RESULTS ON $L(f)$ SPACES

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الخلاصة :

في هذا البحث ندرس فضاءات ل (ق) ونحدد المجموعات الجزئية المحكمة .

ABSTRACT

In this paper, we consider the $L(f)$ spaces and characterize their compact subsets.

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INTRODUCTION

Let f be a real valued function defined on $[0, \infty)$ having the following properties:

(1) $f(x)=0$ if and only if $x=0$.

(2) f is increasing.

(3) $f(x+y) \leqq f(x)+f(y)$ for all $x, y \in [0, \infty)$.

(4) $\lim_{x \to 0+} f(x)=0$.

Such an f is called a modulus; it is clear that a modulus is a continuous function on $[0, \infty)$. The space $L(f)$ consists of all real sequences $(x_n)=X$ satisfying $\Sigma_{n=1}^{\infty} f(|x_n|)<\infty$, this sum being denoted by $|X|_f$. We will show that $(L(f), |\ \ |_f)$ is a complete metric space. The l^p spaces $0<p\leqq 1$ are special cases of $L(f)$ spaces with $f(x)=x^p$.

Let $C=\{f{:}f \text{ is a modulus}\}$ and $H=$the set of all finite sequences; e_n denotes the sequence which is 0 everywhere except for the nth component where it is 1, and $E=\{e_n{:}n=1,2,3,\ldots\}$. In this paper we prove that $L(f)$ is a topological vector space and characterize the compact subsets of $L(f)$, a result which could be considered as an extension of the one given in Reference [1]. We will also show that there exists no $f \in C$ such that $L(f)=H$ and $\bigcap_{p>0} l^p \supseteqq H$.

RESULTS

Lemma 1. Let $f \in C$, then $(L(f), |\ \ |_f)$ is a complete metric space.

Proof. Let X, Y, Z be elements of $L(f)$, $X=(x_n)$, $Y=(y_n)$, $Z=(z_n)$.

(1) If $|X|_f=0=\Sigma f(|x_n|)$ then $f(|x_n|)=0$ so $x_n=0$ for all n.

Now if $X=0$ then $x_n=0$ for all n hence $f(x_n)=0$ for all n so $|X|_f=0$.

(2) $|X-Y|_f=\Sigma f(|x_n-y_n|)=\Sigma f(|y_n-x_n|)=|Y-X|_f$.

(3) $$\begin{aligned}|X-Y|_f &=\Sigma f(|x_n-y_n|)\leqq \Sigma f(|x_n-z_n|+|z_n-y_n|)\\ &\leqq \Sigma f(|x_n-z_n|)+f(|z_n-y_n|)\\ &=|X-Z|_f+|Y-Z|_f.\end{aligned}$$

A proof of the completeness is given in Reference [2], p. 10.

Lemma 2. Let f and g be elements of C, if there exists $\epsilon>0$ such that $f(x)\leqq g(x)$ for all $x\in[0,\epsilon)$ then $L(g)\subseteqq L(f)$.

Proof. Let $X=(x_n)\in L(g)$, since $L(g)\subseteqq l^1$ see Reference [2], then $x_n\to 0$ so there exists N such that $|x_n|\in[0,\epsilon)$ for all $n\geqq N$, hence $f(|x_n|)\leqq g(|x_n|)$ for all $n \geqq N$.

$$\text{Now}\quad \sum_1^{\infty} f(|x_n|)\leqq \sum_1^{N-1} f(|x_n|)+\sum_N^{\infty} g(|x_n|)$$

$$\leqq \sum_1^{N-1} f(|x_n|)+|X|_g<\infty;\ \text{so}\ X\in L(f).$$

Corollary. If $f\in C$, $p\in(0, 1)$ and there exists $\epsilon>0$ such that $f(x)\leqq x^p$ for all $x\in[0,\epsilon)$ then $l^p\subseteqq L(f)$.

Proof. Apply lemma 2 with $g(x)=x^p$.

Lemma 3. If $g\in C$, and f is a real-valued function and there exists $\epsilon>0$ such that $f(x)=g(x)+x$ for all $x\in[0,\epsilon)$ then $f\in C$ and $L(f)=L(g)$.

Proof. The fact that $f\in C$ is obvious. Now since $f(x)\geqq g(x)$ for all $x\in[0,\epsilon)$ then $L(f)\subseteqq L(g)$ by lemma 2.

Let $X=(x_n)\in L(g)$ then $X\in l^1$ so

$$\Sigma f(|x_n|)=\Sigma g(|x_n|)+\Sigma|x_n|<\infty.$$

Lemma 4. Let $f\in C$ and $X=(x_n)\in L(f)$; if (a_n) is a sequence of real numbers such that $a_n\to 0$, then $|a_n X|_f\to 0$.

Proof. We may assume that $|a_n|\leqq 1$. Since f is increasing so $f(|a_n x_n|)\leqq f(|x_n|)$, but $\Sigma f(|x_n|)<\infty$, so by the bounded convergence theorem

$$\lim_{k\to\infty}\sum_{n=1}^{\infty} f(|a_k x_n|)=\sum_{n=1}^{\infty}\lim_{k\to\infty} f(|a_k x_n|)=0$$

because f is continuous at O.

$$\text{Now}\quad \lim_{k\to\infty}|a_k X|_f=\lim_{k\to\infty}\sum_{n=1}^{\infty} f(|a_k x_n|)$$

$$=\sum_{n=1}^{\infty}\lim_{k\to\infty} f(|a_k x_n|)=0.$$

Theorem 1. $L(f)$ with the metric topology is a topological vector space over R.

Proof. It is clear the $L(f)$ is a vector space over R.

Let $T{:}L(f)\times L(f)\to L(f)$ be defined by

$T(X, Y)=X+Y$ and $L:L(f)\times R\to L(f)$ be defined by $L(X, r)=rX$.

We want to show that T and L are continuous. Let A and B be fixed elements of $L(f)$. Let $\epsilon>0$ be given and let $\delta=\frac{1}{2}\epsilon$. Now if $|A-X|_f<\delta$ and $|B-Y|_f<\delta$ then

$$|A+B-(X+Y)|_f\leqq|A-X|_f+|B-Y|_f<2\delta=\epsilon.$$

So T is continuous.

Next, let $A=(a_n)$, r be fixed elements of $L(f)$ and R respectively. Since $|\frac{1}{n}A|_f\to 0$ by lemma 4, then there exists N such that $|\frac{1}{n}A|_f<\frac{\epsilon}{3}$ for all $n\geqq N$. Let $M=[|r|]+1$ where $[|r|]$ is the greatest integer in $|r|$. Also let

$$\delta=\min\left\{1, \frac{\epsilon}{3M}, \frac{1}{N}\right\}.$$

Let $X=(x_n)\in L(f)$, $r^*\in R$ such that

$$|X-A|_f<\delta \quad\text{and}\quad |r-r^*|<\delta.$$

Since

$$r^*\, X-r\, A=(r^*-r)(X-A)+(r^*-r)\, A+r(X-A)$$

we have

$$\begin{aligned}|r^*\, X-rA|_f&\leqq|(r^*-r)(X-A)|_f+|(r^*-r)A|_f+|r(X-A)|_f\\ &=\Sigma f(|r^*-r|\quad|x_n-a_n|)+\Sigma f(|r^*-r|\,|a_n|)\\ &\quad+\Sigma f(|r|\,|x_n-a_n|)\\ &\quad\Sigma f(\delta|x_n-a_n|)+\Sigma f(\delta|a_n|)\\ &\quad+\Sigma f(|r|\,|x_n-a_n|).\end{aligned}$$

By our choice of δ we have

$$\begin{aligned}|r^*\, X-rA|_f&\leqq\Sigma f(|x_n-a_n|)+\Sigma f\left(\frac{1}{N}|a_n|\right)\\ &\quad+\Sigma f(([|r|]+1)\ |x_n-a_n|)\\ &\leqq|X-A|_f+|\frac{1}{N}\ A|_f+([|r|]+1)\ |X-A|_f\\ &<\frac{\epsilon}{3}+\frac{\epsilon}{3}+([|r|]+1)\delta\\ &\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.\end{aligned}$$

So L is continuous.

Lemma 5. $H\subsetneqq L(f)$ or all $f\in C$.

Proof. $H\subseteqq L(f)$ is trivial. Choose $x_1\in(0,\infty)$ such that $f(x_1)<\frac{1}{2}$, choose $x_k\in(0,\infty)$ such that $x_k\neq x_j$ for all $j<k$ and $f(x_k)<\frac{1}{2^k}$, this can be done because f is continuous at O and $f(0)=0$. Let $X=(x_n)$ then

$$\Sigma f(|x_n|)<\Sigma\,\frac{1}{2^n}<\infty \text{ so } X\in L(f) \text{ and } X\notin H.$$

Theorem 2. $H\underset{\neq}{\subset}\bigcap\limits_{p>0} l^p$.

Proof. By Reference [2], $\bigcap\limits_{p>0} l^p$ is an FK space in which E is bounded (see Reference [2] for definition). By theorem 3.2 in Reference [2] there exists $f\in C$ such that $L(f)\subseteqq\bigcap\limits_{p>0} l^p$. So by lemma 5, $H\underset{\neq}{\subset} L(f)\subseteqq\bigcap\limits_{p>o} l^p$.

Theorem 3. If $f\in C$ is such that $\lim\limits_{x\to 0+}\dfrac{f(x)}{x}$ exists and is finite then $L(f)=l'$.

Proof. Assume $\lim\limits_{x\to 0+}\dfrac{f(x)}{x}=M$ and let $X=(x_n)\in l'$. Let $\epsilon>0$ be given then there exists $\delta>0$ such that if $x\in(0,\delta)$ then $|\dfrac{f(x)}{x}-M|<\epsilon$ so

$$|f(x)|<(\epsilon+M)x \text{ for all } x\in(0,\delta).$$

Now $X=(x_n)\in l^1$ so $x_n\to 0$ hence there exists N such that $|x_n|\in(0,\delta)$ for all $n\geqq N$ so

$$f(|x_n|)\leqq(\epsilon+M)|x_n| \text{ for all } n\geqq N$$

and this implies that

$$\Sigma f(|x_n|)\leqq(\epsilon+M)\sum_N^\infty |x_n|+\sum_1^{N-1} f(|x_n|)<\infty.$$

So $X\in L(f)$.

Finally since $L(f)$ and l^1 are FK spaces they must have the same topology, see Reference [4], p. 203.

Theorem 4. Let $f\in C$, $K\subseteqq L(f)$ then K is a compact subset of $L(f)$ if and only if

(1) K is closed and bounded and

(2) Given $\epsilon>0$, there exists $n_0=n_0(\epsilon)$ such that $\Sigma f(x_n|)<\epsilon$, for all $X=(x_n)\in K$, for all $n>n_0$ and,

(3) If $p_k: L(f)\to R$ is given by

$$p_k(X)=x_k,\quad \text{for all } X=(x_k)\in L(f),$$

then $p_k(K)$ is compact, for all $k\geqq 1$.

Proof. Suppose $K\subseteqq L(f)$ is compact, then (1) is clear.

Since p_k is continuous, then $p_k(K)$ is compact. To prove (2), let $\epsilon>0$ be given, for each $a=(a_k)\in K$ consider

$$U(a,\epsilon/2)=\left\{X\in L(f):|X-a|_f<\frac{\epsilon}{2}\right\}$$

so $K \subseteq \bigcup_{a \in K} U(a, \epsilon/2)$, but K is compact, so there exists
$a^1 = (a_k^1)$, $a^2 = (a_k^2), \ldots, a^N = (a_k^N)$, such that
$K \subseteq \bigcup_{j=1}^{N} U\left(a^j, \frac{\epsilon}{2}\right)$, so if $a = (a_k) \in K$

then there exists a^i, $1 \leqq i \leqq N$ such that

$$|a - a^i|_f = \sum_{n=1}^{\infty} f(|a_n - a_n^i|) < \frac{\epsilon}{2}.$$

Since $\sum_{n=1}^{\infty} f(|a_n^i|) < \infty$, for all i, there exists n_1 such that $\sum_{n=n_i}^{\infty} f(|a_n^i|) < \frac{\epsilon}{2}$.

So for $a \in U\left(a^i, \frac{\epsilon}{2}\right)$ we will have

$$\sum_{n_i}^{\infty} f(|a_n|) \leqq \sum_{n_i}^{\infty} f(|a_n - a_n^i|) + \sum_{n_i}^{\infty} f(|a_n^i|)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now taking $n_0 = \max_{1 \leqq i \leqq N} n_i$, then

$$\sum_{n+1}^{\infty} f(|a_i|) < \epsilon, \text{ for all } a \in K, \text{ for all } n > n_0.$$

Conversely. Suppose (1), (2) and (3) hold, since K is closed and $L(f)$ is complete, [2], it suffices to show that K is totally bounded [3].

Let $\epsilon > 0$ be given, then there exist $n_0 = n_0(\epsilon)$ such that

$$\sum_{n+1}^{\infty} f(|a_k|) < \epsilon, \text{ for all } a \in K, \text{ for all } n > n_0.$$

Since f is continuous at 0 and $f(0) = 0$ we can choose $\epsilon^* > 0$ such that $f(\epsilon^*) \leqq \frac{\epsilon}{2n_0}$. Now since $p_k(K)$ is a compact subset of R for all $k \geqq 1$, so it is totally bounded. So for each $k = 1, 2, \ldots, n_0$ there exists

$$a_k^1, a_k^2, \ldots, a_k^{n_k} \in p_k(K) \text{ such that if}$$
$$a_k \in p_k(K) \text{ then } |a_k - a_k^i| < \epsilon^*$$

for some i, $1 \leqq i \leqq n_k$.

Let $K_0 = \{b \colon b = (a_1^{i_1}, a_2^{i_2}, \ldots, a_{n_0}^{i_{n_0}}, 0, 0, \ldots, 0, \ldots) \colon 1 \leqq i_1 \leqq n_1, 1 \leqq i_2 \leqq n_2, \ldots, 1 \leqq i_{n_0} \leqq n_{n_0}\}$

If $a = (a_k) \in K$, then $a_k \in p_k(K)$, for all $k \geqq 1$.

Let $b \in K_0$ be given by
$b = (a_1^{i_1}, a_2^{i_2}, \ldots, a_{n_0}^{i_{n_0}}, 0, 0, \ldots, 0, \ldots)$ where
$|a_k - a_k^{i_k}| < \epsilon^*, \quad k = 1, 2, \ldots, n_0$.

Now

$$|a - b|_f = \sum_{1}^{n_0} f(|a_k - a_k^{i_k}|) + \sum_{n_0+1}^{\infty} f(|a_k|)$$
$$< n_0 f(\epsilon^*) + \frac{\epsilon}{2} \leqq \epsilon$$

So

$$K \subseteq \bigcup_{b \in k_0} U(b, \epsilon)$$

but K_0 is finite, so K is totally bounded.

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