SOME SIMILARITY SOLUTIONS OF EQUATIONS GOVERNING THE STEADY PLANE FLOW ON AN INVISCID COMPRESSIBLE FLUID OF FINITE ELECTRICAL CONDUCTIVITY IN THE PRESENCE OF A TRANSVERSE MAGNETIC FIELD *VIA* ONE-PARAMETER GROUP

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الخلاصة :

تم باستعمال مجموعات التحول أحادية المعيار إيجاد حلول مضبوطة لمعادلات تُمثل التدفق المستوي المستقر لسائلٍ ضَغُوطٍ عديم اللزوجة ذي إيصالية كهربائية σ بوجود حقل مغناطيسي مستعرض .

ABSTRACT

Using one parameter group of transformations, some exact solutions of equations governing the steady plane flow of an inviscid compressible fluid of electrical conductivity σ in the presence of a transverse magnetic field, are determined.

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1. INTRODUCTION

Recently Naeem [1] applied one parameter group of transformations to determine some exact solutions of flow equations of an incompressible fluid of variable viscosity. In the present work, we extend Naeem's approach to determine some exact solutions of equations governing the motion of an inviscid compressible fluid of finite electrical conductivity σ in the presence of a transverse magnetic field. In Section 2, we consider the flow equations and transform them into a new system of equations using one parameter group of transformations. In Section 3, we determine some exact solutions of new system of equations for an arbitrary state equation.

2. FLOW EQUATIONS

The steady flow of a compressible, inviscid, electrically conducting fluid of finite electrical conductivity σ is governed by [2]

$$\begin{aligned} \nabla\cdot(\rho\mathbf{v}) &= 0 \\ \rho(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p &= \mu(\nabla\times\mathbf{H})\times\mathbf{H} \\ \nabla\times(\mathbf{v}\times\mathbf{H})+\frac{1}{\mu\sigma}\nabla^2\mathbf{H} &= 0 \\ \nabla\cdot\mathbf{H} &= 0 \\ \mathbf{v}\cdot\nabla S &= 0 \\ p &= p(\rho,S) \end{aligned} \tag{1}$$

where in $\mathbf{v}$ denotes the velocity, $\mathbf{H}$ the magnetic field, ρ the density of the fluid, S the entropy, p the pressure, and μ the constant magnetic permeability of the fluid.

For plane transverse flows in the (x, y) plane, we have $\mathbf{H} = (0, 0, H)$ and $\partial/\partial z = 0$. For such flows system (1) becomes

$$(\rho u)x + (\rho v)y = 0 \tag{2.1}$$

$$\rho u u_x + \rho v u_y = -P_x \tag{2.2}$$

$$\rho u v_x + \rho v v_y = -P_y \tag{2.3}$$

$$(uH)_x + (vH)_y = \frac{1}{\mu\sigma}\nabla^2 H \tag{2.4}$$

$$uS_x + \vartheta S_y = 0 \tag{2.5}$$

$$p = p(\rho, S) \tag{2.6}$$

wherein

$$P = p + \frac{\mu H^2}{2}. \tag{3}$$

Equation (2.1) implies the existence of a streamfunction ψ such that

$$\rho u = \psi_y, \ \rho\vartheta = -\psi_x. \tag{4}$$

The system of equations (1), employing (4), transforms to the following system of partial differential equations

$$P_\xi = 8\left[R_\eta \psi_\xi \psi_\eta + R\psi_\xi \psi_{\eta\eta} - R_\xi \psi_\eta^2 - R\psi_\eta \psi_{\eta\xi}\right] \tag{5.1}$$

$$P_\eta = 8\left[R_\xi \psi_\xi \psi_\eta + R\psi_\eta \psi_{\xi\xi} - R_\eta \psi_\xi^2 - R\psi_\xi \psi_{\xi\eta}\right] \tag{5.2}$$

$$H_{\xi\xi} + H_{\eta\eta} + \mu\sigma\left[R\psi_\eta H_\xi - R\psi_\xi H_\eta + \left(R_\xi \psi_\eta - R_\eta \psi_\xi\right) H\right] = 0 \tag{5.3}$$

$$\psi_\xi S_\eta - \psi_\eta S_\xi = 0 \tag{5.4}$$

in the variables $\xi = x + y$, $\eta = x - y$. In system (5), the function R is given by

$$R = \frac{1}{\rho}. \tag{6}$$

Once a solution of system (5) is determined, the pressure p and density ρ are determined from Equations (3) and (6), respectively. We now transform the system of partial differential equations (5) into a new system of ordinary differential equations using one parameter group of transformations. We give here only the one parameter group Γ_1 and its variants that are used to obtain exact solutions of system (5). For details of one parameter group theory reader is referred to references [1, 3–4].

If Γ_1 is a group consisting of a set of transformation defined by

$$\bar\xi = \alpha^n \xi, \ \ \bar\eta = \alpha^m \eta, \ \ \bar\psi = \alpha^k \psi, \ \ \bar H = \alpha^j H,$$
$$\bar S = \alpha^r S, \ \ \bar R = \alpha^q R, \ \ \bar P = \alpha^t P$$

with parameter $\alpha \neq 0$, then the invariants of Γ_1 for system (5) are

$$\psi = \eta^{\lambda_1} A(\theta), \ R = \eta^{\lambda_2} B(\theta), \ P = \eta^{\lambda_3} C(\theta), \ S = \eta^{\lambda_4} D(\theta),$$
$$H = \eta^{\lambda_5} E(\theta), \ \theta = \frac{\xi}{\eta} \tag{7}$$

provided

$$\lambda_1 - \lambda_3 = 2 \tag{8.1}$$

$$\lambda_2 = -\lambda_1. \tag{8.2}$$

In the above

$$\lambda_1 = \frac{k}{n}, \ \ \lambda_2 = \frac{q}{n}, \ \ \lambda_3 = \frac{t}{n},$$

$$\lambda_4 = \frac{r}{n}, \ \ \lambda_5 = \frac{j}{n}.$$

The system (5), using invariants (7) of Γ_1, transforms to the following system of ordinary differential equations

$$C' = 8\left[B\left\{\lambda_2 A'(\lambda_1 A - \theta A') + \lambda_1 \theta A A'' + (1-\lambda_1)\theta A'^2\right\} + B'\left\{-\lambda_1^2 A^2 + \lambda_1 \theta A A'\right\}\right] \tag{9.1}$$

$$\lambda_3 C - \theta C' = 8\left[B\left(\lambda_1 A A'' + A'^2\right) + \lambda_1 B' A A'\right] \tag{9.2}$$

$$\left(1+\theta^2\right)E'' + \left\{(2-2\lambda_5)\theta + \mu\sigma\lambda_1 AB\right\}E' + \left[\lambda_5(\lambda_5 - 1) - \lambda_5\mu\sigma BA' + \mu\sigma(\lambda_1 AB' - \lambda_2 BA')\right]E = 0 \tag{9.3}$$

$$\lambda_4 A' D - \lambda_1 A D' = 0 \tag{9.4}$$

for the five unknown functions A, B, C, D, E of θ. In the next section, we determine the solutions of the system (9).

3. SOLUTIONS

Using (9.1) in (9.2), we get

$$\lambda_3 C = 8\left[B\left(\lambda_1 A A'' + A'^2 - \lambda_1^2 \theta A A' + \lambda_1 \theta^2 A A'' + \theta^2 A'^2\right) + B'\left(\lambda_1 A A' - \lambda_1^2 \theta A^2 + \lambda_1 \theta^2 A A'\right)\right]. \tag{10}$$

Equations (10) and (9.1) imply that

$$(Y+\theta M)B'' + \left[X + \theta L + (Y+\theta M)' - \lambda_3 M\right]B' + \left[(X+\theta L)' - \lambda_3 L\right]B = 0 \tag{11.1}$$

wherein

$$\begin{aligned} L &= -\lambda_1^2 A A' + \lambda_1 \theta A A'' + \theta A'^2 \\ M &= -\lambda_1^2 A^2 + \lambda_1 \theta A A' \\ X &= \lambda_1 A A'' + A'^2 \\ Y &= \lambda_1 A A'. \end{aligned} \tag{11.2}$$

Integration of (9.1) and (9.4) yields

$$C = 8\int \left[B\left\{-\lambda_1^2 A A' + \lambda_1 \theta A A'' + \theta A'^2\right\} + B'\left\{-\lambda_1^2 A^2 + \lambda_1 \theta A A'\right\}\right] d\theta + d_1 \tag{12.1}$$

$$D(\theta) = A_1 A^{\lambda_6}(\theta), \quad \lambda_6 = \lambda_4/\lambda_1. \tag{12.2}$$

where A_1 and d_1 are arbitrary constants.

In Equations (12.1–12.2), the functions $A(\theta)$ and $B(\theta)$ are determined from Equations (9.3) and (11.1) using particular methods for determining the solutions of linear differential equations of second order. We know from theory of ordinary differential equations that the first integral of

$$g_1(\theta) Z''(\theta) + g_2(\theta) Z'(\theta) + g_3(\theta) Z(\theta) = g_4(\theta) \tag{13.1}$$

is

$$g_1 Z' + [g_2 - g_1'] Z = \int g_4(\theta) d\theta + \text{Constant} \tag{13.2}$$

provided

$$g_3 - g_2' + g_1'' = 0 \qquad \text{(exactness condition)}. \tag{13.3}$$

We now employ (13.1–13.2) to determine the solutions of Equations (9.3) and (11.1).

Equation (11.1), employing (13.3), give

$$\lambda_3 (L - M') = 0.$$

This leads us to the following two cases:

Case I. $\lambda_3 = 0, \quad L - M' \neq 0.$

Case II. $\lambda_3 \neq 0, \quad L - M' = 0.$

We study the two cases.

Case I. When $\lambda_3 = 0$, the first integral of (11.1) is

$$(Y + \theta M) B' + (X + \theta L) B = C_1 . \tag{14}$$

Assuming $Y + \theta M \neq 0$, the solution of Equation (14) is

$$B(\theta) = e^{-F_1(\theta)} \int \frac{C_1}{Y + \theta M} e^{F_1(\theta)} d\theta + C_2 e^{-F_1(\theta)} \tag{15}$$

where

$$F_1(\theta) = \int \frac{X + \theta L}{Y + \theta M} d\theta$$

and C_1 and C_2 are arbitrary constants. In Equation (15), the function A(θ), on which $F_1(\theta)$ depends, is arbitrary.

If $Y + \theta M = 0$, then

$$\begin{aligned} A(\theta) &= C_3\left(1+\theta^2\right) \\ B(\theta) &= \frac{C_1}{4C_3^2\left(1+\theta^2\right)^2} \end{aligned} \tag{16}$$

where C_3 is an arbitrary constant.

We now solve Equation (9.3) for $\lambda_3 = 0$. Equation (9.3), utilizing (13.3), give

$$\lambda_5(\lambda_5 + 1) + \mu\sigma(-\lambda_5 - \lambda_2 - \lambda_1) BA' = 0.$$

This equation holds $\forall\theta$ provided

$$\lambda_5(\lambda_5 + 1) = 0, \quad \lambda_2 + \lambda_1 + \lambda_5 = 0 \text{ or } \lambda_5(\lambda_5 + 1) = 0, \quad A' = 0.$$

We treat the above two cases separately.

Case a. $\lambda_5(\lambda_5 + 1) = 0$, $\lambda_1 + \lambda_2 + \lambda_5 = 0$. The choice $\lambda_5 = -1$ fails to satisfy (8.2) and is therefore discarded. For choice $\lambda_5 = 0$, the solutions of Equation (9.3) is

$$E = e^{-F_2(\theta)} C_4 \int \frac{e^{F_2(\theta)}}{1+\theta^2} d(\theta) + C_5 e^{-F_2(\theta)} \tag{17}$$

where

$$F_2(\theta) = \mu\sigma\lambda_1 \int \frac{AB}{1+\theta^2} d\theta$$

and C_4 and C_5 are arbitrary constants. In Equation (17) the functions $A(\theta)$ and $B(\theta)$ are given by (15) or (16).

Case b. $\lambda_5(\lambda_5+1)=0$ and $A'=0$ give

$$A=a_1, \quad \lambda_5=0,-1$$

where a_1 is an arbitrary constant.

The solutions of (9.3) for $\lambda_5=0,-1$, respectively, are

$$\begin{aligned} E &= \exp\left(2\mu\sigma a_1 C_7 \tan^{-1}\theta + \frac{\mu\sigma C_1}{2a_1}\int\frac{\ln\theta d\theta}{1+\theta^2}\right) \\ &\quad\times\left\{C_6\int\exp\left(-2\mu\sigma a_1 C_7\tan^{-1}\theta - \frac{\mu\sigma C_1}{2a_1}\int\frac{\ln\theta d\theta}{1+\theta^2}\right)/(1+\theta^2)d\theta + C_8\right\} \\ E &= (1+\theta^2)\exp\left(-2\mu\sigma a_1 C_7\tan^{-1}\theta - \frac{\mu\sigma C_1}{2a_1}\int\frac{\ln\theta d\theta}{1+\theta^2}\right) \\ &\quad\times\left\{C_9\int\exp\left(2\mu\sigma a_1 C_7\tan^{-1}\theta + \frac{\mu\sigma C_1}{2a_1}\int\frac{\ln\theta d\theta}{1+\theta^2}\right)/(1+\theta^2)^2 d\theta + C_{10}\right\}\end{aligned}$$

where $C_6, C_7, C_8, C_9, C_{10}$ are arbitrary constants.

Utilizing $A(\theta)=a_1$ in Equation (14), we find

$$B=-\frac{C_1}{4a_1^2}\ln\theta + C_7.$$

For $C_1=0$,

$$E=\begin{cases} -[C_6]/[2\mu\sigma_1 C_7] + C_8\exp(2\mu\sigma a_1 C_7 \tan^{-1}\theta), & \lambda_5=0 \\ C_4[8(+2\mu^2\sigma^2 a_1^2 C_7^2) + 8\theta^2 + 16\mu\sigma a_1 C_7\theta]/[32\mu\sigma a_1 C_7(1+\mu^2\sigma^2 a_1^2 C_7^2)] & \\ +C_{10}(1+\theta^2)\exp(-2\mu\sigma a_1 C_7\tan^{-1}\theta), & \lambda_5=-1 \end{cases}$$

Case II. $\lambda_3\neq 0, \quad L-M'=0, \quad$ give

$$(\lambda_1-1)(\theta A' - \lambda_1 A)=0.$$

This is satisfied $\forall\theta$ provided

$$\lambda_1=1 \quad \text{or} \quad \lambda_1\neq 1,\ \theta A'=\lambda_1 A \quad \text{or} \quad \lambda_1=1,\ \theta A'=A.$$

Let us consider these cases separately.

Case *a*. $\lambda_1 = 1$. In this case, for $Y + \theta M \neq 0$, the solution of (11.1) is same as given by Equation (15). When $Y + \theta M \neq 0$, the Equation (9.3) is exact provided $\lambda_5 = 0$, and its solution is

$$E = \exp\left[-F_3(\theta)\right] C_{12} \int \frac{\exp\left[F_3(\theta)\right]}{1+\theta^2} d\theta + C_{13} \exp\left[-F_3(\theta)\right]$$

where

$$F_3(\theta) = \mu\sigma \int \frac{AB d\theta}{1+\theta^2}$$

and C_{12}, C_{13} are arbitrary constants.

When $Y + \theta M = 0$, the function $A(\theta)$, $B(\theta)$, and $E(\theta)$ are given by

$$A(\theta) = C_{11}(1+\theta^2)$$

$$B(\theta) = \frac{C_1}{C_{11}^2(1+\theta^2)(1+5\theta^2)}$$

$$E(\theta) = \exp\left(\frac{\mu\sigma C_1}{4C_{11}}\left[(\tan^{-1}\theta - \sqrt{5}\tan^{-1}\sqrt{5}\theta\right]\right)\left\{C_{12}\int \frac{\exp\left[\frac{-\mu\sigma C_1}{4C_{11}}\left(\tan^{-1}\theta - \sqrt{5}\tan^{-1}\sqrt{5}\theta\right)\right]}{1+\theta^2} d\theta + C_{13}\right\}.$$

Case *b*. $\lambda_1 \neq 1, \quad \theta A' = \lambda_1 A$. In this case

$$A = C_{14}\theta^{\lambda_1}$$

$$B = -\frac{C_{15}}{\lambda_1^2 C_{14}^2}\theta^{2\lambda_1}\exp\left(\frac{\lambda_1}{\theta}\right)\int \omega^{4\lambda_1-2}\exp(-\lambda_1\omega)d\omega + C_{16}\theta^{2\lambda_1}\exp\left(\frac{\lambda_1}{\theta}\right)$$

wherein

$$\omega = \frac{1}{\theta}$$

and C_{14}, C_{15}, C_{16} are arbitrary constants. The indefinite integral in the above expression for $B(\theta)$ can easily be evaluated using Tables of indefinite integrals [5] for given $(4\lambda_1 - 2)$. The function $E(\theta)$ can easily be obtained from Equation (9.3) using above expressions for $A(\theta)$ and $B(\theta)$.

Case *c*. When $\lambda_1 = 1$, then

$$A = C_{14}\theta, \qquad B = \frac{C_{17}}{C_{14}^2} + \frac{C_{18}}{\theta}$$

wherein C_{17}, C_{18} are arbitrary constants. In this case, the solution of (9.3) for $\lambda_5 = 0$, is

$$
\begin{aligned}
E &= (1+\theta^2)^{[\mu\sigma C_{17}]/[2C_{14}]} \exp\left(-\mu\sigma C_{18} C_{14} \tan^{-1}\theta\right) C_{12} \int \cos^{[\mu\sigma C_{17}]/[C_{14}]} \omega \exp\{\mu\sigma C_{18} C_{14} \omega\}\, d\omega \\
&\quad + C_{13}(1+\theta^2)^{[\mu\sigma C_{17}]/[2C_{14}]} \exp\left(-\mu\sigma C_{18} C_{14} \tan^{-1}\theta\right)
\end{aligned}
\tag{18}
$$

provided

$$C_{18} \neq 0, \qquad \mu\sigma C_{17} \neq -2C_{14}.$$

In above

$$\omega = \tan^{-1}\theta.$$

The indefinite integral in above can easily be evaluated using reference [5] for given $\mu\sigma C_{17}/C_{14}$. When $C_{18} = 0$ and $C_{18} = 0, \mu\sigma C_{17} = 2C_{14}$, the function $E(\theta)$, respectively, is

$$
\begin{aligned}
E &= -\frac{2C_{12}C_{14}}{\mu\sigma C_{17}} + C_{13}\left(1+\theta^2\right)^{[\mu\sigma C_{17}]/[2C_{14}]}, \quad C_{18} = 0 \\
E &= \frac{C_{12}\theta + C_{13}}{1+\theta^2}, \quad C_{18} = 0, \quad \mu\sigma C_{17} = -2C_{14}.
\end{aligned}
\tag{19}
$$

When $\lambda_5 \neq 0$, the Equation (9.3) is exact provided

$$C_{18} = 0, \lambda_5 = \frac{\mu\sigma C_{17}}{C_{14}} - 1.$$

The first integral of (9.3) is

$$E' + \frac{1}{1+\theta^2}\left(2 - \frac{\mu\sigma C_{17}}{C_{14}}\right)\theta E = \frac{C_{19}}{1+\theta^2}$$

which gives

$$E = \begin{cases} C_{18}/(1+\theta^2)^2 + C_{20}(1+\theta^2)^{[\mu\sigma C_{17}]/[2C_{14}]-1}, & \mu\sigma C_{17} \neq 2C_{14} \\ C_{19}\tan^{-1}\theta + C_{21}, & \mu\sigma C_{17} = 2C_{14} \end{cases}$$

wherein C_{19}, C_{20}, C_{21} are arbitrary constants.

4. CONCLUSIONS

Some exact solutions of equations governing the steady plane flow of an inviscid compressible fluid of finite electrical conductivity in the presence of a transverse magnetic field are determined *via* one parameter group of transformations for an arbitrary state equation.

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