "locally finite network".

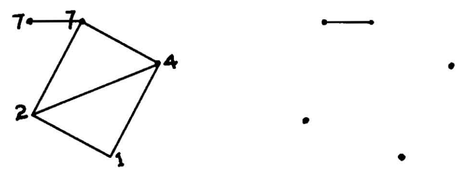
Following standard procedure, for $v \leqslant 0$, the *v-incidence algebra of* N, v-alg(N), is defined as the incidence algebra of v-po(N). The Möbius function of v-alg(N) will be denoted by μ_v.

Example 2:

Let $\vec{Z}+$ be the digraph consisting of the positive integers ordered by divisibility. Then Z+ is the graph naturally associated with $\vec{Z}+$. Note that Z+ is a direct product of the chains $1 \mid p \mid p^2 \mid \ldots$, for p a prime. Regard p as the coordinatizing function $p(n) = n$. Thus μ_v is the classical Möbius function for $0 \leqq v < 1$.

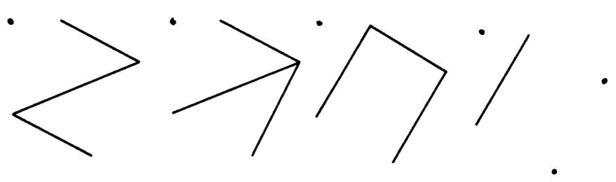
Example 3:

The numbers beside the vertices of the left-hand graph indicate their potential.



network N isopotential graph $\langle V,I\rangle$

The cover graphs below indicate the various v-po(N). A decending line indicates a cover relation.



$0 \leqq v < 1$ $1 \leqq v < 2$ $2 \leqq v < 3$ $3 \leqq v < 5$

For $5 \leqq v$, v-po(N) is discrete.

There are several things to notice in this example. First, the union of these cover graphs and the isopotential graph gives the edges of the original network. Second, given any one edge, e, the set co(e) of non-negative v such that e is in the cover graph of v-po(N) is an interval closed on the left, open on the right. Third, the least upper bound of co(e) is the potential difference between the two end vertices of e.

These observations are true in general, and provide a means of recovering a network from its collection of v-posets, hence of recovering a network from its collection of incidence algebras.

Given a network N, denote the cover graph of v-po(N) by $\text{v-co}(N)=\langle V, C_v\rangle$.

Section 2: Families of Algebras Equivalent to a Network.

We shall see that the collection of the v-cover graphs of N (or equivalently, the collection of the v-po (N)), together with the isopotential graph characterizes the network N.

If $N=\langle V, E, p\rangle$ is a network, let E be the edges of E oriented from higher potential to lower. It the end vertices of $e \in E$ have the same potential then e remains unoriented. Thus $\langle V, E\rangle$ is the union of a digraph and the isopotential graph. Regard an edge

$e \in E$ as an arc by orienting e in both directions.

If ch(x,y) is a chain of E from y to x, then define m ch(x,y) as the minimum of $|p(b) - p(a)|$ where b and a vary over all pairs of adjacent vertices of ch(x,y). Since N is locally finite, m ch(x,y) is finite. For $x, y \in V$, let [x,y] be the union of all chains from y to x. If (x, e, y) is a chain from y to x, then define Mm[x,y] = l.u.b.{ m ch | ch $\neq$ (x, e, y) is a chain of E from y to x). Since N is locally finite, Mm [x,y] $< \infty$.

The following gives an arithmetic characterization of the statement that a is an arc of the cover graph v-co(N).

Lemma:

If $N = \langle V, E, p)$ is a network and if (x, e, y) is a chain of E from t to x, $p(x) < p(y)$, then co(e) is the half closed, half open, interval [Mm[x,y], $p(y) - p(x)$). If $p(y) = p(x)$, then the interval co(e) is singleton {0}.

Proof:

If $p(x) = p(y)$, then the lemma is obvious. If $p(x) < p(y)$, and if ch $\neq$(x, a, y) is a chain from y to x, then m ch $\leqslant p(y) - p(x)$. Hence Mm[x,y] $\leqslant p(y) - p(x)$. For all v such that Mm [x,y] $\leqslant v \leqslant p(y) - p(x)$, we have $a \in C_V$, since clearly a is an arc of v-po(N) and if y does not cover x by a, then v < Mm[x, y]. Hence the half open interval of the lemma is a subset of co(e).

The reverse inclusion follows by assuming $v \in$ co(e). Then $v < p(y) - p(x)$ is clear. If v < Mm[x,y], then there is a chain ch $\neq$ (x, a, y) of $\vec{E}$ from y to x such that $v < |p(w) - p(z)|$ for adjacent vertices w, z of ch. But the existence of such a chain ch denies that y covers x by a.

Theorem:

Let $\mathcal{C}$ be an indexed collection containing some cover digraphs and exactly one finite graph I. There exists a network $N = \langle V, E, p\rangle$ such that $\mathcal{C}$ is the collection of cover graphs of N and the isopotential graph of N if and only if each of the following hold:

1. Each cover digraph of $\mathcal{C}$ and the finite graph $I \in \mathcal{C}$ has the same set of vertices V,
2. The cover digraphs are indexed with $v \geqslant 0$,
3. If the vertex y covers x in some cover digraph in $\mathcal{C}$, then for all of the cover digraphs in $\mathcal{C}$, x never covers y,
4. Denoting the cover digraphs of C by $\langle C, C_V\rangle$, the set of edges of $I \in C$ is disjoint from the graph associated with the digraph $\langle V, \bigcup_{v \geqslant 0} C_V\rangle$,
5. For $a \in C_v$, recall the definition of co(a) = $\{v \geqslant 0 \mid a \in C_V\}$, then co(a) must be an interval of the form $(u_a, v_a]$,
6. $\bigcup \{co(a) \mid a \in C_v, \ v \geqslant 0\} = [0, b)$ for some $b < \infty$, and
7. Kirchoff's law on the sum of voltage differences around a circuit must apply to $\langle V, \bigcup \mathcal{C}\rangle$. More precisely, for every circuit $C \subset \bigcup \mathcal{C}$, we have $\sum_{a \in c} +v_a = 0$ where if $a \in C$ is traversed with its orientation, $+ v_a$ is taken; $-v_a$ otherwise.

Before proving this theorem we state a corollary. If we begin with a network $N = \langle V, E, p\rangle$ and let $\mathcal{C} = \{$v-co(N), $\langle V, I\rangle \mid \geqslant 0\}$, then apply the construction of the above theorem to C we obtain a network $N' = \langle V', E', p')$. The corollary states a type of uniqueness in the fact that $V' = V$, $\vec{E}' = E$ and p' differs from p by a real-valued function c which is constant over any connected component K. Therefore if N is connected, then the edges and potential of N can be recovered from the collection of v-cover graphs of N and the isopotential graph of N, up to an additive constant on p.

Corollary:

Let $N = \langle V, E, p)$ be a network and let $\mathcal{C}$ be the collection of the isopotential graph $\langle V, I\rangle$ and all the v-cover graphs of N. Then:

1. The graph $\langle V, E\rangle$ is the undirected graph associated with the mixed directed and undirected graph $\langle V, \bigcup \mathcal{C}\rangle$. Further, $\vec{E} = \bigcup \mathcal{C}$.
2. Let K be any connected component of $\langle V, E\rangle$, then define $p' : K \to Re$ by the following inductive procedure:

   Select any vertex $x_1 \in K$ and assign $p'(x_1) = c$ where c is any constant. Now, if $p'(x_i)$ is defined, let x_{i+1} be any vertex adjacent to x_i by an arc a of $\bigcup \mathcal{C}$. Then $|p'(x_{i+1}) - p'(x_i)|$ is the least upper bound of co(a) where the orientation of the arc indicates which potential of these vertices is higher. If a is an edge of $\bigcup \mathcal{C}$, then define $p'(x_{i+1}) = p'(x_i)$. This defines p'. Claim: $p - p'$ is constant on K.

Proof of the theorem:

If a network is given, the necessity stated in the theorem is straight-forward. To prove the sufficiency, let an indexed collection $\mathcal{C}$ with the required properties be given. We must construct a network $N = \langle V', E', p' \rangle$. Take $V' = V$ and define (V, E') to be the graph associated with the mixed directed and undirected graph $\cup\mathcal{C}$. Define p' by the inductive procedure of part 2 of the corollary taking $|p'(x_{i+1}) - p'(x_i)| = v_a$ of condition 5. Condition 7 of the theorem is exactly what is needed to show that p' is single-valued. Among other implications, condition 6 implies that $p'(x)$ is always finite . The network N is now constructed.

We must show that $\mathcal{C} = \{\text{v-co(N), isop (N)} \mid v \geqslant 0\}$. Let E be the edges of N oriented from higher potential p' to lower. First we establish $E = \cup\mathcal{C}$. Note $E \supset \cup\mathcal{C}$ is trivial. To see the reverse inclusion, let the edge $e \in E$ connect vertices x, y. If $p'(x) = p'(y)$, then $C \in$ isop (N), hence done. Therefore assume $p'(x) < p'(y)$. The lemma establishes the existence of $v \geqslant 0$ such that $c \in C_v \subset \cup\mathcal{C}$. Therefore $\vec{E} = \cup\mathcal{C}$.

Clearly the finite graph of $\mathcal{C}$ is isop (N). Now let v-co(N) be given. By condition 6 there is a cover digraph $C_v \in \mathcal{C}$. We need v-co(N) $= C_v$, but condition 5 guarantees this. Hence $\{\text{v-co(N), isop(N)} \mid v \geqslant 0\} \subset \mathcal{C}$. To obtain the reverse inclusion let $C_v \in \mathcal{C}$ be given. To see v-co(N) $= C_v$, consider any chain (x,a,y) of C_v from y to x. By condition 5, $v_a < p'(y) - p'(x)$. By the lemma then, (x,a,y) is a chain of v-co(N). Since $\vec{E} = \cup\mathcal{C}$, (x, a, y) is a chain of v-co(N) from y to x. Hence v-co(N) $\supset C_v$.

The reverse inclusion obtains from condition 5 and the definition of p', hence v-co(N) $= C_v$ and $\{\text{v-co(N), isop(N)} \mid v > 0\} = \mathcal{C}$, and this finishes the proof of the theorem.

In addition, the corollary has been proved above, except for the last claim of part 2. We show that for all $z \in K$, $p(z) - p(z) = p(x) - c$ (*)where $x = x_1$ was the initial vertex of the inductive procedure defining p'. Let ch [x, z] be the set of all chains with end vertices x and z. Define l (z) as the minimal length among all chains in ch(x, z). We induct on the range of l as z varies over K.

If $l(z) = 0$, then $x = z$ and (*) holds. Now suppose (*) holds for all minimal lengths, less than the minimal length n. Let $C \in$ ch[x,z] be any chain with this minimal length n and let $y \in C$ be adjacent to z. Then the induction hypothesis holds for the subchain of C with end vertices x and y. Hence $p(y) - p'(y) = p(x) - c$. Observe $p(z) - p(y) = p'(z) - p(y)$, and combine the two equations to obtain. (*).

We have found a network equivalent to some families of convolution algebras.

REFERENCES

(1) Gian-Carlo Rota, "On the foundations of combinatorial theory, I Theory of Möbius functions," *Zeits. für Wahrscheinlichkeitstheorie*, **2.** 1964, 340-368.

(2) Robin J. Wilson, *The Möbius Function in Combinatorial Mathematics; A contribution to Combinatorial Mathematics and its Applications*, edited D.J.A. Welsh. Academic Press (1969) 315-333.

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