ON MIXED BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS IN DOMAINS WITH CORNERS

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الخلاصـــة :

يهتم هذا البحث بحلول المسألة المزدوجة للمعادلة الاهليليجية الخطية في المجال المستـوي Ω ذي الأركان . وقـد أعطيت شروط كـافيـة لجعـل الحـل من المـرتبـة $C_{m+2+\alpha}(\bar{\Omega})$, $0<\alpha<1$, $m\geq 0$.

ABSTRACT

This paper concerns the mixed problem for linear elliptic equations in a plane domain Ω with corners. Conditions, sufficient for the solutions to be of class $C_{m+2+\alpha}(\bar{\Omega})$ are given, $m\geq 0$, $0<\alpha<1$.

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1. INTRODUCTION

In a simply connected bounded domain $\Omega \subset \mathbb{R}^2$, we consider a mixed boundary value problem for the equation

$$Lu = a_{ij}(x) u_{x_i x_j} + a_i(x) u_{x_i} + a(x) u = f(x) \ , \quad (1)$$

where $x = x_1, x_2$, $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and we use the summation convention. General boundary value problems for the uniformly elliptic equation (1) in smooth domains, is thoroughly investigated; see [1]. We state here a known result which we shall need later on. Consider the function $u(x)$ that satisfies (1) in Ω. On $\partial\Omega$, the boundary of Ω, the function u satisfies the additional mixed condition

$$\alpha(s) u + \beta(s) \frac{\partial u}{\partial \nu} = \alpha(s)\phi(s) + \beta(s)\psi(s) \quad (2)$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u. It is known from [1], that if the coefficients and the right hand side of (1) belong to $C_{m+\alpha}(\bar{\Omega})$, $\partial\Omega$ can be represented by $C_{m+2+\alpha}$ functions and if $\alpha(s)$ and $\phi(s) \in C_{m+2+\alpha}(\partial\Omega)$ while $\beta(s)$ and $\psi(s) \in C_{m+1+\alpha}(\partial\Omega)$ then

$$u \in C_{m+2+\alpha}(\bar{\Omega}) \ . \quad (3)$$

If $\partial\Omega$ contains corners, then (3) may not be true, and in this case, $u \in C_{m+2+\alpha}(\Omega_1)$, where $\Omega_1 \subset \bar{\Omega}$ is any region with positive distances from the corner points. To investigate the smoothness of the Dirichlet problem for Equation (1) in domains with corners, a method was introduced in [2]. Let us assume, for simplicity, that there is a single corner point at 0 with interior angle, γ, $0<\gamma<2\pi$. We transform $a_{ij}(0) u_{x_i x_j}$, the principal part of (1) at 0, to the Laplacian. In doing so, the new angle formed at the corner point, is given by

$$\tan \omega = \frac{[a_{11}(0)\, a_{22}(0) \ - \ a_{12}^2(0)]^{1/2}}{a_{22}(0) \cot \gamma \ - \ a_{12}(0)} \ .$$

The regularity properties of the solution depend, among other factors, on the value of this angle. The method used in [2] was then modified to study the Dirichlet and mixed boundary value problems for elliptic equations, as well as initial–Dirichlet and initial–Mixed boundary value problems for parabolic equations. These problems were considered in plane domains with corners as well as in n-dimensional domains with edges, see [2–9] and the references mentioned there. We state here one result that will be needed in proving the main result of this paper.

Theorem 1 [6]. Consider the domain Ω with a single corner point on the boundary, located at 0. Let Γ_1 and Γ_2 be $C_{m+2+\alpha}$-curves that form at 0 the corner of interior angle γ, $0<\gamma<2\pi$. Consider the mixed boundary value problem

$$Lu = f \text{ in } \Omega \quad (4)$$

$$u = 0 \text{ on } \Gamma_1 \quad (5a)$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2 . \quad (5b)$$

If the coefficients of (4) and its right hand side belong to $C_{m+\alpha}(\bar{\Omega})$ then

$$u \in C_\beta(\bar{\Omega}) \quad (6)$$

$$\beta = \min\{m+2+\alpha, \ \pi/(2\omega) - \varepsilon\} \geq 2 \quad (7)$$

$\varepsilon>0$ is arbitrarily small. If moreover $\frac{\partial^k f(0)}{\partial x_1 k_1 \ \partial x_2 k_2} = 0$, $k = 0,1,\ldots,[\beta]-2$, $\beta>2$, then

$$|u| \leq M r^\beta \ .$$

From this theorem, it follows that if ω is small enough such that

$$\pi/(2\omega) > m+2+\alpha \ , \quad (8)$$

then the solution will be as smooth as in the case of a smooth boundary. In this paper, we shall show that, there are other "exceptional" angles, that will allow the solution to belong to $C_{m+2+\alpha}(\bar{\Omega})$. This result is a generalization of the following result that concerns the Poisson equation in a straight sector.

Theorem 2 [10]. Consider the sector $\Omega_{r_o} = \{(r,\theta), \ r<r_o, \ 0<\theta<\omega\}$. In Ω_{r_o} consider the bounded solution u of the mixed problem

$$\Delta u = f \text{ in } \Omega_{r_o} \quad (9)$$

$$u = \phi \text{ on } \Gamma_1 = \{(r,\theta), \ r<r_o, \ \theta = 0\} \quad (10a)$$

$$\frac{\partial u}{\partial \nu} = \psi \text{ on } \Gamma_2 = \{(r,\theta), \ r<r_o, \ \theta = \omega\} \quad (10b)$$

If $f \in C_{m+\alpha}(\bar{\Omega}_{r_o})$, $\phi \in C_{m+2+\alpha}(\bar{\Gamma}_1)$, and $\psi \in C_{m+1+\alpha}(\bar{\Gamma}_2)$, and if $\omega = \pi/(2q)$, $q = 2,3,\ldots.$ then $u \in C_{m+2+\alpha}(\bar{\Omega}_{r_1})$, where $r_1 < r_o$, provided that at the corner, the compatibility conditions imposed by (9)−(10) are satisfied.

We extend this result for problem (4)−(5) as follows.

Theorem 3. Let u be a bounded solution of (4)−(5). Let the assumptions of Theorem 1 be satisfied. If $\omega = \pi/(2q)$, $q = 2,3,\ldots$ then $u \in C_{m+2+\alpha}(\bar{\Omega})$, provided the compatibility conditions imposed by (4)−(5) are satisfied at the corner point.

The plan of proving this theorem is as follows. We first consider the problem in a special setting, we prove the required result there, and then show that problem (4)−(5) in Ω can be transformed to the special setting by a "smooth" invertible transformation.

The Problem in a Special Setting

Consider the sector $\Omega_{r_o} = \{(r,\theta),\ r<r_o,\ 0<\theta<\omega\}$. In Ω_{r_o} we consider the mixed problem

$$Lu = f \text{ in } \Omega_{r_o} \tag{11}$$

$$u = 0 \text{ on } \theta = 0,\ r<r_o \tag{12a}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \theta = \omega,\ r<r_o \tag{12b}$$

where

$$Lu \equiv a_{ij}(x) u_{x_i x_j} + a_i(x) u_{x_i} + a(x) u ,$$

is uniformly elliptic. From Theorem 1, it follows that if a_{ij}, a_i, a, and f belong to $C_{m+\alpha}(\bar{\Omega}_{r_o})$, then $u \in C_\beta(\bar{\Omega}_{r_1})$ where $r_1 < r_o$ and $\beta = \min\{m+2+\alpha,\ \pi/(2\omega)-\varepsilon\}$, $\varepsilon>0$ is arbitrarily small. We are now interested in the case when $\pi/(2\omega) \le m+2+\alpha$, in which case $u \in C_{\pi/(2\omega)-\varepsilon}(\bar{\Omega}_{r_1})$. We would like to improve this last smoothness result when $\pi/(2\omega) = q$, $q = 2,3,\ldots$.

We now state and prove a result analogous to that of Theorem 3, but in a special setting.

Theorem 4. Let u be a bounded solution of (11)−(12) in Ω_{r_o} and assume that a_{ij}, a_i, a, and f belong to $C_{m+\alpha}(\bar{\Omega}_{r_o})$ and assume that:

$$a_{ij}(0) = \delta_{ij},\ i,j = 1,2 , \tag{13}$$

where δ_{ij} is the Kronecker delta. If $\omega = \pi/(2q)$, $q = 2,3,\ldots$ then $u \in C_{m+2+\alpha}(\bar{\Omega}_{r_o})$, provided that the compatibility conditions at 0, imposed by (11) and (12) are satisfied.

To prove this result, we need two lemmas in which more properties of solutions of (11)−(12) will be established.

Lemma 1. Let $u(x)$ be a bounded solution of (11)−(12) in Ω_{2r_o} where a_{ij}, a_i, a, and f belong to $C_{p+\alpha}$ there, and

$$D^k f|_{x=o} = 0,\quad k = 0,1,\ldots,p , \tag{14}$$

where $p \le m$ and $D^k f$ is any partial derivative of f of order k. If for some λ, $\lambda \le p+2+\alpha$ we have

$$|u(x)| \le M r^\lambda,\quad r<2r_o \tag{15}$$

then in Ω_{r_o} we have

$$|D^k u(x)| \le K r^{\lambda-k};\quad r<r_o,\ k = 1,2,\ldots,p+2 . \tag{16}$$

Proof. In the proof we use a method introduced in [5]. In Ω_{2r_o} consider the domains

$$\Omega_n = \{(r,\theta)\colon \frac{r_o}{2^{n+1}} \le r \le \frac{r_o}{2^n},\ 0\le\theta\le\omega\}$$

$$\Omega_n' = \Omega_{n-1} \cup \Omega_n \cup \Omega_{n+1} ;\quad n = 1,2,\ldots .$$

We denote by Γ_n' and Γ_n'' the straight parts of the boundary of Ω_n' with $\theta = 0$ and $\theta = \omega$ respectively.

Consider the transformation:

$$x_i = 1/2^n\, y_i ,\quad i = 1,2 . \tag{17}$$

This transformation maps Ω_n and Ω_n' to Ω_o and Ω_o' respectively. In Ω_o', the function $U(y) = u(1/2^n y)$; $y = y_1, y_2$, satisfies the uniformly elliptic equation:

$$A_{ij}(y) U_{y_i y_j} + 2^{-n} A_i(y) U_{y_i} + 2^{-2n} A(y) U = 2^{-2n} F(y),$$

where $A_{ij}(y) = a_{ij}(1/2^n y)$ and similarly A_i, A, and F may be defined in terms of a_i, a, and f. The function U satisfies the boundary conditions:

$$U = 0 \text{ on } \Gamma_o'$$

$$\frac{\partial U}{\partial \nu} = 0 \text{ on } \Gamma_o'' .$$

In Ω_o and Ω_o' we apply the Schauder inequality to get:

$$\|U\|^{\Omega_o}_{p+2+\alpha} \le C[\|U\|^{\Omega_o'}_o + 2^{-2n}\|F\|^{\Omega_o'}_{p+\alpha}]. \tag{18}$$

We now estimate the right hand side of this inequality. It is clear that:

$$\|U\|^{\Omega_o'}_o = \|u\|^{\Omega_n'}_o \le M_1 r^\lambda, \text{ (see (15))}.$$

Since $F(y) = f(1/2^n y)$, then $D^k_y F = (1/2^n)^k D^k_x f$ and it follows from (14) that

$$|D_x^p f(x)| = |D_x^p f(x) - D_x^p f(0)| \le K_p r^\alpha$$

$$|D_x^{p-1} f(x)| \le \int_0^x |D^p f(t)| \mathrm{d}t \le K_{p-1} r^{1+\alpha}$$

and generally

$$|D_x^k f(x)| \le K_k r^{p-k+\alpha}, \quad k = 0,1,\ldots,p \tag{19}$$

Thus in Ω'_o we have

$$|D_y^k F(y)| = |(1/2^n)^k D_x^k f(x)|, \quad k = 0,1,\ldots,p. \tag{20}$$

If $P(r_1,\theta_1)$ and $Q(r_2,\theta_2)$ are any two points in Ω'_n and their images are $P_o(r'_1,\theta_1)$ and $Q_o(r'_2,\theta_2)$, then $d(P_o, Q_o) = 2^n d(P,Q)$ and

$$\frac{D_y^p F(P_o) - D_y^p F(Q_o)}{[d(P_o, Q_o)]^\alpha} = (1/2^n)^{p+\alpha} \frac{D_x^p f(P) - D_x^p f(Q)}{[d(P,Q)]^\alpha}. \tag{21}$$

Thus from (20), (21), and from

$$\|F\|_{p+\alpha}^{\Omega'_o} = \sum_{k=o}^{p} \|D_y^k F\|_o + H_\alpha(D_y^p F),$$

where $H_\alpha^\Omega(g)$ is the Holder coefficient of g of exponent α in Ω, it follows that

$$\|F\|_{p+\alpha}^{\Omega'_o} \le C_2 (1/2^n)^{p+\alpha} \|f\|_{p+\alpha}^{\Omega'_n} \le C_3 (1/2^n)^{p+\alpha}.$$

Thus, noting that $\lambda \le p+2+\alpha$, we get from (18)

$$\|U\|_{p+2+\alpha}^{\Omega_o} \le C_4[M_1 r^\lambda + 2^{-2n} K_o (1/2^n)^{p+\alpha}] \le C_5 (1/2^n)^\lambda.$$

Since

$$|D_y^k U(y)| \le \|U\|_{p+2+\alpha}^{\Omega_o}, \quad k \le p+2$$

and

$$D_y^k U(y) = (1/2^n)^k D_x^k u(x)$$

then

$$(1/2^n)^k |D_x^k u(x)| \le C_5 (1/2^n)^\lambda.$$

Thus

$$|D_x^k u(x)| \le C_5 (1/2^n)^{\lambda-k}$$

or equivalently

$$|D_x^k u(x)| \le K r^{\lambda-k}. \ \square$$

This proves the lemma.

In the next lemma we are going to show that the singularity of the solutions of (11)−(12) is of pole-type; multiplying the solution by $O(r^\mu)$ functions, removes "some" of its singularities at 0.

Lemma 2. Let $u(x)$ be a bounded solution of the mixed problem

$$Lu = f(x) \text{ in } \Omega_{r_o} \text{ (see (11))}$$

$$u = \phi(r) \text{ on } \Gamma'_1\text{: } \theta = 0,\ r < r_o$$

$$\frac{\partial u}{\partial \nu} = \psi(r) \text{ on } \Gamma'_2\text{: } \theta = \omega,\ r < r_o,$$

where a_{ij}, a_i, a, and f belong to $C_{m+\alpha}(\bar{\Omega}_{r_o})$, $\phi \in C_{m+2+\alpha}(\Gamma'_1)$ and $\psi \in C_{m+1+\alpha}(\Gamma'_2)$. Assume also that for some integer p, $0 \le p \le m$ we have

$$\frac{d^k}{dr^k} \phi(r)|_{r=o} = 0, \quad k = 0,1,\ldots,p+2$$

$$\frac{d^k}{dr^k} \psi(r)|_{r=o} = 0, \quad k = 0,1,\ldots,p+1$$

$$D^k f(x)|_{x=o} = 0, \quad k = 0,1,\ldots,p.$$

Assume that $u \in C_{p+2-\varepsilon}(\bar{\Omega}_{r_o})$ and $|D^{p+2}u(x)| \le M r^{-\varepsilon}$, where $0 < \varepsilon \le 1$. Then for any function $h(x) \in C_\mu(\bar{\Omega}_{r_o})$, $\varepsilon \le \mu \le 1$ with $h(0) = 0$, we have

$$h(x) D^{p+2} u(x) \in C_{\mu_o}(\bar{\Omega}_{r_o})$$

where

$$\mu_o = \begin{cases} \mu - \varepsilon & \text{if } p < m \\ \min(\alpha, \mu - \varepsilon) & \text{if } p = m \end{cases}$$

Proof. Consider any two points $P(r_1,\theta_1)$ and $Q(r_2,\theta_2)$ in Ω_{r_o} and suppose that $0 \le r_2 \le r_1 \le r_o$. If $r_2 \le ½ r_1$, then $d(P,Q) \ge ½ r_1$ and:

$$\frac{|h(P) D^{p+2}u(P) - h(Q) D^{p+2}u(Q)|}{[d(P,Q)]^{\mu_o}} \le \frac{M_2 r_1^\mu r_1^{-\varepsilon} + M_2 r_2^\mu r_2^{-\varepsilon}}{(½ r_1)^{\mu_o}} \le M. \tag{20}$$

To prove an inequality of the form (20) for the case when $r_2 > ½ r_1$, we first prove that in this case

$$\frac{|D^{p+2}u(P) - D^{p+2}u(Q)|}{[d(P,Q)]^{\mu_o}} \le M r^{-\mu_o - \varepsilon}. \tag{21}$$

Consider the transformation $x_i = \dfrac{2r_1}{r_o} y_i$, $i = 1,2,.$ This transformation maps Ω_o and Ω'_o to Ω_1 and Ω'_1 respectively, where:

$$\Omega_o = \{(r,\theta):\ r_1/2 \le r \le r_1,\ 0 \le \theta \le \omega\}$$

$$\Omega'_o = \{(r,\theta):\ r_1/4 \le r \le 2r_1,\ 0 \le \theta \le \omega\}$$

$$\Omega_1 = \{(r,\theta):\ \tfrac14 r_o \le \rho \le \tfrac12 r_o,\ 0 \le \theta \le \omega\}$$

$$\Omega'_1 = \{(r,\theta):\ \tfrac18 r_o \le \rho \le r_o,\ 0 \le \theta \le \omega\}\ ,$$

and $\rho = \dfrac{r_o}{2r_1} r$. In Ω'_1, the function $V(y) = u\left(\dfrac{2r_1}{r_o} y\right)$ satisfies the elliptic equation

$$B_{ij}(y) V_{y_i y_j} + \left(\frac{2r_1}{r_o}\right) B_i(y) V_{y_i} + \left(\frac{2r_1}{r_o}\right)^2 B(y) V = \left(\frac{2r_1}{r_o}\right)^2 F(y)$$

and also satisfies the boundary conditions

$$V = \phi_o \qquad \text{on } \tilde{\Gamma}_1:\ 0 \le \rho \le 2r_1,\ \theta = 0$$

$$\frac{\partial V}{\partial \nu} = \frac{2r_1}{r_o}\psi_o \quad \text{on } \tilde{\Gamma}_2:\ 0 \le \rho \le 2r_1,\ \theta = \omega$$

In Ω_1 and Ω'_1, Schauder's inequality yields

$$\|V\|^{\Omega_1}_{p+2+\mu_o} \le C\left[\|V\|^{\Omega'_1}_o + \left(\frac{2r_1}{r_o}\right)^2 \|F\|^{\Omega'_1}_{p+\mu_o} + \|\phi_o\|^{\tilde{\Gamma}_1}_{p+2+\mu_o} + \left(\frac{2r_1}{r_o}\right)\|\psi_o\|^{\tilde{\Gamma}_2}_{p+1+\mu_o}\right). \quad (22)$$

As in the proof of Lemma 1, we get

$$\|V\|^{\Omega'_1}_o = \|u\|^{\Omega'_o}_o \le M_5 r_1^{p+2-\varepsilon}$$

and

$$\|F\|^{\Omega'_1}_{p+\mu_o} \le M_6 r_1^{p+\alpha}$$

and similarly we can prove that

$$\|\phi_o\|^{\tilde{\Gamma}_1}_{p+2+\mu_o} \le M_7 r_1^{p+2+\mu_o}$$

$$\|\psi_o\|^{\tilde{\Gamma}_2}_{p+1+\mu_o} \le M_8 r_1^{p+1+\mu_o}.$$

Thus (22) gives

$$\|V\|^{\Omega_1}_{p+2+\mu_o} \le C_o[r_1^{p+2-\varepsilon} + r_1^{p+2+\mu_o}] \le C_1 r_1^{p+2-\varepsilon}.$$

We now return to the x-coordinates. We first note that

$$\left(\frac{2r_1}{r_o}\right)^{p+2} |D_x^{p+2} u| = |D_y^{p+2} V| \le \|V\|^{\Omega_1}_{p+2+\mu_o}$$

and that

$$\left(\frac{2r_1}{r_o}\right)^{p+2+\mu_o} H^{\Omega_o}_{\mu_o}(D_x^{p+2} u) = H^{\Omega_1}_{\mu_o}(D_y^{p+2} V) \le \|V\|^{\Omega_1}_{p+2+\mu_o}$$

where by $H^{\Omega}_{\mu}(W)$ we denote the Hölder coefficient of exponent μ of the function W in Ω. Thus

$$|D_x^{p+2} u| \le c_2 r_1^{-\varepsilon}$$

and

$$H^{\Omega_o}_{\mu_o}(D_x^{p+2} u) \le c_3 r_1^{-\mu_o-\varepsilon},$$

and finally we get in the case when $r_2 > \tfrac12 r_1$

$$\frac{|h(P)\, D_x^{p+2}u(P) - h(Q)\, D_x^{p+2}u(Q)|}{[d(P,Q)]^{\mu_o}} \le \frac{|h(P)|\,|D_x^{p+2}u(P) - D_x^{p+2}u(Q)|}{[d(P,Q)]^{\mu_o}} + |D_x^{p+2}u(Q)|\left\{\frac{|h(P)-h(Q)|}{[d(P,Q)]^{\mu}}\right\}^{\mu_o/\mu} \times |h(P)-h(Q)|^{1-(\mu_o/\mu)}$$

$$\le M_9 r_1^{\mu} \,.\, c_3 r_1^{-\mu_o-\varepsilon} + c_2 r_2^{-\varepsilon} \,.\, M_{10}(r_1^{\mu} + r_2^{\mu})^{(\mu-\mu_o)/\mu} \le M_{11}\ ,$$

since $\quad \mu_o \le \mu - \varepsilon$ and $\tfrac12 r_1 < r_2 < r_1$.

Thus $h\, D_x^{p+2} u \in C_{\mu_o}(\bar{\Omega}_{r_o})$. This concludes the proof of the lemma.

Proof of Theorem 4

We first consider the case when $\pi/(2\omega) = q = 2$. Thus it follows from Theorem 1 that $u \in C_{2-\varepsilon}(\bar{\Omega}_{r_o})$. We shall prove that $u \in C_{m+2+\alpha}(\bar{\Omega}_{r_o})$. Consider the function $v = u - u_o$, where

$$u_o = \tfrac12 f(0) x_2(x_2 + 2x_1).$$

The function u_o satisfies the conditions

$$u_o = 0 \qquad \text{on } \Gamma_1:\ \theta = 0$$

$$\frac{\partial u_o}{\partial \nu} = 0(r) \quad \text{on } \Gamma_2:\ \theta = \omega$$

and

$$\Delta u_o = f(0).$$

Thus

$$Lu_o = \Delta u_o + [a_{ij}(x) - \delta_{ij}]\frac{\partial^2 u_o}{\partial x_i \partial x_j} + a_i(x)\frac{\partial u_o}{\partial x_i} + a(x) u_o = f(0) + g(x), \text{ where } g(0) = 0.$$

Thus v is a solution of the mixed problem

$$Lv = F_1(x) \tag{23}$$

$$v = 0 \text{ on } \Gamma_1 \tag{24a}$$

$$\frac{\partial v}{\partial \nu} = 0(r) \text{ on } \Gamma_2, \tag{24b}$$

where

$$F_1(x) = f(x) - f(0) - g(x), \quad F_1(0) = 0.$$

Thus it follows from [6], that $v \in C_{2-\varepsilon}(\bar{\Omega}_{r_o})$ and that $|v| \leq M r^{2-\varepsilon}$ in $\bar{\Omega}_{r_o}$. Applying Lemma 1, we get

$$|D^k v| \leq M r^{2-\varepsilon-k}, \quad k = 1,2.$$

We rewrite (23) in the form

$$\Delta v = f_1(x) = F_1(x) - [a_{ij}(x) - \delta_{ij}] v_{x_i x_j} - a_i(x) v_{x_i} - a(x) v. \tag{25}$$

To prove that $v \in C_{k+2+\alpha}(\Omega_{r_o})$, it follows from Theorem 2 that, it is sufficient to show that $f_1 \in C_{k+\alpha}(\bar{\Omega}_{r_o})$, $k \leq m$. It is clear that $F_1 - a(x)v - a_i(x) v_{x_i} \in C_\alpha(\bar{\Omega}_{r_o})$. It remains to show that $[a_{ij}(x) - \delta_{ij}] v_{x_i x_j} \in C_\alpha$. This we do in two steps. It is clear that $h_{ij}(x) \equiv a_{ij}(x) - \delta_{ij} \in C_\alpha$ and $h_{ij}(0) = 0$. Thus applying Lemma 2, we get $h_{ij}(x) v_{x_i x_j} \in C_{\alpha-\varepsilon}(\bar{\Omega}_{r_o})$. Thus $f_1 \in C_{\alpha-\varepsilon}(\bar{\Omega}_{r_o})$ and from Theorem 2, it follows that $v \in C_{2+\alpha-\varepsilon}(\bar{\Omega}_{r_o})$. Substituting v in the right hand side of (25), we can show, as before, that now $f_1 \in C_\alpha(\bar{\Omega}_{r_o})$ which gives $v \in C_{2+\alpha}(\bar{\Omega}_{r_o})$. Thus $u \in C_{2+\alpha}(\bar{\Omega}_{r_o})$.

We now use mathematical induction to complete the proof of the theorem when $q = 2$. Assume that u has been proved to belong to $C_{p+\alpha}(\bar{\Omega}_{r_o})$; $2 \leq p \leq m+1$, $|D^k u| \leq M r^{p+\alpha-k}$, $k = 0,1,\dots,p+1$. Let $u = T_p(x) + R_p(x)$, where T_p is the Mclaurin expansion of u up to and including powers of degree p and R_p is the remainder. The remainder $R_p(x)$ belongs to $C_{p+\alpha}$ and $|D^k R_p(x)| \leq M_o r^{p+\alpha-k}$, $k = 0,1,\dots,p+1$. The function R_p satisfies in Ω_{r_o} an equation of the form (11) which can be written as

$$\Delta R_p = f_p = F_{p-1} - [a_{ij}(x) - \delta_{ij}] \frac{\partial^2 R_p}{\partial x_i \partial x_j} - a_i(x) \frac{\partial R_p}{\partial x_i} - a(x) R_p$$

where

$$D^k f_p(x)|_{x=o} = 0, \quad k = 0,1,\dots,p-2$$

and

$$R_p|_{\Gamma_1} = \phi_p$$

$$\frac{\partial}{\partial \nu} R_p|_{\Gamma_2} = \psi_p$$

with

$$D_o^k \phi_p = 0, \quad k = 0,1,\dots,p$$

$$D_\omega^k \psi_p = 0, \quad k = 0,1,\dots,p-1.$$

As before to prove that $R_p \in C_{p+1+\alpha}$, it is required to show that $[a_{ij}(x) - \delta_{ij}] \dfrac{\partial^2 R_p}{\partial x_i \partial x_j} \in C_{p-1+\alpha}$. The other terms of f_p belong to $C_{p-1+\alpha}$. Since $a_{ij} \in C_{p-1+\alpha}$ then it remains to show that $D^{p-1}[a_{ij} - \delta_{ij}] \dfrac{\partial^2 R_p}{\partial x_i \partial x_j} \in C_\alpha$. This is equivalent to showing that $[a_{ij}(x) - \delta_{ij}] D^{p+1} R_p \in C_\alpha$. This follows from Lemma 2 since $|D^{p+1} R_p(x)| \leq M r^{\alpha-1}$ and $h_{ij} = a_{ij}(x) - \delta_{ij} \in C_1$, $h_{ij}(0) = 0$. Thus $f_p \in C_{p-1+\alpha}$ and Theorem 2 gives $R_p \in C_{p+1+\alpha}(\bar{\Omega}_{r_o})$. Thus $u \in C_{p+1+\alpha}$, $p = 2,3,\dots,m+1$. Theorem 4 is proved for the case $q = 2$. If $q > 2$, then it follows from Theorem 1 that $u \in C_{q-\varepsilon}(\bar{\Omega}_{r_o})$, where $\varepsilon > 0$ is arbitrarily small. As in the previous case, we can show first that $u \in C_{q+\alpha}(\bar{\Omega}_{r_o})$, then step by step we can reach $u \in C_{m+2+\alpha}(\bar{\Omega}_{r_o})$. The theorem is proved.

We now prove Theorem 3.

Proof of Theorem 3. Without loss of generality, we assume that the corner point is located at the origin $x = 0$ and we also assume that the two curves that form at 0 the corner of angle γ, are $x_2 = g_1(x_1)$ and $x_1 = g_2(x_2)$ where $g_i(x_i) \in C_{m+2+\alpha}$ and $g_2(0) = g_1(0) = g_1'(0) = 0$. To transform the equation $a_{ij}(0) \dfrac{\partial^2 u}{\partial x_i \partial x_j} = 0$ to canonical form, we use the transformation

$$y_1 = \frac{1}{\Lambda \sqrt{\alpha_{11}}} \{[\alpha_{12}[x_1 - g_2(x_2)] + \alpha_{11}[x_2 - g_1(x_1)]\}$$

$$y_2 = \frac{1}{\sqrt{\alpha_{11}}} [x_1 - g_2(x_2)]$$

where

$$\alpha_{11} = a_{11}(0) - 2g_2'(0) a_{12}(0) + g_2'^2(0) a_{22}(0)$$

$$\alpha_{12} = a_{22}(0) g_2'(0) - a_{12}(0)$$

$$\alpha_{22} = a_{22}(0)$$

$$\Lambda = [a_{11}(0) a_{22}(0) - a_{12}^2(0)]^{1/2}.$$

The domain N will be transformed to a domain G bounded by two straight segments Γ_1 and Γ_2 and a curve joining them. The new angle ω will be given by $\tan \omega = \dfrac{\Lambda}{\alpha_{12}}$. In G, the transformed function $v(y) = u(x)$ will satisfy an elliptic equation of the form (11) and will satisfy boundary conditions of the form (12), with all the conditions of Theorem 4 being satisfied. Thus, it can be proved that $u \in C_{m+2+\alpha}(\overline{\Omega}_{r_0})$ where $\Omega_{r_0} \subset G$ is a sector with vertex at 0 and radius $r_0 > 0$. Noting that the transformation used is of class C_{m+2+a} and its Jacobian at 0 has the value $-1/\Lambda$, we conclude that $u \in C_{m+2+a}(\overline{N})$, where $N = \{(x\colon x \in \Omega,\ |x| < \sigma_1\},\ \sigma_1 < \delta$. This proves the theorem.

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Condition", *Comm. Pure Appl. Math.*, **N12** (1959), p. 623.

[2] A. Azzam, "On the First Boundary Value Problem for Elliptic Equations in Regions Corners", *Arabian Journal for Science and Engineering*, **4** (1979), p. 129.

[3] A. Azzam, "Smoothness Properties of Bounded Solutions of Dirichlet Problem for Elliptic Equations in Regions with Corners on the Boundary", *Canadian Math. Bull.*, **23(2)** (1980), p. 213.

[4] A. Azzam, "On Differentiability Properties of Solutions of Elliptic Differential Equations", *Journal of Mathematical Analysis and Applications,* **75(2)** (1980), p. 431.

[5] A. Azzam, "Schauder's Estimations of the Solution of Dirichlet's Problem of Second Order Elliptic Equations in Sectionally Smooth Domains", *Bull. Moscow Univ. Math.*, **5** (1981), p. 29.

[6] A. Azzam, "Smoothness Properties of Solutions of Mixed Boundary Value Problems in Sectionally Smooth *n*-Dimensional Domains", *Annal. Polon. Math.*, **XL** (1981), p. 81.

[7] A. Azzam, "On Mixed Boundary Value Problems for Parabolic Equations in Singular Domains", *Osaka J. Math.*, **22** (1985), p. 691.

[8] P. Grisvard, *Elliptic Problems in Nonsmooth Domains.* Boston: Pitman, 1985.

[9] V. A. Kondratev and O. A. Oleinik, "Boundary Value Problems for Partial Differential Equations in Nonsmooth Domains", *Uspekh Mat. Nauk,* **38(2)** (1985), p. 3.

[10] E. A. Volkov, "Differential Properties of Solutions of Boundary Value Problems for the Laplace Equation on Polygons", *Trudy Mat. Inst. Steklov,* **77** (1965), p. 113.

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