THE ρ-QUASI-CENTER OF A BANACH ALGEBRA

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الخلاصـــة :

اعطينا في هذا البحث مثلا عن جبر باناخ A وعنصر شبه مركزي من نوع σ ولكنـة ليس شبه مركزي من نوع ρ ونتيجة لذلك حصلنا على الاشتـقاق الداخلي لجبر باناخ .

ABSTRACT

An example of a Banach algebra A and a σ-quasi central element of A which is not ρ-quasi central is given; also a result on inner derivation on A is obtained.

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INTRODUCTION

The purpose of this paper is to study the ρ-quasi center of a Banach algebra. The results obtained are applied to the study of inner derivation on a Banach algebra.

Throughout the paper, all linear spaces and algebras are assumed to be defined over the field C of complex numbers. If A is a Banach algebra we shall denote the center of A by $Z(A)=\{a\in A: ax=xa$ for every x in $A\}$. The resolvent set of an element x in A is $\rho_A(x)=\{\lambda\in C: (x-\lambda e)^{-1}$ exists$\}$. Also $C-\rho_A(x)$ is the spectrum of x and is denoted by $\sigma_A(x)$.

Lepage [1; proposition 3] proved the following result. If $a\in A$ satisfies

$\|x(\lambda-a)\|\leqslant\|(\lambda-a)x\|$
for all $x\in A$ and $\lambda\in C$,

then a lies in the center of A. (1)

From this result, Rennison [2–4] studied elements which satisfy a condition superficially very similar to (1).

In fact he called an element a of A quasi-central if, for some $k\geqslant 1$,

$(\|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$
for all $x\in A$ and $\lambda\in C$.

$Q(K,A)=\{a\in A: \|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$
for all x in A and $\lambda\in C\}$ and
$Q(A)=\bigcup_{K\geqslant 1} Q(K,A)$.

Similarly we define the set $Q_\sigma(A)$ of all σ-quasi central elements as follows:

$Q_\sigma(K,A)=\{a\in A: \|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$
for all $x\in A$ and $\lambda\in\rho_A(a)\}$ and
$Q_\sigma(A)=\bigcup_{K\geqslant 1} Q_\sigma(K,A)$.

This paper stems from these ideas. An element a of A will be called ρ-quasi central if for some $K\geqslant 1$,

$\|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$
for all $x\in A$ and $\lambda\in\sigma_A(a)$.

When $K\geqslant 1$, we shall write

$Q_\rho(K,A)=\{a\in A: \|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$
for every $x\in A$ and $\lambda\in\sigma_A(a)\}$ and
$Q_\rho(A)=\bigcup_{K\geqslant 1}(K,A)$.

Thus $Z(A)\subset Q(A)\subset Q_\rho(A)$.

THE ρ-QUASI CENTRAL ELEMENT

We start by noting an elementary property of the definition of Q_ρ.

Proposition 1.

$Q_\rho(K,A)\cap Q_\sigma(K,A)=Q(K,A)$.

Proof:

Clearly $Q(K,A)\subset Q_\rho(K,A)\cap Q_\sigma(K,A)$.
If $a\in Q_\rho(K,A)\cap Q_\sigma(K,A)$, then

$\|x(\lambda-a)\|\leqslant K\|(\lambda-a)x\|$ for all $x\in A$ and

$\lambda\in\sigma_A(a)\cup\rho_A(a)=C$ and so $a\in Q(K,A)$.

Corollary: If $\sigma_A(a)$ is countable for every $a\in Q_\sigma(A)$, then $Q_\sigma(A)\subset Q_\rho(A)$.

Proof: Since $\sigma_A(a)$ is a countable compact set, it has zero analytic capacity [5]. Making use of [2, Theorem 3.7] we have $Q_\sigma(A)=Q(A)$.

Since $Q(A)\subset Q_\rho(A)$, it follows that $Q_\sigma(A)\subset Q_\rho(A)$.

We give an example of a complex Banach algebra and a σ-quasi central element of A which is not ρ-quasi central.

EXAMPLE

Let B be the closed unit disc in the complex plane C.

Let $C(B)$ be the algebra of all continuous complex functions on B with the supremum norm.

Let $A=M_2(C(B))$ be the algebra of all 2×2 matrices over $C(B)$.

For $X=\begin{pmatrix} f & g \\ h & k\end{pmatrix}\in A$, define

$\|X\|=\max\{\|f\|+\|g\|, \|h\|+\|k\|\}$.

Choose any u and v in $C(B)$ with $v\neq 0$ and $uv=0$.

Let

$$Y=\begin{pmatrix} u & 0 \\ 0 & u\end{pmatrix},\quad Z=\begin{pmatrix} v & -v \\ v & -v\end{pmatrix},\quad W=\begin{pmatrix} 0 & v \\ 0 & v\end{pmatrix}$$

and

$T=Y+Z$.

Then $TZ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so T^{-1} does not exist.

This implies that $(0-T)^{-1}$ does not exist and hence $0 \in \sigma_A(T)$. Moreover,

$$WT = \begin{pmatrix} v^2 & -v^2 \\ v^2 & -v^2 \end{pmatrix} \neq 0 \text{ since } v \neq 0$$

and

$$\|WT\| = \left\| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = 0 \text{ since } uv = 0.$$

Moreover,

$$\|W(0I-T)\| = \|WT\| = \left\| \begin{pmatrix} v^2 & -v^2 \\ v^2 & -v^2 \end{pmatrix} \right\|$$

$$> 0 = K\|(0I-T)W\| = K\|TW\|.$$

Hence $T \notin Q(A)$.
Finally we show that $T \in Q_\sigma(A)$.
Observe that $Z^2 = YZ = 0$.
Thus for any non zero complex number λ we have

$$(\lambda I - Y)(I - \lambda^{-1} Z) = \lambda I - Z - Y = \lambda I - T.$$

Also,

$$(I - \lambda^{-1} Z)(I + \lambda^{-1} Z) = I - \lambda^{-2} Z^2 = I.$$

Therefore,

$$(I - \lambda^{-1} Z)^{-1} = (I + \lambda^{-1} Z).$$

Now by computation,

$(\lambda I - Y)^{-1}$ does not exist iff $(\lambda I - u)^2 = 0$,

and

$(\lambda I - T)^{-1}$ does not exist iff $(\lambda I - u)^2 = 0$.

We see that $\lambda \in \sigma_A(T)$ iff $\lambda \in \sigma_A(Y)$.

Let $\lambda \in \rho_A(T)$,

$$(\lambda I - T) = (\lambda I - Y)(I - \lambda^{-1} Z)$$

since

$$0 \notin \rho_A(T)(0 \in \sigma_A(T)).$$

So

$$(\lambda I - T)^{-1}(\lambda I - T)(I + \lambda^{-1} Z)(\lambda I - Y)^{-1}$$

$$= (\lambda I - T)^{-1}(\lambda I - Y)(I - \lambda^{-1} Z)(I + \lambda^{-1} Z)(\lambda I - Y)^{-1}$$

$$= (\lambda I - T)^{-1}(\lambda I - Y)(\lambda I - Y)^{-1} = (\lambda I - T)^{-1}.$$

Therefore,

$(\lambda I - T)^{-1} = (I + \lambda^{-1} Z)(\lambda I - Y)^{-1})$
for all $\lambda \in \rho_A(T) = \rho_A(Y)$.

Let $X = \begin{pmatrix} f & g \\ h & k \end{pmatrix} \in A$. Then

$YX = XY$ and so $Y \in Z(A)$, the center of A.

By computation,

$$(\lambda I - T)^{-1} x (\lambda I - T) = (I + \lambda^{-1} Z) x (I - \lambda^{-1} Z).$$

Hence

$$\|(\lambda I - T)^{-1} x (\lambda I - T)\| \leq (1 + |\lambda^{-1}| \|Z\|)$$

$$\times \|x\| \left(1 + \frac{\|Z\|}{|\lambda|} \right) \leq K \|X\|,$$

where $K = (1 + |\lambda^{-1}| \|Z\|)^2$ and so $T \in Q_\sigma(A)$.

INNER DERIVATIONS

For a in a Banach algebra A, let D_a be the inner derivation corresponding to a, defined by

$$D_a x = ax - xa (x \in A).$$

Proposition 2

Let A be a Banach algebra and $a \in Q_\rho(K, A)$. Then,

$$\|D_a T\| \leq (K+1) \|(\lambda I - L_a) T\|$$

for every bounded linear operator T on A, $\lambda \in \sigma_A(a)$, and

$$L_a x = xa (x \in A).$$

Proof: Let $R_a x = ax$ for every $x \in A$.

Let $a \in Q_\rho(K, A)$. Then for every $x \in A$ and $\lambda \in \sigma_A(a)$ we have,

$\|x(\lambda e - a)\| \leq K \|(\lambda e - a) x\|$,
where e is the unit element of A, so that

$$\|x(\lambda e - a)\| = \|x\lambda - xa\| = \|\lambda x - R_a x\|$$

$$= \|\lambda e - R_a) x\|$$

Similarly

$$\|(\lambda e - a) x\| = \|(\lambda e - L_a) x\|,$$

so that

$$\|(\lambda e - R_a) x\| \leq K \|(\lambda e - L_a) x\|.$$

Hence

$$\|D_a x\| \leq (K+1) \|(\lambda e - L_a) x\|.$$

Let $T: A \to A$ be any bounded linear operator on A. Then for every $x \in A$ we have

$$\|D_a Tx\| \leqslant (K+1)\|(\lambda I - L_a)Tx\|, \ \lambda \in \sigma_A(a).$$

This implies that

$$\|D_a T\|(K+1)\|(\lambda I - L_a)T\|.$$

Proposition 3

Let $a \in Q_p(K, A)$ and $0 \in \sigma_A(a)$. Then for each integer n we have $\|D_a^n\| \leqslant (K+1)^n \|a\|^n$.

Proof: We prove this result by induction on n. Since $0 \in \sigma_A(a)$ we have,

$\|xa\| \leqslant K\|ax\|$ and so

$$\|D_a x\| \leqslant \|ax\| + \|xa\| \leqslant (K+1)\|ax\|$$
$$\leqslant (K+1)\|a\|\,\|x\| \text{ for every } x \text{ in } A.$$

Hence,

$$\|D_a\| \leqslant (K+1)\|a\|.$$

Next, assume that for some integer n:

$$\|D_a^n\| \leqslant (K+L)^n \|a\|^n.$$

Then

$$\|D_a^{n+1}x\| = \|D_a^n D_a x\| \leqslant (K+1)^n \|a\|^n \|D_a x\|$$
$$\leqslant (K+1)^{n+1}\|a\|^{n+1}\|x\|$$

for every x in A.

This proves that

$$\|D_a^{n+1}\| \leqslant (K+1)^{n+1}\|a\|^{n+1}$$

and so by induction the proof is completed.

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