FLOW THROUGH A SPARSELY-PACKED POROUS LAYER SANDWICHED BETWEEN TWO FLUID LAYERS

N. Rudraiah*, R. Sheela

and

J. K. Shivashankara Murthy

UGC-DSA Centre in Fluid Mechanics
Department of Mathematics
Central College, Bangalore University
Bangalore 560 001, India

الخلاصـــة :

تمت دراسة الانسياب الطبقي الثابت للمائع في الطبقة المتناثرة المسامية المحصورة بين طبقتين يحدهما من الأعلى والأسفل صفيحتين متوازيتين قاسيتين . وتتكون الصياغة من معادلة (برنكمان) للطبقة المسامية ، ومعادلة (ستوكس) لطبقة المائع مع استخدام شرط (ب . ج . آر) الإنزلاقي على السطح البيئي بين طبقة المائع والطبقة المسامية .

وقد انتُهِيَ إلى الحل الناجع (الصحيح) للطبقات المعنية باعتبار عوامل اللزوجة الفعالة والإنفاذية . واقترح البحث الطرق التحليلية لايجاد هذه المقادير وبحث في تأثيرات متغير اللزوجة والمسامية وسماكة الطبقة المسامية على معدل الإندفاق الطبقي ومعامل الاحتكاك للحالات التي تكون فيها سماكة الطبقة المسامية صغيرة أو كبيرة بالمقارنة مع طبقات المائع .

وانتهى البحث أن سماكة الطبقة المسامية تلعب دورا رئيسيا في التحكم في اندفاق طبقات المائع وأن توزيع السرعة في الوسط المسامي يشبه ذلك الذي في الطبقة الجدارية المجاورة للسطح .

*To whom correspondence should be addressed.

ABSTRACT

The steady laminar flow in a sparsely-packed horizontal porous layer sandwiched between two fluid layers is studied with the top and bottom boundaries bounded by rigid parallel plates. The formulation consists of the Brinkman equation for the porous layer and the Stokes equation for the fluid layer with the BJR-slip condition at the interface between the fluid layer and the porous layer. Exact solutions are obtained for the respective layers involving effective viscosity and permeability. Analytical methods to determine these quantities are proposed. The effects of viscous parameter, porous parameter and the thickness of the porous layer on the mass flow rate and friction factor are investigated for the cases in which the thickness of the porous layer is small or large compared to that of the fluid layers. We found that the thickness of the porous layer plays a significant role in controlling the flow in fluid layers and the velocity distribution in the porous medium exhibits a boundary layer nature of the flow very near the surface.

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NOMENCLATURE

c = $\frac{4}{3}\pi d_p^3 N$, Volume fraction
C_f = Friction factor
d_p = Average radius of the particles
h_1, h_2, h_3 = Thickness of the fluid and porous layer (Figure 1)
k = Darcy permeability of the porous medium
$\bar{k}$ = λk, Effective permeability of the porous medium
M_1, M_3 = Mass flow rate in the regions 1 and 3
N = Number density
P_1 = $\frac{1}{\mu}\frac{dp}{dx}$, Constant pressure gradient
P_2 = $\phi\frac{\mu}{\bar{\mu}}P_1$
Q = $-kP_1$ Darcy velocity in the porous medium
Re_i = $\frac{2\rho u_i h_1}{\mu}$ $(i = 1, 3)$, Reynolds numbers in regions 1 and 3
u_{B_1} = Slip velocity at the nominal surface at $y = h_2$
U_{B_1} = $\frac{u_{B_1}}{P_1 h_1^2/2}$
u_{B_2} = Slip velocity at the nominal surface at $y = h_1$
U_{B_2} = $\frac{u_{B_2}}{P_1 h_1^2/2}$

Greek Symbols

α = $\surd\lambda$ Slip parameter
λ = Viscosity factor
δ = $\frac{1}{(\lambda k)^{1/2}}$, $\delta_1 = \frac{\varepsilon_1\sigma}{\surd\lambda}$, $\delta_2 = \frac{\sigma}{\surd\lambda}$,
$\delta_3 = \frac{\varepsilon\delta_2}{2}$, $\delta_4 = \sigma\surd\lambda \tanh\delta_3$
η = $\frac{y}{h_1}$, Dimensionless y co-ordinate
ϕ = $1-c$, Porosity of the medium
σ = $\frac{h_1}{\surd k}$, Porous parameter
ρ = Density of the fluid
μ = Dynamic viscosity
$\bar{\mu}$ = Effective viscosity
ε_1 = $\frac{h_2-h_1}{h_1}$, $\varepsilon_2 = \frac{h_2-h_1}{h_3-h_2}$

1. INTRODUCTION

The study of flow past a porous medium has gained importance in recent years because of its natural occurrence and of its importance in bio-mechanics [1–3], in engineering problems like porous bearings [4–8], porous rollers [9], porous layer insulation [10] consisting of solids and pores, and in flow through fractures within petroleum reservoir formations. For an effective use of a porous layer in these technological problems, the structure of the porous layer should be viewed from all angles. In particular, it is desirable to minimize the solid conduction and maximize the porosity in such a way as to reduce the apparent viscosity, thermal conductivity and the mass diffusivity. This may be achieved using a special type of sparsely-packed porous medium of small volume fraction and high porosity.

Many authors [11–18] have shown that in such a medium the flow is governed by the Brinkman equation, with an effective viscosity which takes care of the boundary layer, called the Brinkman boundary layer, of the order $\surd k$. Freed and Muthukumar [19], Koplik *et al.* [20], and Kim and Russel [18] have obtained the ratio of an effective viscosity to the dynamic viscosity as a monotonically increasing function of c, valid for small value of c, Neale and Nader [21], Rudraiah [17], and Kim and Russel [18] have suggested a method to determine the effective viscosity with the help of the slip coefficient proposed by Beavers and Joseph [22].

In this paper, we study the flow in a sparsely-packed porous layer sandwiched between two fluid layers of different depths. A typical example of this is a situation where porous insulation material occupies only a small fraction of the space between two walls. The fluid motion in the porous layer is governed by the Brinkman equation and that in the fluid layers is governed by the Stokes equation, and the interaction between the two takes place through a proper boundary condition at the permeable interface. The crux of the problem is therefore to specify a proper boundary condition depending on the depth and nature of the porous layer.

When the thickness of the porous layer is large, Beavers and Joseph [22] (hereafter called BJ) have used the slip boundary condition known as the BJ-slip condition. This BJ-condition has to be modified suitably in the case of a sparsely-packed porous layer of finite thickness.

Recently, Rudraiah [17] has shown, in the case of a porous layer of finite thickness, that the BJ-slip condition has to be modified, and has derived a modified BJ-condition (hereafter called BJR-slip condition) incorporating the thickness of the porous layer. This condition, since it depends on the depth of the porous layer, may be used to study the practical problems cited earlier.

The object of this paper is, therefore, to use the BJR-slip condition to study the interaction of fluid motion through and past the porous layers under the assumption of laminar flow. The basic equations and the boundary conditions for the two configurations, namely:

(*i*) a thin sparsely-packed porous layer sandwiched between two thick fluid layers (Figure 1*a*);

(*ii*) a thick sparsely packed porous layer sandwiched between two thin fluid layers (Figure 1*b*).

are given in Section 2. In Sections 3 and 4 the exact solutions are obtained for the above two configurations. In Section 5, the results are discussed and some important conclusions are drawn.

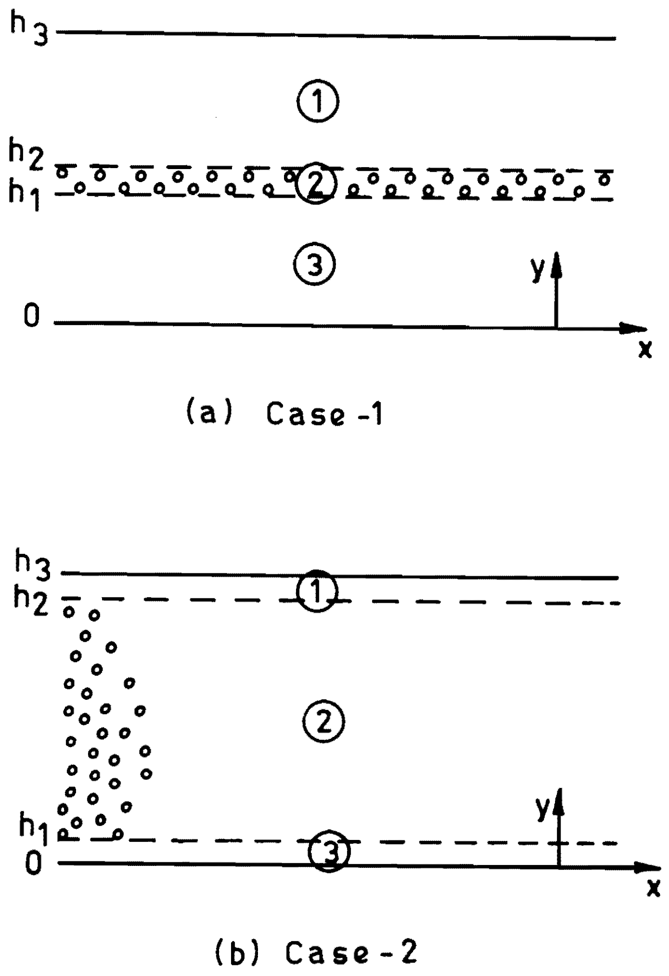

Figure 1. Physical Configuration.

2. FORMULATION OF THE PROBLEM

The physical configurations illustrating the problem under consideration are shown in Figure 1*a* and 1*b*. In Figure 1*a*, a thin porous layer is sandwiched between two thick layers of fluid (case 1), while in Figure 1*b*, a thick porous layer is sandwiched between two thin layers of fluid (case 2), the whole unit lying between the two rigid horizontal boundaries. We choose a Cartesian coordinate system with the x-axis parallel to the channel and the y-axis vertically upwards with the origin at the lower rigid wall. The porous medium occupies region 2 ($h_1 \leq y \leq h_2$) and the fluid, identical with that saturating the medium, occupies regions 1 and 3, ($h_2 \leq y \leq h_3$ and $0 \leq y \leq h_1$) respectively.

The equations of motion for a steady laminar flow of a viscous incompressible fluid are the well-known Stokes equations

$$\nabla . \mathbf{V}_i = 0 \,, \tag{2.1a}$$

$$-\nabla p + \mu \, \nabla^2 \mathbf{V}_i = 0 \;\; (i = 1, 3) \tag{2.1b}$$

and those for the porous medium are the Brinkman equations

$$\nabla . \mathbf{V}_i = 0 \,, \tag{2.2a}$$

$$-\nabla p + \bar{\mu} \, \nabla^2 \mathbf{V}_i - \frac{\mu}{k} \mathbf{V}_i = 0 \;\; (i = 2) \,. \tag{2.2b}$$

These equations are valid only when the Reynolds number, based on this pore velocity and the permeability, is sufficiently small. The effective permeability $\bar{k}$ is related to the Darcy permeability k by the relation

$$\phi^2 \, \frac{\mu}{k} = \frac{\bar{\mu}}{\bar{k}} \tag{2.3}$$

which differs from the relation given by Lundgren [23] and Kim and Russel [18] by the multiplication factor ϕ^2. The effective viscosity $\bar{\mu}$ arises while using the Brinkman equation to govern the flow in a porous medium. The main reason in using the Brinkman equation in a sparsely-packed porous medium is to provide an information as to what is happening within the boundary layer region beneath the surface of the permeable channel wall. Since the porous medium involves multiparticle interaction, it cannot be predicted exactly the relationship between μ and $\bar{\mu}$. Consequently a number of approximate methods have emerged among which the more popular are the mixture theory model (Williams [24] and references therein), cell models (Happel and Brenner [26]), self-consistent-field models (Hashin [27]), suspended particles model (Lundgren [23]), pairwise additive interaction models (Glendinning and Russel [25]) and averaged equations model for dilute systems (Kim and Russel [18]). If we know λ, $\bar{\mu}$ may be determined using (2.4) because μ is the dynamic viscosity. We present here a method to determine λ based on the shear flow model.

In this paper we assume that the effective viscosity in the Brinkman model is due to the fact that the porous matrix may present less window for a fluid to react on itself or may enhance the drag. So following the mixture theory (Williams [24]) we are led to guess

$$\bar{\mu} = \lambda\, \phi^2\, \mu\, . \qquad (2.4)$$

Here the factor ϕ^2 reflects that both the averaged stress and the fluid sees only a fraction ϕ of the fluid. Since $\lambda > 0$, $\lambda\, \phi^2$ increases with increase in porosity ϕ. A method to determine λ will be discussed in section 5.

The required equations, from (2.1) and (2.2), and the boundary conditions for different regions are as follows.

Region 1.

The Stokes equation for unidirectional flow in the region 1, which is the fluid region above the porous layer, is

$$\frac{d^2u_1}{dy^2} = P_1 \qquad (2.5)$$

with the boundary conditions (see Appendix).

$$u_1 = 0 \text{ at } y = h_3 \qquad (2.6)$$

$$\frac{du_1}{dy} = \lambda\delta\left[\frac{-(U_{B_2}-Q)}{\sinh\delta(h_2-h_1)} + (U_{B_1}-Q)\coth\delta(h_2-h_1)\right] \text{ at } y = h_2 \qquad (2.7)$$

in case 1, and

$$\frac{du_1}{dy} = \frac{\lambda(u_{B_1}-Q)}{\sqrt{k}} \text{ at } y = h_2 \qquad (2.8)$$

in case 2.

The conditions (2.7) and (2.8) are the BJR-slip conditions which are obtained from the matching conditions

$$u_1 = \phi u_2\, , \qquad (2.9a)$$

$$\phi\mu\, \frac{du_1}{dy} = \bar{\mu}\, \frac{du_2}{dy} \text{ at } y = h_2 \qquad (2.9b)$$

The first condition here represents the mass balance, and the second represents the balance of shear stress at the interface between the fluid region and the porous medium.

Region 2.

The flow in the porous medium, driven by the same pressure gradient $\frac{dp}{dx}$, is governed by the Brinkman equation

$$\frac{d^2u_2}{dy^2} - \delta^2 u_2 = P_2 \qquad (2.10)$$

with the boundary conditions

$$u_2 = u_{B_1} \text{ at } y = h_2 \qquad (2.11)$$

$$u_2 = u_{B_2} \text{ at } y = h_1 \qquad (2.12)$$

in case 1 and

$$u_2 = -\phi\, \frac{\bar{k}}{\bar{\mu}}\frac{dp}{dx} \text{ as } (h_2-h_1) \to \infty \qquad (2.13)$$

in case 2.

Region 3.

In region 3 which is the fluid region below the porous layer, the flow is governed by the Stokes equation

$$\frac{d^2u_3}{dy^2} = P_1 \qquad (2.14)$$

with the boundary conditions (see Appendix)

$$u_3 = 0 \text{ at } y = 0 \qquad (2.15)$$

$$\frac{du_3}{dy} = -\lambda\delta\left[\frac{-(U_{B_1}-Q)}{\sinh\delta(h_2-h_1)} + (U_{B_2}-Q)\coth\delta(h_2-h_1)\right] \text{ at } y = h_1 \qquad (2.16)$$

in case 1, and

$$\frac{du_3}{dy} = -\frac{\lambda}{\sqrt{k}}(u_{B_2} - Q) \text{ at } y = h_1 \quad (2.17)$$

in case 2.

We note that the BJR conditions (2.7), (2.8), (2.16), and (2.17) reduce to

$$\frac{du_1}{dy} = \frac{\lambda^{1/2}(u_{B_1} - Q)}{\sqrt{k}} \quad (2.18a)$$

and

$$\frac{du_3}{dy} = -\frac{\lambda^{1/2}(u_{B_2} - Q)}{\sqrt{k}} \quad (2.18b)$$

in the limit of $(h_2 - h_1) \to \infty$ (i.e. large thickness of the porous layer). These coincide with the BJ-slip condition when $\sqrt{\lambda} = \alpha$. In other words, the solutions obtained from the BJR slip condition are general in the sense that the solutions tend to those obtained using the BJ-slip condition when the thickness of the porous layer is large compared to the thickness of the fluid layer.

3. SOLUTIONS FOR CASE 1

Solving (2.5), (2.10), and (2.14) separately using the conditions (2.6), (2.7), (2.11), (2.12), (2.15), and (2.16), we obtain [see Appendix (A)]:

$$u_1 = \frac{P_1 h_1^2}{2}\left[\eta^2 - \left(2 + 2\varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2}\right)\eta + (1+\varepsilon_1)\left(1 + \varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2}\right) + \phi U_{B_1}\left(1 + \varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2} - \eta\right)\frac{\varepsilon_2}{\varepsilon_1}\right] \quad (3.1)$$

$$u_2 = \frac{P_1 h_1^2}{2}\left[\frac{1}{\sinh \delta_1} U_{B_2} \sinh \delta_2(1 + \varepsilon_1 - \eta) - U_{B_1} \sinh \delta_2(1-\eta) - \frac{2}{\phi\sigma^2}\{\sinh \delta_2(1+\varepsilon_1-\eta) - \sinh \delta_2(1-\eta) - \sinh \delta_1\}\right] \quad (3.2)$$

$$u_3 = \frac{P_1 h_1^2}{2}[\eta^2 - \eta + \phi U_{B_2}\eta] \quad (3.3)$$

$$U_{B_1} = \frac{1}{\Delta}\left[\sigma\sqrt{\lambda} \operatorname{cosech} \delta_1 + \frac{\varepsilon_2}{\varepsilon_1}(1 + \sigma\sqrt{\lambda} \coth \delta_1) + \frac{2\sqrt{\lambda}}{\sigma} \tanh \frac{\delta_1}{2}\left(1 + \sigma\sqrt{\lambda} \coth \frac{\delta_1}{2}\right)\right] \quad (3.4)$$

$$U_{B_2} = \frac{1}{\Delta}\left[\sigma\sqrt{\lambda} \coth \delta_1 + \frac{\varepsilon_2}{\varepsilon_1}\sigma\sqrt{\lambda} \operatorname{cosech} \delta_1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{2}{\delta_2} \tanh \frac{\delta_1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2} + \sigma\sqrt{\lambda} \coth \frac{\delta_1}{2}\right)\right] \quad (3.5)$$

$$\Delta = -\phi\left[1 + \lambda\sigma^2 + 2\sqrt{\lambda}\sigma \coth \delta_1 + (1 + \sqrt{\lambda}\sigma \coth \delta_1)\left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right)\right].$$

Solutions (3.1) to (3.3) are valid when the depth of the fluid layers in the regions 1 and 3 are different (i.e. $\varepsilon_1 \neq \varepsilon_2$). These solutions will be simplified for the equal depths of flow (i.e. $\varepsilon_1 = \varepsilon_2 = \varepsilon$) because $u_{B_1} = u_{B_2}$. In this case the solutions (3.1) to (3.5) take the form

$$u_1 = \frac{P_1 h_1^2}{2}[2 - 4f_0 + (\eta - \varepsilon + 2f_0 - 3)(\eta - \varepsilon)] \quad (3.6)$$

$$u_2 = -\frac{\mu}{\bar{\mu}}\frac{P_1 h_1^2}{\sigma^2}\left[1 + \sinh \delta_3(1 + \varepsilon - \eta) - \sinh \delta_2(1-\eta) \times \frac{(\sigma^2 f_0 - 1)}{\sinh \delta_2 \varepsilon}\right] \quad (3.7)$$

$$u_3 = \frac{P_1 h_1^2}{2}[\eta^2 - 2(1 + f_0)\eta] \quad (3.8)$$

$$f_0 = \frac{1}{2\sigma^2}\frac{(\sigma^2 + 2\delta_4)}{(1 + \delta_4)} \cdot \quad (3.9)$$

From these solutions it is possible to obtain the relations that involve quantities which may be measured in the laboratory. Among these are the mass flow rate and friction factor in regions 1 and 3. If M_1 and M_3 are the mass flow rate in regions 1 and 3 respectively, then

$$M_1 = \rho\bar{u}_1(h_3 - h_2) = \rho\int_{h_2}^{h_3} u_1 dy = -\frac{\rho P_1 h_1^3}{12}\frac{\varepsilon_1}{\varepsilon_2}\left[\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 - 3\phi U_{B_1}\right] \quad (3.10)$$

$$M_3 = \rho\bar{\mu}_3 h_1 = \rho\int_0^{h_1} u_3 dy = -\frac{\rho P_1 h_1^3}{12}[1 - 3\phi U_{B_2}] \cdot \quad (3.11)$$

If M^* is the mass flow rate in the channel, of depth h_1, bounded by the rigid impermeable walls, then

$$M^* = -\frac{\rho P_1 h_1^3}{12} \quad (3.12)$$

where the asterisk (*) is used to denote the quantity evaluated in the absence of a porous layer. For the condition of equal pressure gradient, the ratios of (3.10) and (3.11) respectively to (3.12) yield

$$\frac{M_1}{M^*} = \left[1 + \frac{3(-\phi\, U_{B_1})}{(\varepsilon_1/\varepsilon_2)^2}\right]\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^3 \tag{3.13}$$

and

$$\frac{M_3}{M^*} = 1+3(-\phi U_{B_2}) \,. \tag{3.14}$$

Since $(-\phi U_{B_1})$ and $(-\phi U_{B_2})$ are always positive, it follows that the presence of a porous layer sandwiched between the two fluids results in a higher mass flow rate. Further, when $\varepsilon_1 = \varepsilon_2 = \varepsilon$, (3.13) and (3.14) coincide because $u_{B_1} = u_{B_2}$ due to symmetry and we have

$$\frac{M_1}{M^*} = \frac{M_3}{M^*} = 1 + \frac{3(\sigma + 2\sqrt{\lambda}\,\tanh\delta_3)}{\sigma(1+\sigma\sqrt{\lambda}\,\tanh\delta_3)} \,. \tag{3.15}$$

We note that when $\varepsilon \to \infty$ and $\sqrt{\lambda} = \alpha$, (3.15) reduces to

$$\frac{M_1}{M^*} = 1 + \frac{3(\sigma+2\alpha)}{\sigma(1+\alpha\sigma)} \tag{3.16}$$

which coincides with the one given by BJ. This means that when the depth of the porous layer is very large, the results of Darcy equation with the BJ-slip condition are similar to those obtained from the Brinkman equation with the BJR-slip condition.

The results obtained in this paper are based on the assumption of laminar flow. Therefore, to know up to what values of the Reynolds number the laminar flow is valid, we calculate the friction factor C_f defined by

$$(C_f)_i = \frac{4\,\dfrac{dp}{dx}\,h_1}{\bar{\mu}\;\rho \bar{u}_i^2}\; (i = 1, 3) \,. \tag{3.17}$$

From this, using (3.6) and (3.7), we find that

$$(C_f)_1\, Re_1 = \frac{96}{\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 - 3\phi\, U_{B_1}} \tag{3.18}$$

$$(C_f)_3\, Re_3 = \frac{96}{1-3\phi U_{B_2}} \tag{3.19}$$

From physical considerations it is clear that the porous layer diminishes the friction factor relative to that in the absence of porous layer for which

$$(C_f\, Re)^* = 96 \tag{3.20}$$

Thus

$$\frac{(C_f)_1 Re_1}{(C_f\, Re)^*} = \frac{1}{\left(\dfrac{\varepsilon_1}{\varepsilon_2}\right)^2 - 3\phi\, U_{B_1}} \tag{3.21}$$

$$\frac{(C_f)_3 Re_3}{(C_f\, Re)^*} = \frac{1}{1-3\phi U_{B_2}} \,. \tag{3.22}$$

In the case of $\varepsilon_1 = \varepsilon_2 = \varepsilon$, these expressions take the form

$$(C_f)_1\, Re_1 = (C_f)_3\, Re_3 = \frac{96}{1+6f_0} \tag{3.23}$$

and

$$\frac{(C_f)_1\, Re_1}{(C_f\, Re)^*} = \frac{(C_f)_3\, Re_3}{(C_f\, Re)^*} = \frac{1}{1+6f_0} \,. \tag{3.24}$$

From (3.18) and (3.19) it is clear that for laminar flow $(C_f)_i Re_i$, $(i = 1, 3)$, is independent of the Reynolds number for a region of fixed height and for a given porous layer. From (3.21) to (3.24) it also follows that the porous layer diminishes the friction factor compared to that for a channel in the absence of a porous layer.

4. SOLUTIONS FOR THE CASE 2

In this case a porous layer of large thickness is sandwiched between two thin fluid layers as shown in the Figure 1(*b*). The solutions of (2.5), (2.10), and (2.14) satisfying the boundary conditions stated in section 2 are [see Appendix (B)]

$$u_1 = -\frac{P_1 h_1^2}{2}\left(1+\varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2} + \eta - A_1\right)\left(1+\varepsilon_1+\frac{\varepsilon_1}{\varepsilon_2} - \eta\right) \tag{4.1}$$

$$u_2 = -\frac{P_1 h_1^2}{2\phi}\,\frac{2\phi^2\mu}{\sigma^2\bar{\mu}} + A_2 \exp[-\sigma(1+\varepsilon_1 - \eta)] \tag{4.2}$$

$$u_3 = -\frac{P_1 h_1^2}{2\phi}\,[A_3 - \eta]\eta \,, \tag{4.3}$$

$$A_1 = \frac{\sigma^2\left(2+2\varepsilon_1+\dfrac{\varepsilon_1}{\varepsilon_2}\right) - 2(1-\sigma\sigma-\varepsilon_1)\,\dfrac{\phi^2\mu}{\bar{\mu}}\,\dfrac{\varepsilon_2}{\varepsilon_1}}{\sigma\left(\sigma + \dfrac{\phi^2\mu}{\bar{\mu}}\,\dfrac{\varepsilon_2}{\varepsilon_1}\right)}$$

$$A_2 = \frac{\varepsilon_1}{\varepsilon_2}\left(2+2\varepsilon_1+\frac{\varepsilon_1}{\varepsilon_2} - A_1\right) - \frac{2\phi^2\mu}{\bar{\mu}\sigma^2}$$

$$A_3 = \frac{\sigma^2 + 2\phi^2\,\dfrac{\mu}{\bar{\mu}}\,(\sigma+1)}{\sigma\left(\sigma+\phi^2\,\dfrac{\mu}{\bar{\mu}}\right)} \,.$$

The condition of mass balance (i.e. $u_3 = \phi u_2$) leads to the equation

$$a\,\lambda^2 + b\lambda + c = 0 \,, \tag{4.4}$$

where

$$a = \sigma^3 \frac{\varepsilon_1}{\varepsilon_2}\left[1-\frac{\varepsilon_1}{\varepsilon_2}\exp(-\sigma\,\varepsilon_1)\right]$$

$$b = \sigma^2\left[1-\frac{\varepsilon_1^2}{\varepsilon_2^2}\exp(-\sigma\,\varepsilon_1)\right]+2\sigma\left[\exp(-\varepsilon_1)\,\frac{\varepsilon_1}{\varepsilon_2}\right]$$

$$c = 2[\exp(\sigma\,\varepsilon_1)-1]\,.$$

The value of λ (=0.01) computed from (4.4) for different values of σ, ε_1, and ε_2 coincides with the experimental value of BJ.

The ratios of the mass flow rates, as in Section 3, in the different zones are:

$$\frac{M_1}{M^*} = \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\left[\frac{\varepsilon_1}{\varepsilon_2}+\frac{3\left(\sigma+2\dfrac{\varepsilon_2}{\varepsilon_1}\right)\dfrac{\phi^2\mu}{\bar{\mu}}}{\sigma\left(\sigma+\dfrac{\phi^2\mu}{\bar{\mu}}\dfrac{\varepsilon_2}{\varepsilon_1}\right)}\right] \tag{4.5}$$

$$\frac{M_3}{M^*} = 1+\frac{3(\sigma+2)\dfrac{\phi^2\mu}{\bar{\mu}}}{\sigma\left(\sigma+\phi^2\dfrac{\mu}{\bar{\mu}}\right)} \tag{4.6}$$

These, using (2.3) and (2.4), may be written as

$$\frac{M_1}{M^*} = \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\left[\frac{\varepsilon_1}{\varepsilon_2}+\frac{3\left(\sigma+2\dfrac{\varepsilon_2}{\varepsilon_1}\sqrt{\lambda}\right)}{\sigma\left(\dfrac{\varepsilon_1}{\varepsilon_2}+\sigma\sqrt{\lambda}\right)}\right] \tag{4.7}$$

$$\frac{M_3}{M^*} = 1+\frac{3(\sigma+2\sqrt{\lambda})}{\sigma(1+\sigma\sqrt{\lambda})} \tag{4.8}$$

If $\varepsilon_1 = \varepsilon_2$, we have

$$\frac{M_1}{M^*} = \frac{M_3}{M^*} = 1+\frac{3(\sigma+2\sqrt{\lambda})}{\sigma(1+\sigma\sqrt{\lambda})} \tag{4.9}$$

which is identical with (3.16) when $\sqrt{\lambda} = \alpha$. Thus the two approaches, the Darcy equation with the BJ-slip condition or the Brinkman equation with the BJR-slip condition, lead to the similar end results in the case of a porous layer of large thickness sandwiched between two fluid layers, each of equal thickness.

To know the value of the Reynolds number upto which the laminar flow is valid, we calculate, as in the case 1, the friction factor in the two regions 1 and 3, and when $\varepsilon_1 = \varepsilon_2 = \varepsilon$, we have

$$\frac{(C_f)_1 Re_1}{(C_f\,Re)^*} = \frac{(C_f)_1 Re_1}{(C_f\,Re)^*} = \frac{1}{1+\dfrac{3(\sigma+2\sqrt{\lambda})}{\sigma(1+\sigma\sqrt{\lambda})}} \tag{4.10}$$

The results are as depicted in Figure 7.

5. DISCUSSION AND CONCLUSIONS

The steady incompressible Newtonian flow through a sparsely-packed porous layer sandwiched between two fluid layers is studied, using the Brinkman equation satisfying the BJR-slip condition. Exact solutions, in different regions, are obtained in terms of the ratios $\frac{\varepsilon_1}{\varepsilon_2}$, $\frac{\bar{\mu}}{\mu}$ and $\frac{\bar{k}}{k}$. The velocity profiles in different regions with finite depth of the porous layer (*i.e.* case 1) are drawn in Figure 2, and for large thickness of the porous layer (*i.e.* case 2) in Figure 3, for different values of ε_1 and ε_2. The thick lines correspond to the velocity distribution in the fluid layers with a slip at the surface of the porous layer and exhibits a boundary layer nature (called the Brinkman boundary layer in the porous layer—see Appendix C) near the interface. Outside the boundary layer region, called the core, the effect of the viscous shear term in the Brinkman equation becomes negligible because of the resistance offered to the flow by the elements in unit volume of the medium is small, thereby indicating the validity of the Darcy law in the core. These figures depict the symmetrical nature of the velocity distribution when the thickness of the fluid layers are equal (*i.e.* $\varepsilon_1 = \varepsilon_2$). We found that the decrease in the width of the porous layer increases the velocity compared to that in the absence of porous layer. Our analysis is valid only for laminar flow (*i.e.* the Stoke's and Brinkman equations are valid only for sufficiently small value of the Reynolds number based on the pore velocity and permeability). The friction factor relations discussed in the Sections 3 and 4, however, predict the value of the Reynolds number up to which the laminar flow is valid. In other words, the laminar flow can be controlled by adjusting the width of the porous layer. The effective viscosity, called the Brinkman viscosity, $\bar{\mu}$, and the dynamic viscosity, μ, are related to each other through (2.4) which can be guessed by the analogy of mixture theory [24]. This relation involves the unknown viscosity parameter λ.

One way of finding the λ as explained in the Section 2, could be by relating it to the slip coefficient α measured by BJ which is valid for large volume concentration. In recent years there have been a number of attempts to derive an analogue to Einstein's expression

$$\frac{\bar{\mu}}{\mu} = 1+5/2\,c \tag{5.1}$$

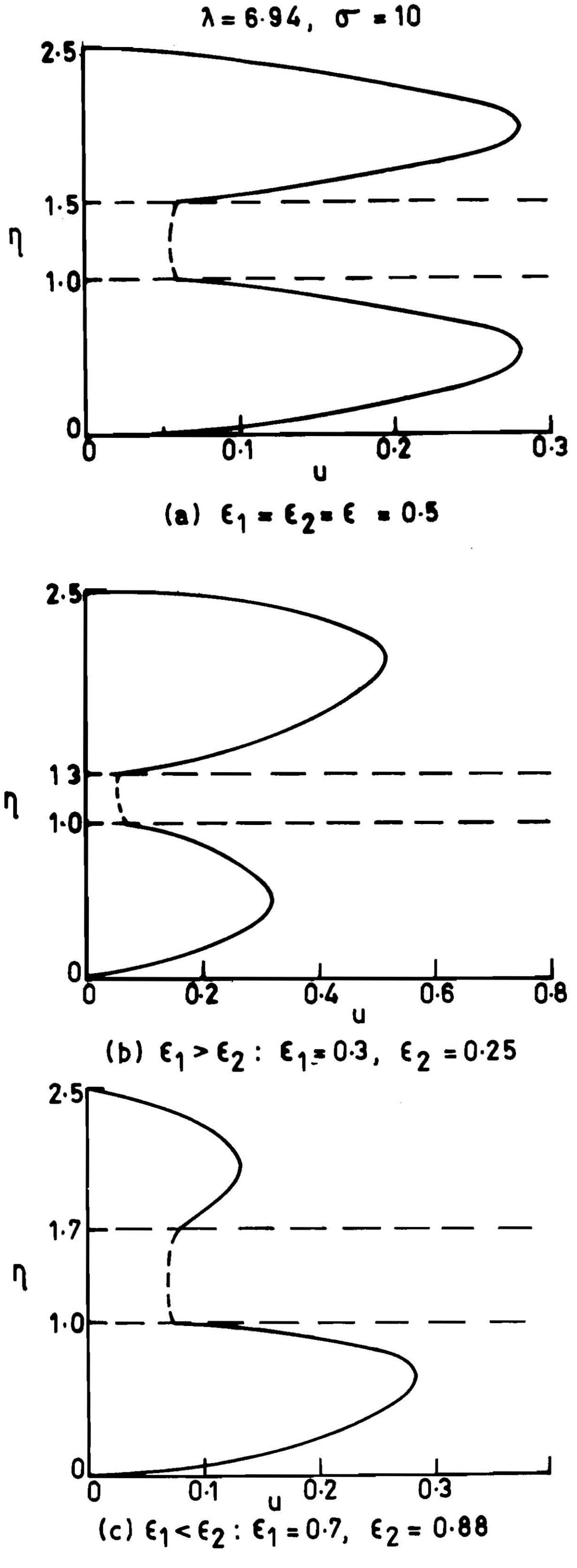

Figure 2. Velocity Profiles for Case 1.

when the volume fraction is small. Brinkman [11] suggested the possible use of (5.1) in addition to providing *ad hoc* momentum transfer arguments favouring the use of the dynamic viscosity μ (*i.e.* $\bar{\mu} = \mu$). The theory of Freed and Muthukumar [19] predicts the effective viscosity for dilute arrays in the form

$$\frac{\bar{\mu}}{\mu} = 1 + \tfrac{5}{2}\, c - \frac{9}{2\sqrt{2}}\, c^{3/2} \qquad (5.2)$$

a value greater than that for the pure solvent, but is less than the Einstein result. Koplik *et al.* [20] have also obtained, from the energy dissipation in an external flow about the isolated stationary sphere, an expression for $\frac{\bar{\mu}}{\mu}$ in the form

$$\frac{\bar{\mu}}{\mu} = 1 - \tfrac{1}{2}\, c\,, \qquad (5.3)$$

a value less than that for the dynamic viscosity.

Recently, Kim and Russel [18] have derived an expression for the Brinkman viscosity through a rigorous calculation of the bulk stress to settle the disagreement in the expression for $\frac{\bar{\mu}}{\mu}$ obtained by various authors. They [18] have obtained

$$\frac{\bar{\mu}}{\mu} = \tfrac{5}{2}\, c + (\tfrac{81}{32} \log c + 19.66)c^2 + 6.59 c^{5/2} \log c\,, \qquad (5.4)$$

a value greater than the solvent viscosity. To the order of c, this agrees with that of Freed and Muthukumar [19] and not with Koplik *et al.* [20].

The matching process employed in this paper leads to the quadratic equation (4.4) in λ. The values of λ computed from this equation are compared with those obtained from equations (5.1) to (5.4) in Figure 4 and a good agreement is found. We see that $\lambda\phi^2$ increases with increasing porosity, except in the case of Einstein formula (5.2).

We can also measure λ directly using the simple viscometers as explained below.

For this we first obtain (see [17]) the simple shear flow solutions in the form

$$u_1 = U_0 + \lambda\phi\delta[U_{B_1} \coth \delta(h_2 - h_1) - U_{B_2} \operatorname{cosech} \delta(h_2 - h_1)](y - h_3) \qquad (5.5)$$

$$u_3 = U_0' + \lambda\phi\delta[U_{B_1} \operatorname{cosech} \delta(h_2 - h_1) - U_{B_2} \coth \delta(h_2 - h_1)]y \qquad (5.6)$$

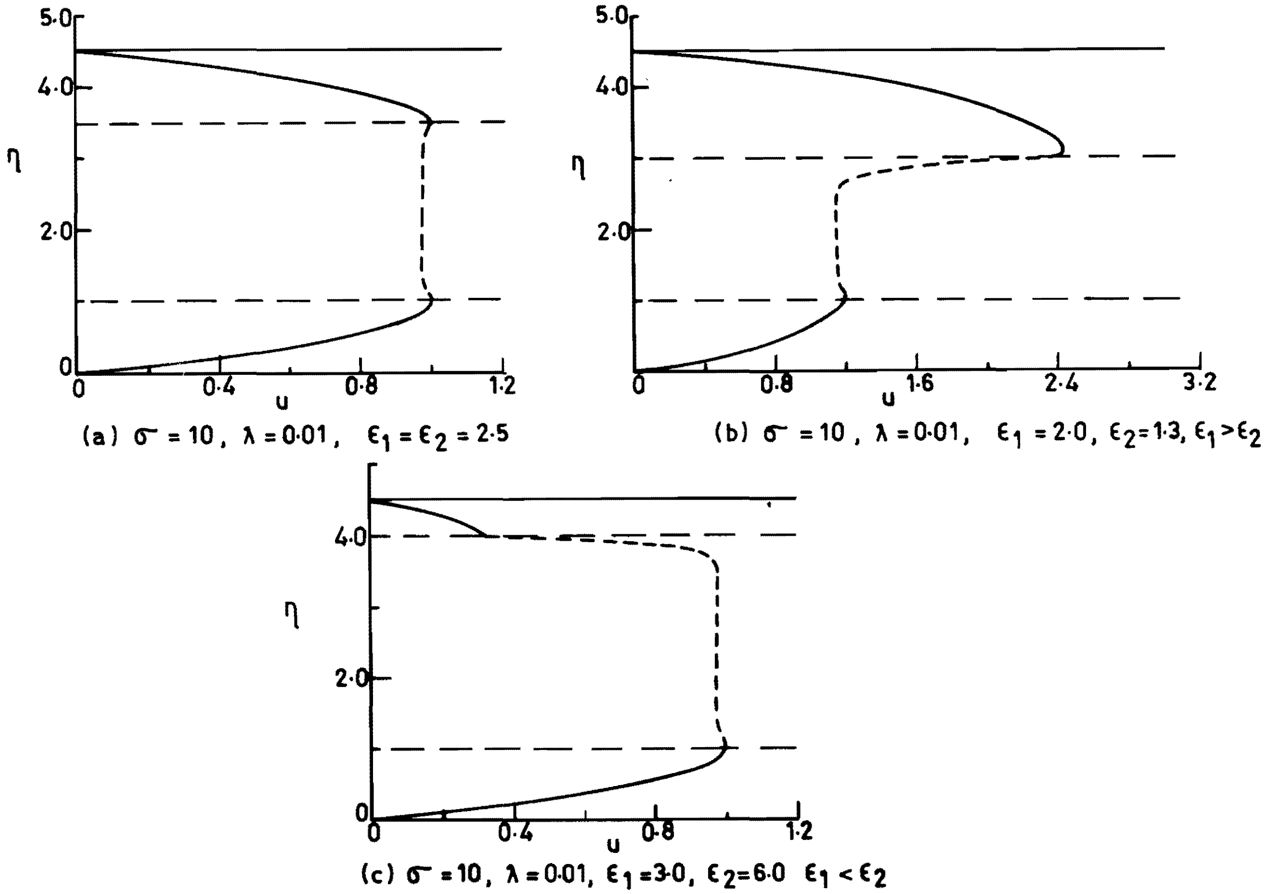

Figure 3. Velocity Profiles for Case 2.

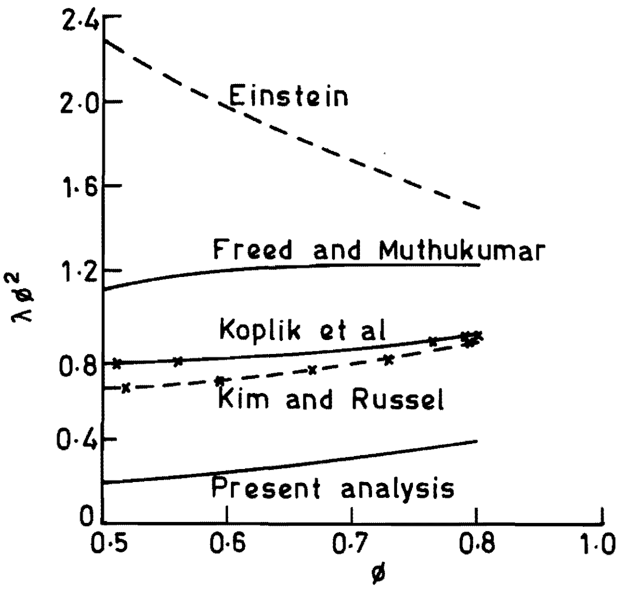

Figure 4. Viscosity Factor versus ϕ.

where the upper and lower plates are moving with different uniform velocities U_0 and U_0' respectively,

$$U_{B_1} = \frac{\sigma\sqrt{\lambda}\ \frac{\varepsilon_1}{\varepsilon_2}\ U_0' \operatorname{cosech} \delta_1 + (\sigma\sqrt{\lambda} \coth \delta_1 + 1) U_0}{\phi \Delta_1}$$

$$U_{B_2} = \frac{\sigma\sqrt{\lambda}\ U_0 \operatorname{cosech} \delta_1 + (\sigma\sqrt{\lambda} \coth \delta_1 + 1) U_0'}{\phi \Delta_1}$$

and

$$\Delta_1 = 1 + \sigma\sqrt{\lambda}\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right) \coth \delta_1 + \sigma^2 \lambda \frac{\varepsilon_1}{\varepsilon_2}.$$

The shear force per unit area required to drive the motion in the region 1 is

$$\frac{\mu\lambda\delta}{\Delta_1}\ [U_0(\sigma\sqrt{\lambda} + \coth \delta_1) - U_0' \operatorname{cosech} \delta_1] \quad (5.7)$$

and that in the region 3 is

$$\frac{\mu\lambda\delta}{\Delta_1}\left[U_0 \operatorname{cosech}\delta_1 - U_0'\left(\sigma\sqrt{\lambda}\,\frac{\varepsilon_1}{\varepsilon_2}+\coth\delta_1\right)\right]. \qquad (5.8)$$

We note that if $\varepsilon_1 = \varepsilon = \varepsilon_2$ and $U_0 = U_0'$,

$$U_{B_1} = U_{B_2} = \frac{U_0}{\phi(1+\sigma\sqrt{\lambda}\tanh\delta_3)} = U_B \qquad (5.9)$$

so that

$$\frac{U_1}{U_0} = 1 + \frac{\lambda\delta[\coth\delta(h_2-h_1) - \operatorname{cosech}\delta(h_2-h_1)](y-h_3)}{(1+\sigma\sqrt{\lambda}\tanh\delta_3)} \qquad (5.10)$$

$$\frac{U_3}{U_0} = 1 + \frac{\lambda\delta[\operatorname{cosech}\delta(h_2-h_1) - \coth\delta(h_2-h_1)]y}{(1+\sigma\sqrt{\lambda}\tanh\delta_3)}. \qquad (5.11)$$

In this case, the shear force per unit area required to drive the motion in region 1 is equal and opposite to that in region 3 and is given by

$$\frac{\mu\lambda\delta\, U_0 \tanh\delta_3}{1+\sigma\sqrt{\lambda}\tanh\varepsilon_3}. \qquad (5.12)$$

The measurement of (5.12) yields λ. We can also measure λ either from M_1/M^* or from M_3/M^* where all the quantities except λ are known experimentally.

The other quantity to be determined is the effective permeability $\bar{k}$. Brinkman [11] pioneered the modeling of porous media via fixed arrays of uniform spheres and determined the effective permeability $\bar{k}$ in the form

$$\left(\frac{a}{\sqrt{\bar{k}}}\right)^2 = 9/2\, c\, B_0\left(\frac{a}{\sqrt{\bar{k}}}\right) \qquad (5.13)$$

where $B_0(x) = 1+x+1/3\,x^2$ and a is the radius of the sphere. By equating the total force on the spheres contained in the column of the medium to the Darcy drag on that column, he [11] obtained a relationship between a and the porosity and hence an expression for B_0 in the form,

$$B_0 = \left\{1+3/4\,c\left[1-\left(\frac{8}{c}-3\right)^{1/2}\right]\right\}^{-1} \qquad (5.14)$$

where $c = 1-\phi$. This requires that the effective permeability $\bar{k}$ be given by

$$\bar{k} = k\,B_0\,. \qquad (5.15)$$

According to (5.14), B_0 becomes unbounded as $\phi \rightarrow 1/3$ and hence we must assume that $1/3 < \phi < 1$.

Using the method of reflection, Kim and Russel [18] have also obtained the permeability parameter for an effective porous medium in the form

$$\left(\frac{a}{\sqrt{\bar{k}}}\right)^2 = 9/2\, c\, B_0\left(\frac{a}{\sqrt{\bar{k}}}\right) \qquad (5.16)$$

which resembles the Brinkman relation (5.13) with the exception that (5.16) applies to both the single- and two-particle fixed problem.

The effective permeability may also be obtained by computing the average volume flux, $\bar{u}$, in the effective porous layer where

$$\bar{u}_2 = -\frac{P_1}{\phi}\,k(1+\Delta_2) \qquad (5.17)$$

$$\Delta_2 = \frac{\tanh\frac{\delta_1}{2}}{\delta_1}\left[\frac{\sigma^2}{2\Delta}\left\{2\,\frac{\varepsilon_1}{\varepsilon_2} + \sigma\sqrt{\lambda}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)(\coth\delta_1 + \operatorname{cosech}\delta_1) + \frac{2\sqrt{\lambda}}{\sigma}\tanh\frac{\delta_1}{2}\left(1+\frac{\varepsilon_1}{\varepsilon_2} - 2\sigma\sqrt{\lambda}\coth\frac{\delta_1}{2}\right)\right\} - 2\right]. \qquad (5.18)$$

Then $\bar{k} = \frac{k}{\phi}(1+\Delta_2)$

In particular, when $\varepsilon_1 = \varepsilon_2$, (5.17) becomes

$$\bar{u}_2 = \frac{-P_1 k}{\phi}\left[1+\frac{(\sigma^2-2)\tanh\delta_3}{2\delta_3(1+\sigma\sqrt{\lambda}\tanh\delta_3)}\right] \qquad (5.19)$$

and hence

$$\bar{k} = \frac{\mathrm{k}}{\phi}\left[1+\frac{(\sigma^2-2)\tanh\delta_3}{2\delta_3(1+\sigma\delta\lambda\tanh\delta_3)}\right]. \qquad (5.20)$$

The results obtained from (5.20) for $k = 12.7\times10^{-5}$ are compared with (5.15) in Figure 5. This reveals that the results obtained in this paper are in good

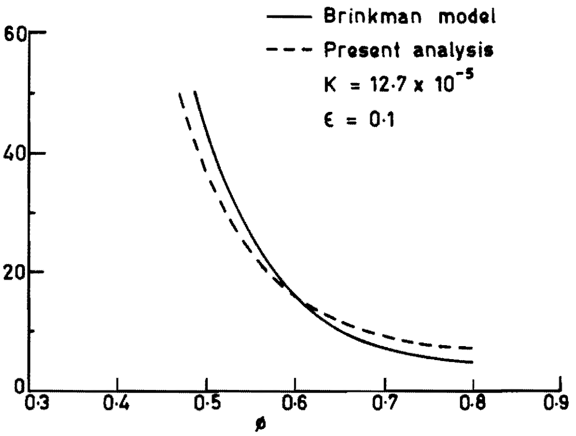

Figure 5. Permeability versus ϕ.

agreement with the Brinkman result for values of ϕ (> 0.5).

The values for the ratio of mass flow rate M_1/M^*, M_3/M^* given by (3.13)–(3.15) and (4.7)–(4.9), and the ratio of friction factors given by (3.20), (3.21), and (3.23) are obtained for different values of $\frac{\varepsilon_1}{\varepsilon_2}$, λ and σ and the results shown in Figures 6 and 7. We see that the effect of the presence of a porous layer of finite thickness is to increase the mass flow rate and to decrease the friction factor relative to those for laminar flow in channel in the absence of a porous layer. Figure 6 reveals that the decrease in the thickness of the porous layer increases the mass flow rate and a slight increase in the value of λ has no significant effect on the mass flow rate. From Figure 6(*a*), we see that when the thickness of the fluid layers are the same (i.e. $\varepsilon_1 = \varepsilon_2$), the increase in σ and ε decreases the mass flow rate. Figure 6(*b*) reveals that the mass flow rate can be controlled by varying the values of ε_1 and ε_2 (i.e. by varying the thickness of the porous layer and the fluid layer in region 1). In the case of a large thickness of the porous layer compared to the width of the flow, we observe that if $\alpha = \sqrt{\lambda}$, and the thickness of the fluid layers are equal, the expressions (4.7) and (4.8) agree identically with those of Beavers *et al.* [22] using the BJ-slip condition. From this we conclude that in the case of large thickness of this porous layer with equal depth of flow, the BJR-slip condition (2.16) provides a mathematical model with $\alpha = \sqrt{\lambda}$ for hydrodynamic studies within the channel.

From Figure 7, it is clear that the increase in the thickness of the porous layer increases the friction factor because the resistance offered to the flow by the elements in unit volume of the medium is sufficiently small.

Finally we conclude that the thickness of the porous layer greatly influence the velocity distribution in different zones. The decrease in the thickness of the porous layer increases the velocity distribution in the channel compared to that in the channel bounded by impermeable walls and hence decreases the pressure considerably, which is useful in the design of a gas-cooled reactor to reduce the pressure distribution. These results are valid only for laminar flow regime (i.e. sufficiently small Reynolds number).

ACKNOWLEDGEMENT

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APPENDIX: THE BOUNDARY CONDITIONS AND SOLUTIONS

A. Case 1

Solution of (2.5) satisfying (2.6) is

$$u_1 = -\frac{P_1}{2}(h_3^2-y^2)-C(h_3-y) \qquad (A.1)$$

where C is a constant of integration to be determined. Similarly, the solution to (2.10) is

$$u_2 = C_1 \cosh \delta y + C_2 \sinh \delta y - \frac{P_2}{\delta^2} \qquad (A.2)$$

Where $P_2 = \phi \frac{\mu}{\bar{\mu}} P_1$. The constants C_1 and C_2 satisfying (2.11) and (2.12) are

$$C_1 = \frac{1}{\sinh \delta(h_2-h_1)}[(u_{B_2} \sinh \delta h_2 - u_{B_1} \sinh \delta h_1) + \frac{P_2}{\delta^2}(\sinh \delta h_2 - \sinh \delta h_1)]$$

$$C_2 = \frac{1}{\sinh \delta(h_1-h_2)}[(u_{B_2} \cosh \delta h_2 - u_{B_1} \cosh \delta h_1) + \frac{P_2}{\delta^2}(\cosh \delta h_2 - \cosh \delta h_1)] \qquad (A.3)$$

The constant C in (A.1) is determined using the condition (2.9a) and is of the form

$$C = -\frac{P_1}{2}(h_3+h_2) - \frac{\phi}{(h_3-h_2)}u_{B_1} \qquad (A.4)$$

The BJR-condition (2.7) follows from

$$\frac{du_1}{dy} = \lambda\phi \frac{du_2}{dy}$$

by keeping $\frac{du_1}{dy}$ as it is and finding $\frac{du_2}{dy}$ using (A.2)

Similarly, the solution of (2.14) satisfying (2.15) is

$$u_3 = \frac{P_1 y^2}{2} C'y \qquad (A.5)$$

The constant C' is determined by using $u_3 = \phi u_2$ at $y = h_1$, and is of the form

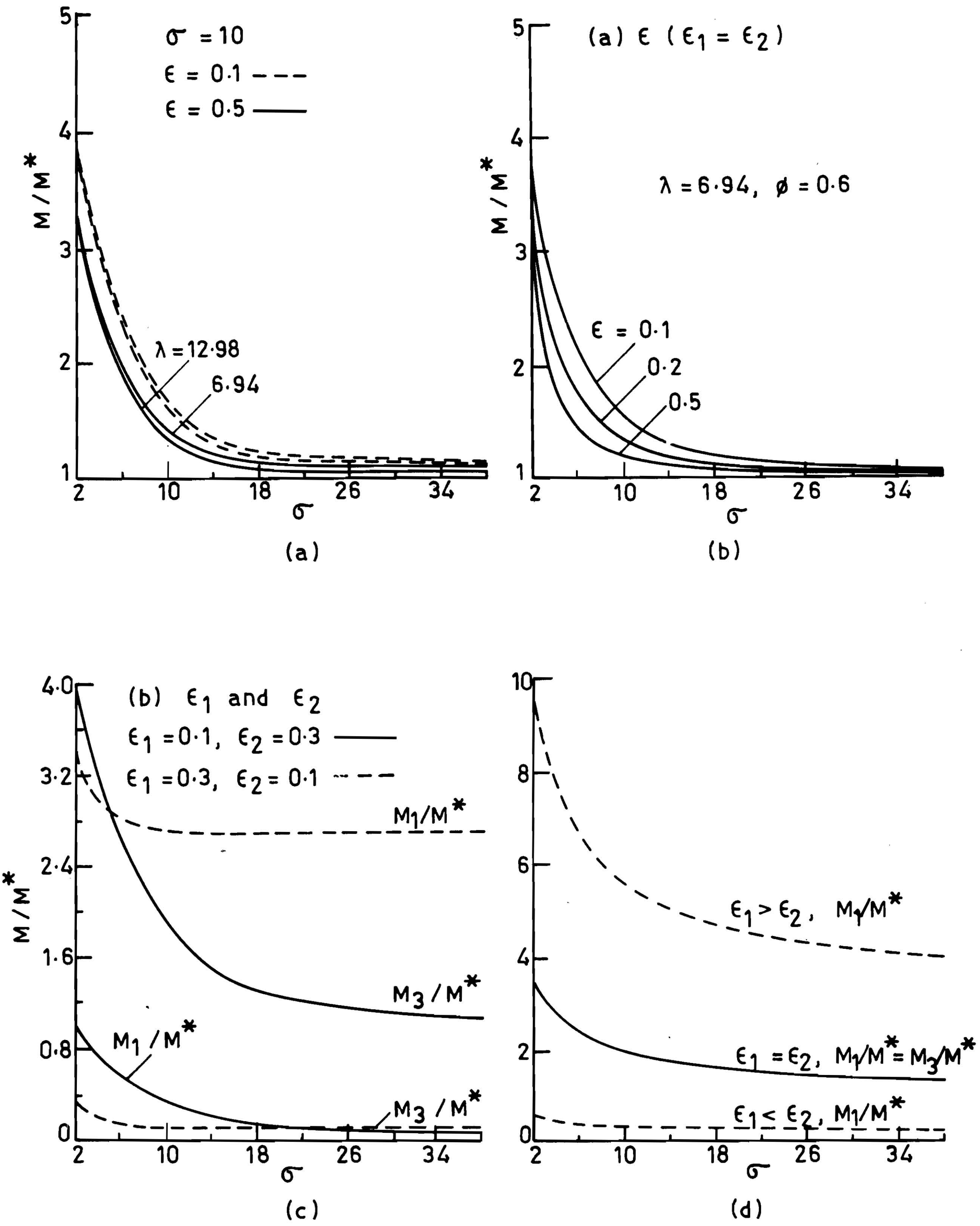
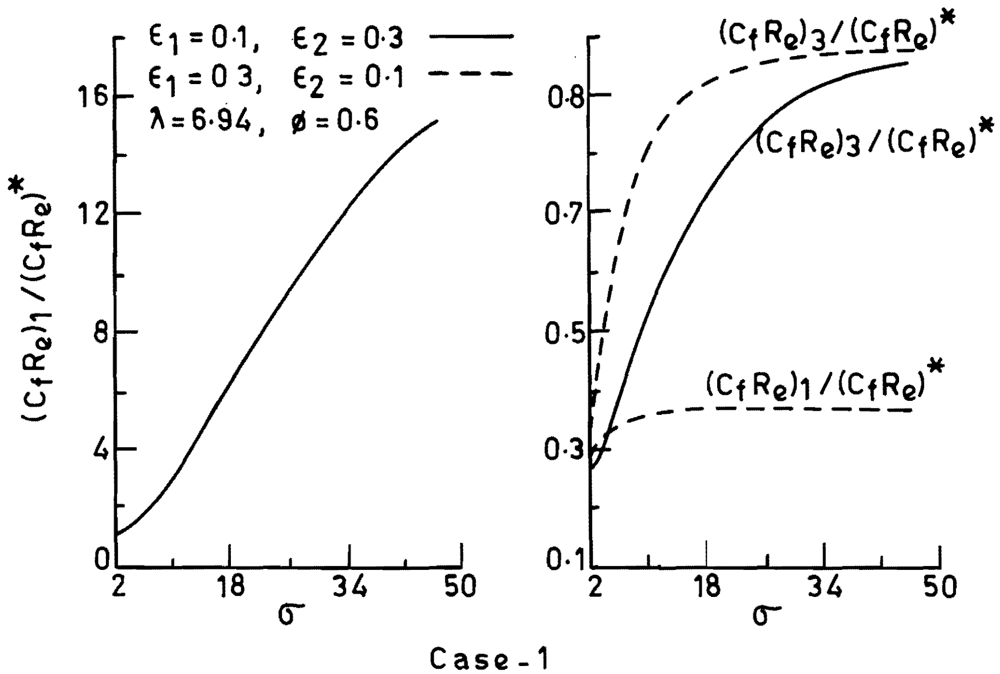


Figure 6. Mass Flow Rate versus *Porous Parameter (for* a, b, c, *Refer to Case 1).*



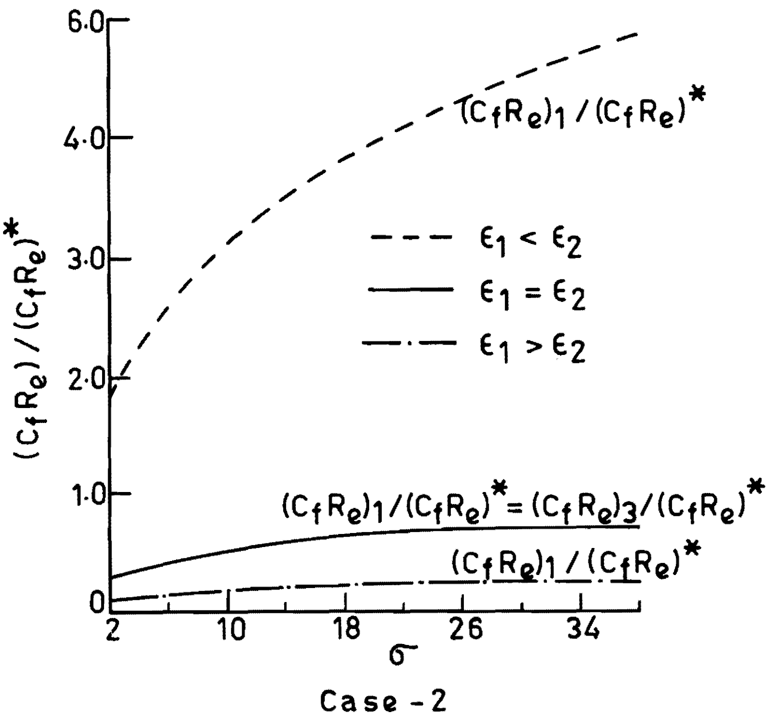


Figure 7. Friction Factor versus *Porous Parameter.*

$$C' = -\frac{P_1 h_1}{2} + \frac{\phi}{h_1} u_{B_2} \quad (A.6)$$

The BJR-condition (2.16) follows from

$$\frac{du_3}{dy} = \lambda\phi \frac{du_2}{dy}$$

after using (A.2).

Substituting the constants in the general solutions, and after some simplifications, we get the exact solutions (3.1) to (3.3).

B. Case 2

In this case, the solutions in the free flow regions 1 and 3 are the same as (A.1) and (A.5) with

$$C = -\frac{P_1 h_1}{2} A_1 \quad (B.1)$$

$$C' = -\frac{P_1 h_1}{2} A_3 \quad (B.2)$$

while the solution for the porous region 2 satisfying $u_1 = \phi u_2$ is

$$u_2 = -\frac{k}{\phi} P_1 - \frac{P_1 h_1^2}{2\phi} \exp\left[-\frac{(h_2-y)}{\sqrt{k}}\right] A_2 . \quad (B.3)$$

The solution (B.3) is obtained in the limit $(h_2-y)\rightarrow\infty$ and it represents the boundary layer solution, where the first term, which is the Darcy velocity, is valid in the core outside the boundary layer of thickness $O(\sqrt{k})$, [see Appendix C] and the second term is the inner solution valid in the boundary layer region, adjacent to the nominal surface.

The BJR-slip conditions (2.8) and (2.17) follows respectively from

$$\frac{du_1}{dy} = \lambda\phi \frac{du_2}{dy} \text{ at } y = h_2 \quad (B.4)$$

and

$$\frac{du_3}{dy} = \lambda\phi \frac{du_2}{dy} \text{ at } y = h_1 \quad (B.5)$$

using (B.3) on the right hand side. As in Appendix A, we get the exact solutions (4.1) to (4.3).

C. Boundary Layer

The boundary layer development will depend on the entry flow. Therefore, to find the boundary layer thickness in the porous medium, we use the following basic equations

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\mu}{k} u + \mu \frac{\partial^2 u}{\partial y^2} \quad (C.1)$$

$$\frac{\partial u}{\partial x} + \frac{dv}{dy} = 0 . \quad (C.2)$$

The boundary layer thickness for the present problem can be obtained from the solution of (C.1) as $x\rightarrow\infty$.

The equation (C.1), using (C.2), may be written as

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) = -\frac{\partial p}{\partial x} - \frac{\mu}{k} u + \mu \frac{\partial^2 u}{\partial y^2} \quad (C.3)$$

Integrating this from 0 to δ, where δ is the boundary layer thickness, and using $[\int uv]_{y=\delta} = 0$, we get

$$\frac{d}{dx}\int_0^\delta \rho u(u-u_\infty)dy + \frac{du_\infty}{dx}\int_0^\delta \rho u dy + \frac{\partial p}{\partial x}\delta = -\frac{\mu}{k}\int_0^\delta u dy - \mu \left.\frac{\partial u}{dy}\right|_{y=0} \quad (C.4)$$

where $u = u_\infty(x)$ is the velocity for the main flow and satisfies the equation

$$\rho u_\infty \frac{du_\infty}{dx} = -\frac{dp_\infty}{dx} - \frac{\mu}{k} u_\infty . \quad (C.5)$$

Assuming the boundary layer approximation $p = p_\infty$, and combining (C.4) and (C.5), we get

$$\frac{d}{dx}\int_0^\delta \rho u(u-u_\infty)dy + \frac{du_\infty}{dx}\int_0^\delta \rho(u-u_\infty)dy = -\frac{\mu}{k}\int_0^\delta (u_\infty u)dy - \mu \left.\frac{\partial u}{dy}\right|_{y=0} \quad (C.6)$$

Neglecting the variations in u_∞ in (C.6) and assuming the velocity profile as

$$\frac{u}{u_\infty} = 3/2\left(\frac{y}{\delta}\right) - 1/2\left(\frac{y}{\delta}\right)^3 \quad (C.7)$$

the equation (C.6) takes the form

$$\frac{d}{dx}(\delta^2) = B - A\delta^2 \quad (C.8)$$

where

$$A = \frac{70}{13}\frac{\gamma}{u_\infty}\frac{1}{k}$$

$$B = \frac{280}{13}\frac{\gamma}{u_\infty}$$

Solving (C.8), we get

$$\delta^2 = 4k\left[1-\exp\left\{-\frac{70}{13}\ \frac{\gamma}{u_\infty}\ \frac{x}{k}\right\}\right]. \qquad (C.9)$$

This is for the entry region. In the present problem, we deal with the fully developed flow. For this we get the boundary layer thickness as $x \to \infty$. Hence, the boundary layer thickness, denoted by δ_∞, is given by

$$\delta_\infty^2 = 4\ k$$

$$\delta_\infty = 2\sqrt{k}$$

In other words, the boundary layer thickness is of order $\sqrt{k}$, as stated in the text.

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