ON THE EXACT SOLUTION OF A SYSTEM OF LINEAR HOMOGENEOUS EQUATIONS *via* A PROJECTIVE ALGORITHM

Mustafa Akgül

Department of Industrial Engineering
Bilkent University
P.O.B. 8, Maltepe 06572 Ankara, Turkey

In a recent paper De Ghellinck and Vial [1] proposed a projective algorithm to solve a system of linear homogeneous equations: find $x \in \mathfrak{R}^n$ satisfying:

$$(P1)\qquad \begin{aligned} Ax &= 0 \\ ex &= n \\ x &\geq 0\,, \end{aligned}$$

where $A \in \mathfrak{R}^{m\times n}$, $x \in \mathfrak{R}^n$ and e is the row vector of ones. Let L be the length of the input as defined in [2]. Their main algorithm, which is a variant of the projective algorithm [3] and will be called $\mathcal{A}_1$, takes $\varepsilon > 0$ as part of the input and delivers in time polynomial in m, n, L, and $\log 1/\varepsilon$, one of the following:

(*i*) an exact solution of $P1$,

(*ii*) a proof that $P1$ has no solution,

(*iii*) an ε-approximate solution, *i.e.*, an x satisfying

$$|a_i x| < \varepsilon \ \forall i \tag{1}$$

$$ex = n \tag{2}$$

$$x \geq 0\,. \tag{3}$$

Their algorithm is based on the perturbation lemma of Gács–Lovász [4].

Lemma: For $\varepsilon > 0$ sufficiently small, the system

$$a_i x \leq b_i,\ \forall i \tag{4}$$

has a solution if and only if the system

$$a_i x < b_i + \varepsilon,\ \forall i \tag{5}$$

has a solution.

In order to determine an exact solution, more precisely a basic feasible solution (bfs), of $P1$, they call the algorithm $\mathcal{A}_1$ n times. At each iteration they attempt to fix a variable x_j to zero.

It is interesting to note that the De Ghellinck–Vial algorithm for finding an exact solution to a system of linear homogeneous equations is exactly Hachiyan's algorithm [2, 5] for finding an exact solution to a system of linear inequalities. Recall that Hachiyan solved the decision problem (Is $S = \{x : Ax \leq b\} \neq \emptyset$, a yes or no question), in proving $LP \in \mathcal{P}$. Since Hachiyan did not have Gács–Lovász's Perturbation Lemma, he used the ellipsoidal algorithm as a subroutine to find an exact solution of a system of linear equations. Similarly, De Ghellinck–Vial used their main algorithm to find an exact solution of a system of homogeneous linear equations.

Earlier, we presented [6] an interpretation of the perturbation lemma within the simplex algorithm framework. Given $\bar{x}$ satisfying $\{x : a_i x \leq b_i + \varepsilon\ \forall i\}$, one finds a bfs $\tilde{x}$ of it in at most n simplex pivots (see also [7]). Basis or hyperplanes defining $\tilde{x}$ will give a bfs $\hat{x}$ of the unperturbed system (4). Actually, this idea is first used by Maurras *et al.* [8], to find an exact solution from an ε-approximate solution for totally unimodular systems.

As usual we assume that A is an integral matrix. Consequently there exists a positive integer $q = O(2^L)$ satisfying:

(*i*) For every basis B of $\left[\frac{A}{e}\right]$ we have

$$|(B^{-1})_{i,j}| \leq q\,,\quad |\det B| \leq q \tag{6}$$

and

$$(B^{-1})_{i,j} \neq 0 \Rightarrow |(B^{-1})_{i,j}| \geq 1/q . \quad (7)$$

(ii) For any basic solution $\bar{x}$ of

$$Ax = 0 , \; ex = n ,$$

we have

$$|\bar{x}_j| \leq q \text{ and } \bar{x}_j \neq 0 \Rightarrow |\bar{x}_j| \geq 1/q . \quad (8)$$

Given $\varepsilon > 0$ and $\bar{x} \in \mathfrak{R}^n$ satisfying (1–3), let us transform (1–3) to

$$a_i x + u_i = \varepsilon \qquad i = 1 \ldots m \quad (9)$$

$$-a_i x + v_i = \varepsilon \qquad i = 1 \ldots m \quad (10)$$

$$ex = n \quad (11)$$

$$z = (x, u, v) \geq 0 . \quad (12)$$

Let us rewrite (9–12) compactly as

$$\tilde{A} z = \tilde{b}, \quad z \geq 0 \quad (13)$$

with $\tilde{b}^T = (\varepsilon\bar{e}, \varepsilon\bar{e}, n) \in \mathfrak{R}^{2m+1}$ and $\bar{e}$ is an m-vector of ones. Clearly, $\exists \bar{u} \in \mathfrak{R}^m$, $\bar{v} \in \mathfrak{R}^m$ such that $\bar{z} = (\bar{x}, \bar{u}, \bar{v})$ is a solution of (13).

Lemma: Given a solution $\bar{z}$ of (13) there exists a polynomial algorithm, say $\mathcal{A}_2$, which finds a basis $\tilde{B}$ of $\tilde{A}$ and a basic feasible solution $\tilde{z}$ such that $\tilde{z}_{\tilde{B}} = (\tilde{x}, \tilde{v}, \tilde{v})_{\tilde{B}} = (\tilde{B})^{-1}\tilde{b}$.

This result is, in fact, a constructive proof of the fundamental theorem of linear programming and goes back to Dantzig [9]. The algorithm $\mathcal{A}_2$ can be viewed as a simplex-type algorithm or a projection algorithm. For a formal proof of this result see reference [7]. $\mathcal{A}_2$ requires, in general, dim $z = 2m + n$ pivots. However, for ε sufficiently small it requires n pivots.

Let $b^T = (0,0,n)$ and define a basic solution of $\tilde{A}z = b$ by

$$\hat{z} = (\hat{x}, \hat{u}, \hat{v}) = (\tilde{B})^{-1} b , \quad (14)$$

and, of course, $\hat{z}_{\tilde{N}} = 0$, where $\tilde{N}$ is the set of all non-basic variables. Then we have:

Theorem: For $\varepsilon < \dfrac{1}{2mq}$,
$\hat{x}$ is a basic feasible solution of $P1$.

Proof: It suffices to show that $\hat{z}$ is a basic feasible solution of $\tilde{A}z = b$ and $\hat{u} = \hat{v} = 0$. It is easy to see that $\hat{x}$ or $\hat{z}$ satisfies the equation $ex = n$. Clearly for j non-basic $\tilde{z}_j = \hat{z}_j = 0$. For j basic, it suffices to show that

$$|\hat{z}_j - \tilde{z}_j| < 1/q_o , \quad (15)$$

where q_o is $|\det B|$. This follows since if $\hat{z}_j < 0$, then $\hat{z}_j \leq -1/q_o$, by integrality of $\hat{A}$, b and definition of q_o. But, this contradicts (15) since $\tilde{z}_j \geq 0$. Let $B_j^{\#}$ denote the matrix obtained from B by replacing j'th column with $\tilde{b} - b$. By Cramer's rule,

$$\alpha_j \equiv (\det B)(\tilde{z}_j - \hat{z}_j) = \det B_j^{\#} . \quad (16)$$

Then, by expanding $B_j^{\#}$ along jth column, and using (6) we have

$$|\alpha_j| \leq 2m\,\varepsilon\, q < 1 \quad (17)$$

by definition of ε. Clearly (17) implies (15) *via* (6). Thus $\hat{z}$ is a basic feasible solution of $\tilde{A}z = b$.

We need to show that for basic u_j (similarly for v_j) $\hat{u}_j = 0$. Since $\hat{z}$ is a basic feasible solution of $\hat{A}z = b$, we have $u_j \geq 0$, $\hat{v}_j \geq 0$. But, in every solution of $\tilde{A}z = b$, $u_j + v_j = 0$, (by adding (9) and (10)), which implies $\hat{u}_j = \hat{v}_j = 0$. This completes the proof of the theorem.

Thus it suffices to call $\mathcal{A}_1$ once with $\varepsilon < 1/2mq$, followed by a call to $\mathcal{A}_2$, to obtain an exact solution of $P1$.

REFERENCES

[1] G. De Ghellinck and J. -PH. Vial, "An Extension of Karmarkar's Algorithm for Solving a System of Linear Homogeneous Equations on the Simplex", *Mathematical Programming,* **39** (1987), p. 79.

[2] L. G. Hachiyan, "Polynomial Algorithms in Linear Programming", *U.S.S.R. Computational Mathematics and Mathematical Physics,* **20** (1980), p. 191.

[3] N. Karmarkar, "A New Polynomial Algorithm for Linear Programming", *Combinatorica,* **4** (1984), p. 373.

[4] P. Gács and L. Lovász, "Khachiyan's Algorithm for Linear Programming", *Mathematical Programming Study,* **14** (1981), p. 61.

[5] L. G. Hachiyan, "On Exact Solution of Systems of Linear Inequalities and lp Problems", *U.S.S.R. Computational Mathematics and Mathematical Physics,* **22** (1982), p. 999.

[6] M. Akgül, *Topics in Relaxation and Ellipsoidal Methods.* London: Pitman, 1984; Research Notes in Mathematics, vol. 97.

[7] M. Akgül, "An Algorithmic Proof of the Polyhedral Decomposition Theorem", *Naval Research Log. Quarterly,* **35** (1988), p. 463.

[8] J. F. Maurras, K. Truemper, and M. Akgül, "Polynomial Algorithms for a Class of Linear Programs", *Mathematical Programming,* **21** (1981), p. 121.

[9] G. B. Dantzig, *Linear Programming and Extensions.* Princeton University Press, 1963.

Paper Received 11 November 1989; Revised 1 May 1990.