CONSTITUTIVE EQUATIONS FOR WORK-HARDENING PLASTIC MATERIALS

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الخلاصة

هذا البحث يقترح مجموعة من المعادلات التكوينية خاصة بالمعادن في مرحلة المرونة – اللدونة المعرضة لدرجات كبيرة من التشويه .

هذه المعادلات عند كتابتها في النظام اللاجرانجي للإحداثيات سوف تشمل كل النماذج الحركية (Kinematic) والتصلد المحوري (Isotropic Hardening) . وقد أجريت تجربة على سبائك الألمونيوم للحصول على المتغيرات (parameters) الخاصة بهذه السبائك .

ABSTRACT

A system of constitutive equations applicable to elasto-plastic metals subjected to large deformations has been proposed. These equations, written in the material form, combine the kinematic and the isotropic hardening models. A method of experimental determination of the necessary material parameters has been demonstrated for the case of an aluminum alloy.

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1. INTRODUCTION

An extensive research effort has been devoted in recent years to the analysis of elasto-plastic deformations of various structural elements and solid bodies. Numerous successful solutions have been obtained, mainly with the use of the finite element method and electronic computers. A large majority of the published results deal with the small-deformation problems. (An excellent review of the basic approaches and a comprehensive bibliography of the subject can be found in [1, Ch. 18]). Although there are several important contributions to the problem of large elasto-plastic deformations [2-4], it appears that a thorough investigation of the implications of the assumption of arbitrarily large strains and rotations has not been made. In fact, even the foundations of the theory of plasticity at large strains have been formulated only recently and are not devoid of certain controversial aspects (see, for example [5, 6 and 7]).

This paper deals with the problem of large elasto-plastic deformations of materials which initially display linear elasticity followed, at increasing loading, by plastic strains with no rate, or viscous effects. This kind of mechanical behavior is typical for structural metals. The topics included in this paper are: (a) a system of constitutive equations; (b) an experimental procedure for the determination of a small number of material parameters.

The constitutive equations formulated in this paper have been selected for possible applications in two areas: large deformations of elastoplastic structures and - perhaps more demanding - mechanics of cold forming of metals.

2. PRELIMINARY DEFINITIONS AND RELATIONS

The following is a brief summary of the concepts of continuum mechanics which are used in this work. This section is based on the monograph by Truesdell and Toupin [8].

Displacement

Let B_o be the initial, or undeformed, state at $t=0$, and B the current, or deformed, state at some time t

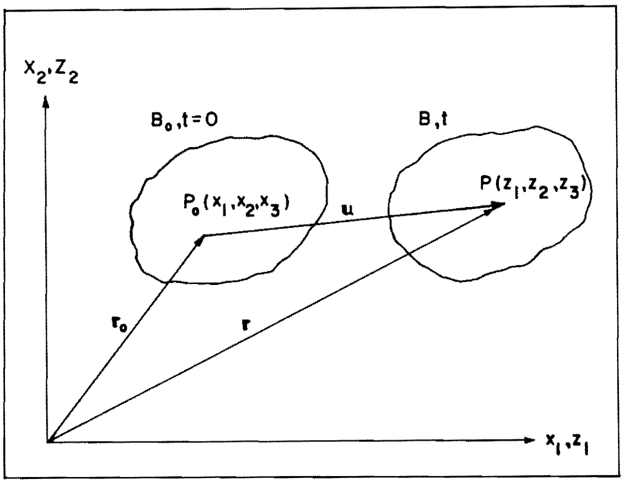

Figure 1. Coordinate systems and description of displacement.

(Figure 1). The displacement of the body is described by

$$z_k = z_k(x_1, x_2, x_3; t), \qquad k = 1,2,3 \tag{2.1}$$

$$\text{or } x_A = x_A(z_1, z_2, z_3; t), \qquad A = 1,2,3 \tag{2.2}$$

The usual assumptions of singlevaluedness and continuity with

$$0 < \det\left|\frac{\partial z_k}{\partial x_A}\right| \equiv J < \infty \tag{2.3}$$

are made with regard to Equations (2.1) and (2.2).

Under the conditions shown in Figure 1, the components of **u** are related to z_k and x_A by

$$u_1 = z_1 - x_1, \quad u_2 = z_2 - x_2, \quad u_3 = z_3 - x_3 \tag{2.4}$$

The material and spatial forms of the displacement vector are $u_A = u_A(x_1 x_2, x_3, t)$, $A = 1,2,3$, and $u_k = u_k(z_1, z_2, z_3, t)$, $k = 1,2,3$, respectively.

The velocity vector **v** is defined as

$$v_A = \frac{\partial u_A(\mathbf{x}, t)}{\partial t} \tag{2.5}$$

or, equivalently,

$$v_k = \frac{\partial z_k(\mathbf{x}, t)}{\partial t} \tag{2.6}$$

The expression (2.5) results in the material form of **v**, if in Equation (2.6), the variable **x** is replaced by **z** (using Equation 2.2), the spatial form of the velocity vector is obtained.

Strain

The *material strain tensor* is defined as

$$e_{AB}=\tfrac{1}{2}\left(\delta_{kl}\frac{\partial z_k}{\partial x_A}\frac{\partial z_l}{\partial x_B}-\delta_{AB}\right) \quad (2.7)$$

or, in terms of the displacement vector,

$$e_{AB}=\tfrac{1}{2}\left(\frac{\partial u_A}{\partial x_B}+\frac{\partial u_B}{\partial x_A}+\frac{\partial u_C}{\partial x_A}\frac{\partial u_C}{\partial x_B}\right) \quad (2.8)$$

The changes of length of a line segment dl_0 at P_0 can be computed as

$$dl^2-dl_0^2=2e_{AB}\,dx_A\,dx_B$$

or

$$\frac{dl^2-dl_0^2}{dl_0^2}=2e_{AB}\,l_{0A}\,l_{0B} \quad (2.9)$$

where l_{0A} are the components of the unit vector along dl_0.

The *spatial strain tensor* is defined as

$$h_{kl}=\tfrac{1}{2}\left(\delta_{kl}-\frac{\partial x_A}{\partial z_k}\frac{\partial x_B}{\partial z_l}\delta_{AB}\right) \quad (2.10)$$

$$\text{or } h_{kl}=\tfrac{1}{2}\left(\frac{\partial u_k}{\partial z_l}+\frac{\partial u_l}{\partial z_k}-\frac{\partial u_m}{\partial z_k}\frac{\partial u_m}{\partial z_l}\right) \quad (2.11)$$

The terms of h_{kl}, the length changes are

$$dl^2-dl_0^2=2h_{kl}\,dz_k\,dz_l \quad (2.12)$$

or

$$\frac{dl^2-dl_0^2}{dl^2}=2h_{kl}\,l_k\,l_l \quad (2.13)$$

where l_k are the components of the unit vector along **dl** at P.

The measure of extension defined as

$$\varepsilon=\frac{dl-dl_0}{dl_0} \quad (2.14)$$

is frequently used in describing the results of unixial testing of various materials. It can be related to the components of the material or spatial strain tensors. For example, for a line segment $\mathbf{dl_0}$ whose initial direction at P_0 was parallel to x_1,

$$\varepsilon=\sqrt{(1+2e_{11})}-1 \quad (2.15)$$

The *volumetric strain* dV/dV_0 is equal to the Jacobian determinant J defined in (2.3).

Strain rates

The *material strain-rate tensor* is defined as

$$\dot{e}_{AB}=\frac{\partial e_{AB}}{\partial t} \quad (2.16)$$

whence

$$\frac{D}{Dt}dl^2=2\dot{e}_{AB}\,dx_A\,dx_B \quad (2.17)$$

The *spatial strain-rate tensor* is defined as

$$d_{kl}=\tfrac{1}{2}\left(\frac{\partial v_k}{\partial z_l}+\frac{\partial v_l}{\partial z_k}\right) \quad (2.18)$$

whence

$$\frac{D}{D_t}dl^2=2d_{kl}\,dz_k\,dz_l \quad (2.19)$$

The condition of incompressibility has a simple form in terms of d_{kl}, namely,

$$d_{kk}=0 \quad (2.20)$$

The following relation exists between the material and the spatial strain-rate tensors:

$$\dot{e}_{AB}=d_{kl}\frac{\partial z_k}{\partial x_A}\frac{\partial z_l}{\partial x_B} \quad (2.21)$$

Stress

Let $\mathbf{t}_{(n)}$ denote the *stress vector*, or *surface traction*, acting on the area element at P with the unit normal vector **n** (note: $\mathbf{t}_{(n)}$ is force per unit area of the deformed body). In terms of the *spatial*, or *Cauchy*, *stress tensor* t_{kl}, the components of $\mathbf{t}_{(n)}$ are

$$t_{(n)l}=t_{kl}n_k \quad (2.22)$$

In this work, the *material*, or *Piola-Kirchhoff*, *stress tensor* will be used. Its definition is

$$s_{AB}=t_{kl}\,J\,\frac{\partial x_A}{\partial z_k}\frac{\partial x_B}{\partial z_l} \quad (2.23)$$

The components of the stress vector $\mathbf{p}_{(n)}$ defined as the surface force per unit area in the undeformed body, i.e.

$$\mathbf{p}_{(n)}=\mathbf{t}_{(n)}\frac{dA}{dA_0}$$

can be expressed in terms of t_{kl} or S_{AB} in the following manner

$$p_{(n)l}=t_{kl}\,J\,\frac{\partial x_A\,n_{0A}}{\partial z_k} \quad (2.24)$$

$$=s_{AB}\frac{\partial z_l}{\partial x_B}n_{0A}$$

It is understood that $\mathbf{p}_{(n)}$ acts on an area element whose unit normal in the initial state is $\mathbf{n}_0$.

Stress-rates

In terms of the material stress tensor S_{AB}, an objective stress-rate tensor is

$$\dot{s}_{AB} = \frac{\partial s_{AB}}{\partial t} \tag{2.25}$$

provided S_{AB} in (2.25) is given as a function of the material coordinates x and time t.

Principle of Virtual Work

Let f_A be the components of the body forces per unit volume of the undeformed body, and p_A the components of the surface forces per unit area of the undeformed body. The condition of equilibrium can be written in the form of the following Principle of Virtual Work (material form):

$$\int_V s_{AB}\, \delta e_{AB} dV - \int_V f_A\, \delta u_A\, dV - \int_S P_A\, \delta u_A\, dS = 0 \tag{2.26}$$

where δu_A are the virtual displacements and δe_{AB} the corresponding variations of the material strain tensor. In (2.26), V is the volume and S the bounding surface of B_0.

Matrix notation

In addition to the indicial notation used above, the matrix notation appears to be helpful in some applications. The displacement, body force, and surface forces matrices are column matrices , 3 x 1, or vectors with the components:

$$\begin{aligned} \mathbf{u}^T &\equiv (u_1,u_2,u_3) \equiv (u,v,w) \\ \mathbf{f}^T &\equiv (f_1,f_2,f_3) \\ \mathbf{p}^T &\equiv (p_1,p_2,p_3) \end{aligned} \tag{2.27}$$

The strain matrix is the 6 x 1 column matrix

$$\mathbf{e}^T \equiv (e_{11}, e_{22}, e_{33}, 2e_{12}, 2e_{23}, 2e_{31}) \tag{2.28}$$

and, similarly, the stress matrix.

$$\mathbf{s}^T \equiv (s_{11}, s_{22}, s_{33}, s_{12}, s_{23}, s_{31}) \tag{2.29}$$

Note that the expressions $S_{AB}\ \delta e_{AB}$, $f_A\ \delta u_A$, etc., read now:

$\delta \mathbf{e}^T \mathbf{s} \equiv \mathbf{s}^T \delta \mathbf{e}$, $\delta \mathbf{u}^T\, \mathbf{f} \equiv \mathbf{f}^T\, \delta \mathbf{u}$ etc.

3. CONSTITUTIVE EQUATIONS

The constitutive equations used in this work are based on the general theory of plasticity at large strain presented in [5].

The material strain tensor e_{AB} will be decomposed into two parts: e'_{AB} which will be called the elastic strain and e''_{AB}, the plastic strain. Thus,

$$e_{AB} = e'_{AB} + e''_{AB} \tag{3.1}$$

The assumption is made that e'_{AB} and e''_{AB} have the same invariance properties as e_{AB}; however, only e_{AB} corresponds to a continuous displacement field.

For the elastic strain e'_{AB}, it is postulated that it is a linear function of stress, i.e.,

$$s_{AB} = E_{ABCD}\ e'_{CD} \tag{3.2}$$

or in the matrix rotation

$$\mathbf{s} = \mathbf{E}\, \mathbf{e}' \tag{3.3}$$

where **E** is a 6 x 6 symmetric matrix. The form of the elastic moduli matrix is well known for isotropic materials.

The plastic behavior of the material is specified in the following manner. The existence of the yield surface in the form.

$$f(s_{AB}, e''_{AB}, \varkappa) = 0 \tag{3.4}$$

is postulated. The function $f(s_{AB}, e''_{AB}, \varkappa)$ is the *loading* or *yield function*, while $\varkappa$ is the *hardening parameter*, which is a functional of plastic strain. Furthermore, if

$$f = 0 \text{ and } \frac{\partial f}{\partial s_{AB}}\ \dot{s}_{AB} > 0 \tag{3.5}$$

then $e''_{AB} \neq 0$ and $\varkappa \neq 0$ and it is assumed

$$e''_{AB} = e''_{AB}\,(s_{CD}, s_{CD}, e''_{CD}) \tag{3.6}$$

If

$$f = 0 \text{ and } \frac{\partial f}{\partial s_{AB}}\ \dot{s}_{AB} < 0 \tag{3.7}$$

or, if

$$f < 0 \tag{3.8}$$

then

$$\dot{e}''_{AB} = 0,\ \dot{\varkappa} = 0 \tag{3.9}$$

In selecting the form of the function f, the parameter $\varkappa$, and the relation (3.6), certain generally recognized facts of the plastic deformation of metals, especially such as low-carbon steel and aluminum should be utilized. They appear to be the following:

1. The Mises yield condition and the associated flow rule are satisfactory forms of (3.4) and (3.6) in the small deformation plasticity theory of metals.
2. The hydrostatic state of stress has no effect on the plastic behavior of metals even at large strains.
3. There are no plastic volumetric strains.

Certain difficulties exist in establishing a model for the changes in the yield surface caused by past histories of plastic strains. Reference [9] contains an exhaustive review and bibliography of 131 works on this subject. In addition to the known experimental problems connected with the generation of complex states of stress and the measurement of the corresponding strains, the results are extremely sensitive to the particular way of defining the yield point and, for a given material, they depend on the mechanical and thermal treatment of the specimens.

A systematic study of the subsequent yield surfaces is a serious research topic for itself and clearly beyond the scope of this work. Also, an important constraint against too complicated material relations is their adaptability in a workable computing scheme. For these reasons, in formulating the yield function, a simple hypothesis involving a possibly small number of material parameters has been assumed. It allows for a relatively simple determination of the material parameters from a uniaxial tension-compression test.

Two frequently discussed models of subsequent yield surfaces have been considered. They are the

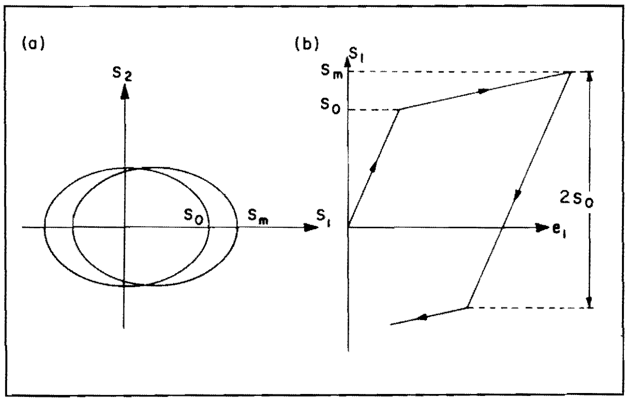

Figure 2. Kinematic hardening: (a) changes of the yield surface under uniaxial stress; (b) uniaxial stress-strain relation.

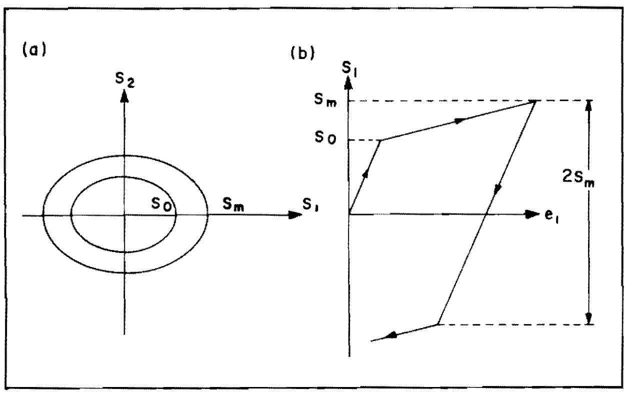

Figure 3. Isotropic hardening: (a) changes of the yield surface under uniaxial stress; (b) uniaxial stress-strain relation.

kinematic hardening model, shown in Figure 2, and the *isotropic hardening* model, shown in Figure 3. The proposed hypothesis, expressed by Equation (3.10) contains both these models. The kinematic hardening, is controlled by the constant c_1, and the isotropic hardening by c_2.

The experimental data obtained in this research seem to indicate, however, that for large plastic strains in a stress reversal, the isotropic hardening model leads to reasonable approximations. The typical form of the uniaxial stress-strain relation obtained for an aluminum alloy is shown in Figure 4; detailed results of the tests performed are discused in the second part of this section.

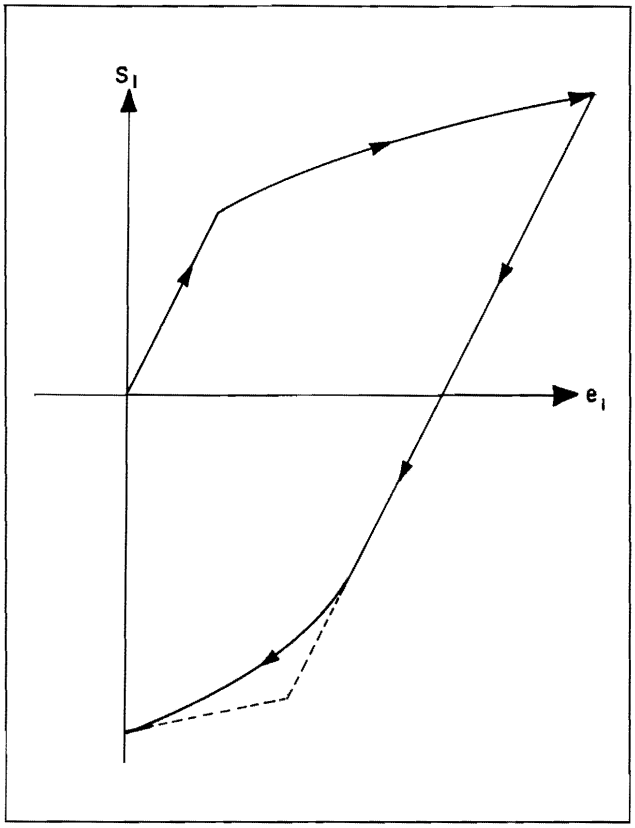

Figure 4. Typical uniaxial stress-strain relation under stress reversal (continuous line) and isotropic hardening model (broken line).

In the spatial form, the loading function and the flow rule which possess the above features are, respectively,

$$f=\tfrac{1}{2}(\bar{t}_{kl}-c_1 h''_{kl})(\bar{t}_{kl}-c_1 h''_{kl})-k^2-c_2\varkappa = 0 \quad (3.10)$$

and

$$d''_{kl}=\bar{\Lambda}\ \frac{\partial f}{\partial t_{kl}}\ =\bar{\Lambda}\bar{t}_{kl} \qquad (3.11)$$

where

$$\bar{t}_{kl}=t_{kl}-\tfrac{1}{3}\,t_{mm}\,\delta_{kl} \qquad (3.12)$$

in the deviatoric stress.

$$\dot{\varkappa}=t_{kl}\,d''_{kl} \qquad (3.13)$$

is the rate of plastic work, and $\bar{\Lambda}$ is a scalar function. The constants k, c_1, and c_2 describe the initial yield stress, kinematic hardening, and isotropic hardening, respectively. The absence of the hydrostatic stress in (3.10) is evident; similarly, the lack of plastic volummetric strain can be verified in (3.11)

$$d''_{kk}=\bar{\Lambda}\,\bar{t}_{kk}=0 \qquad (3.14)$$

In order to express the loading function and the flow rule in terms of the material stress and material strain, use must be made of Equations (2.21) and (2.23). The following expression results for f

$$f\equiv\tfrac{1}{2}\Big[\Big(s_{AB}\frac{\partial z_k}{\partial x_A}\frac{\partial z_l}{\partial x_B}J^{-1}s_{CD}\frac{\partial z_k}{\partial x_C}\frac{\partial z_l}{\partial x_D}J^{-1}\Big)-$$
$$-\tfrac{1}{3}\Big(s_{EF}\frac{\partial z_m}{\partial x_E}\frac{\partial z_m}{\partial x_F}J^{-1}s_{GH}\frac{\partial z_n}{\partial x_G}\frac{\partial z_n}{\partial x_H}J^{-1}\Big)\Big]-$$
$$-c_1\,s_{IJ}\,e''_{IJ}\,J^{-1}$$
$$+\frac{c_1}{3}\,s_{PQ}\frac{\partial z_m}{\partial x_p}\frac{\partial z_m}{\partial x_Q}J^{-1}e''_{SR}\frac{\partial x_S}{\partial z_n}\frac{\partial x_R}{\partial z_n}+$$
$$+\frac{c_1^2}{2}e''_{CD}\frac{\partial x_C}{\partial z_k}\frac{\partial x_D}{\partial z_l}e''_{AB}\frac{\partial x_A}{\partial z_k}\frac{\partial x_B}{\partial z_l}$$
$$-k^2-c_2\varkappa=0, \qquad (3.15)$$

with

$$\dot{\varkappa}=\dot{s}_{AB}\,\dot{e}''_{AB}\,J^{-1} \qquad (3.16)$$

If the kinematic hardening is neglected, i.e., $c_1=0$ the function f becomes

$$f\equiv\tfrac{1}{2}\Big[s_{AB}\frac{\partial z_k}{\partial x_A}\frac{\partial z_l}{\partial x_B}J^{-1}s_{CD}\frac{\partial z_k}{\partial x_C}\frac{\partial z_l}{\partial x_D}J^{-1}-$$
$$-\tfrac{1}{3}s_{EF}\frac{\partial z_m}{\partial x_E}\frac{\partial z_m}{\partial x_F}J^{-1}s_{GH}\frac{\partial z_n}{\partial x_G}\frac{\partial z_n}{\partial x_H}J^{-1}\Big]- \qquad (3.17)$$
$$-k^2-c_2\varkappa=0$$

The flow rule (3.11) is transformed as follows

$$d''_{kl}=\Lambda\frac{\partial f}{\partial t_{kl}}=\Lambda\frac{\partial f}{\partial s_{CD}}\frac{\partial s_{CD}}{\partial t_{kl}}$$
$$=\Lambda\frac{\partial f}{\partial s_{CD}}J\frac{\partial x_C}{\partial z_k}\frac{\partial x_D}{\partial z_l}$$

Multiplication of both sides of the above equation by $\frac{\partial z_k}{\partial x_A}\frac{\partial z_l}{\partial x_B}$ yields

$$\dot{e}''_{AB}=\Lambda\frac{\partial f}{\partial s_{AB}} \qquad (3.18)$$

(with $\Lambda=J\bar{\Lambda}$).

It should be noted that the function Λ is not independent. Its value must be such that for any plastic deformation the stress point remains on the yield surface, i.e., Equation (3.4) is satisfied. The following steps result in the elimination of Λ and a system of relations between the stress-rate tensor $\dot{s}_{AB}$ and the strain-rate tensor e_{AB}.

Combining (3.2) and (3.18),

$$\dot{s}_{AB}=E_{ABCD}\Big(\dot{e}_{CD}-\Lambda\frac{\partial f}{\partial s_{CD}}\Big) \qquad (3.19)$$

Differentiation of f=0 with respect to time yields

$$\frac{\partial f}{\partial s_{AB}}\dot{s}_{AB}+\frac{\partial f}{\partial e''_{AB}}\dot{e}''_{AB}+\frac{\partial f}{\partial\varkappa}\dot{\varkappa}=0$$

or, with (3.16) and (3.18),

$$\frac{\partial f}{\partial s_{AB}}\dot{s}_{AB}+\Big(\frac{\partial f}{\partial e''_{AB}}+\frac{\partial f}{\partial\varkappa}s_{AB}\Big)\Lambda\frac{\partial f}{\partial s_{AB}}=0 \qquad (3.20)$$

The above equation could be solved for Λ. The resulting expression, however, would become indeterminate for a perfectly plastic solid. Furthermore, the resulting relation between the stress-rate tensor and the strain-rate tensor would be implicit rather than explicit. To avoid these shortcomings, Equation (3.19) is multiplied by $\partial f/\partial s_{AB}$, with summation in A and B,

$$\frac{\partial f}{\partial s_{AB}}\dot{s}_{AB}=E_{ABCD}\Big(\dot{e}_{CD}-\Lambda\frac{\partial f}{\partial s_{CD}}\Big)\frac{\partial f}{\partial s_{AB}}$$

and subtracted from (3.20)

$$\therefore\Big(\frac{\partial f}{\partial e_{AB}''}+\frac{\partial f}{\partial\varkappa}s_{AB}J^{-1}\Big)\Lambda\frac{\partial f}{\partial s_{AB}}=\Lambda E_{ABCD}\frac{\partial f}{\partial s_{AB}}\frac{\partial f}{\partial s_{CD}}-$$
$$-E_{ABCD}\frac{\partial f}{\partial s_{AB}}\dot{e}_{CD}$$

The function Λ is now

$$\therefore\Lambda=\frac{E_{ABCD}\dfrac{\partial f}{\partial s_{AB}}\dot{e}_{CD}}{\Big(E_{EFGH}\dfrac{\partial f}{\partial s_{EF}}\dfrac{\partial f}{\partial s_{GH}}\Big)-\Big(\dfrac{\partial f}{\partial e''_{MN}}+\dfrac{\partial f}{\partial\varkappa}S_{MN}J^{-1}\Big)\dfrac{\partial f}{\partial S_{MN}}} \qquad (3.21)$$

with the above Λ, Equation (3.19) becomes

$$S_{AB} = E_{ABCD}\left[\dot{e}_{CD} - \frac{E_{OPQR}\,\dfrac{\partial f}{\partial s_{OP}}\,\dot{e}_{RQ}\,\dfrac{\partial f}{\partial s_{CD}}}{\left(E_{EFGH}\,\dfrac{\partial f}{\partial s_{EF}}\,\dfrac{\partial f}{\partial s_{GH}}\right)-\left(\dfrac{\partial f}{\partial e''_{MN}}+\dfrac{\partial f}{\partial \varkappa}\,s_{MN}\,J^{-1}\right)\dfrac{\partial f}{\partial s_{MN}}}\right] \tag{3.22}$$

which is the sought relation between $\dot{s}_{AB}$ and $\dot{e}_{AB}$ In matrix notation, Equation (3.22) reads

$$\dot{s} = D\,\dot{e} \tag{3.23}$$

where D is the *elastic-plastic moduli* matrix.

A series of unixial tension and compression tests have been performed for the aluminum alloy 2024, temper T4. The purpose of the tests was: (a) the exposition of the character of the hardening model (isotropic vs. kinematic); (b) a verification of the general validity of the proposed form of the constitutive equations; (c) the determination of the material parameters to be used in numerical examples.

Tubular specimens (0.5 in. outer diameter, 0.25 in inner diameter, 1.0 in. gage length) were prepared from slabs of 2 in. thickness. The relatively small gage length - outer diameter ratio was selected to reduce the effect of possible eccentricities in the compression test.

The testing machine was a Universal Testing Instron, model TTD, of 20,000 lb. loading capacity. The strains were measured with electrical, resistance type gages (maximum strains up to 4%) and with an optical gage (strains up to 8%).

The results of tension tests including unloading and loading in compressions are shown in Figure 5,

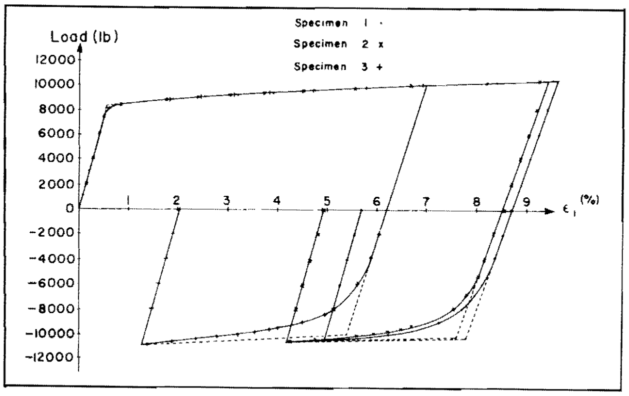

Figure 5. Load-elongation curves for aluminum alloy 2024 *T4.*

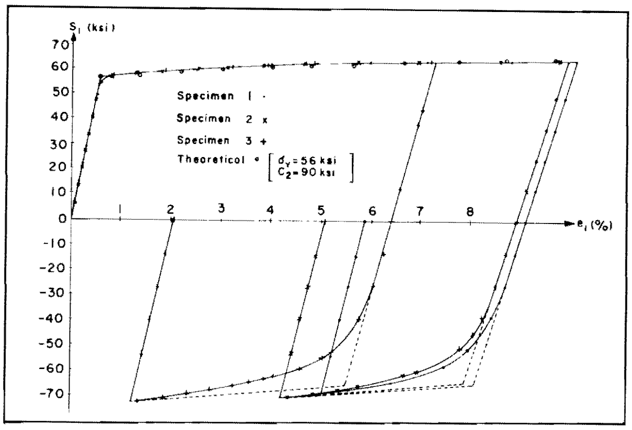

Figure 6. Theoretical and experimental relations between material strain e_1 and Piola-Kirchhoff stress s_1.

where the load is plotted vs. elongation $\varepsilon = (L-L_0)/L_0$. Figure 6 shows the theoretical and the experimental relations between the material strain component e_1 and the Piola-Kirchhoff stress component s_1.

SUMMARY AND CONCLUSIONS

A system of constitutive equations applicable to elasto-plastic metals subjected to large deformations has been proposed. These equations represent the simplest generalization for the large-deformation conditions of the classical theory of plasticity of metals based on the Von Mises yield conditions and the associated flow rule. The material formulation of the kinematical and dynamical relations has been used. The proposed equations combine the kinematic and the isotropic strain hardening models. A method of experimental determination of the material parameters (the initial yield stress, the kinematic hardening coefficient, and the isotropic hardening coefficient) has been demonstrated for the case of the aluminum alloy 2024 T4. It has been found that for sufficiently large stress reversals, resulting in the strain recoveries of at least two to three percent , the isotropic hardening model represents a fairly reasonable approximation of the actual behavior.

The work described here seems to demonstrate that certain topics require further extensive investigations:

1. Development of more general hardening rules. The yield function containing a linear term of plastic work, as in this paper, can hardly be expected to cover a large variety of metals.

2. Experimental data on the elastic-plastic behavior of metals in complex states of stress and at large strains.

NOTATION

B_0 = initial, or undeformed, state at time (t=0)

B = current, or deformed, state at some time t

z_k = spatial description of the displacement of the body.

x_A = material description of the displacement of the body.

u_A = material form of the displacement vector

u_k = spatial form of the displacement vector

v_A = velocity vector

$\dot{e}_{AB}$ = material strain tensor

h_{kl} = spatial strain tensor

dl_0 = changes of length of a line segment

l_{0A} = components of the unit vector along dl_0 at P_0

l_k = components of the unit vector along dl at P.

ε = measure of extension

dV/dV_0 = volumetric strain

$\dot{e}_{AB}$ = material strain-rate tensor

d_{kl} = spatial strain-rate tensor

t_{kl} = spatial or Cauchy, stress tensor

$\dot{s}_{AB}$ = material, or Piola-Kirchhoff, stress tensor

$p(n)$ = surface force per unit area in the undeformed body

s_{AB} = Stress-rate in terms of the material stress tensor

f_A = components of the body forces per unit volume of the undeformed body

p_A = components of the surface forces per unit area of the undeformed body

δu_A = virtual displacements

δe_{AB} = variations of the material strain tensor

u^T = displacement matrix

f^T = body force matrix

p^T = surface force matrix

e^T = strain matrix

s^T = Stress matrix

e'_{AB} = elastic strain

E = 6x6 elastic moduli matrix

$\varkappa$ = hardening parameter

$\Lambda, \bar{\Lambda}$ = scalar functions

k, c_1, c_2 = constants

J = Jacobian

D = elastic-plastic moduli matrix

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