HIGHER ORDER PLATE EQUATIONS BASED ON REISSNER'S FORMULATION OF THE PLATE PROBLEM

M. H. Baluch and G. Z. Voyiadjis

Department of Civil Engineering, University of Petroleum and Minerals, Dhahran, Saudi Arabia

الـخلاصـة :

يعرض هذا البحث المعادلات المتتالية ذات الدرجات العالية الخاصة بانحناء الالواح والمبنية على نظرية رايسنر (Reissner) للالواح . وقد عبر عن تأثير كل من أجهاد القص المستعرض (transverse shear stress) والاجهاد العمودى المستعرض (transverse normal stress) بدلالة الازاحة المستعرضة (transverse displacement) التي أصبحت المجهول المتغير الوحيد . وقد لوحظ نمط معين لهذه المعادلات يمكن عن طريقة كتابة المعادلات بواسطة سابقتها في الدرجة .

ومن مميزات هذا البحث أنه يمكن أستخدام طريقة نافير (Navier) لحل هذه المعادلات في حالة الالواح ذات الارتكاز البسيط المعرضة لا حمال مختلفة .

ABSTRACT

Successive higher order equations of plate bending are presented, based on the Reissner formulation of the plate problem. The effect of transverse shear and transverse normal stress is included, and the equations are formulated in terms of the transverse displacement as the single unknown variable. A certain pattern is observed in the resulting equations of successive higher order theories, enabling one to write down equations of each succeeding theory by inspection. The equations lend themselves readily to a Navier type solution, enabling one to solve simply supported plate problems with different loading conditions.

0377–9211/80/0205–0075$01.00

HIGHER ORDER PLATE EQUATIONS BASED ON REISSNER'S FORMULATION OF THE PLATE PROBLEM

1. INTRODUCTION

Reissner [1, 2] formulated a theory of plate bending in terms of two variables, the transverse displacement w and a stress function ψ, with the governing equations of the system being of the sixth order as opposed to the fourth-order system of the classical plate bending problem. Reissner's formulation included the effect of the transverse shear and the transverse normal stress on the deformation of the plate.

Speare and Kemp [3] presented a simplified Reissner theory, describing the governing equations solely in terms of the transverse displacement w, citing such a need in order to make Reissner's formulation more attractive both from a computational and a physical point of view.

In this paper, an alternative approach to formulating the Reissner problem in terms of w alone is presented and the Speare–Kemp equations are shown to be a special case of the more general formulation.

NOTATION

$D=\dfrac{Eh^3}{12(1-v^2)}$	flexural rigidity of plate
E	Young's modulus
h	plate thickness
M_x, M_y	bending moments per unit length
M_{xy}	twisting moment per unit length
q	distributed load per unit area
Q_x, Q_y	shear forces per unit length
w	transverse displacement
x, y, z	orthogonal coordinates
Δ^n	$\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)^n$
v	Poisson's ratio
σ_x, σ_y, σ_z	normal components of stress
τ_{xy}, τ_{xz}, τ_{yz}	shear components of stress

2. THEORETICAL PRELIMINARIES

Reissner [1, 2] formulated the equations describing plate behavior, inclusive of transverse shear and transverse normal stress effects, by initially assuming a linear variation of bending stresses over the thickness of the plate:

$$\sigma_x=\frac{M_x}{h^2/6}\,\frac{z}{h/2} \tag{1}$$

$$\sigma_y=\frac{M_y}{h^2/6}\,\frac{z}{h/2} \tag{2}$$

$$\tau_{xy}=-\frac{M_{xy}}{h^2/6}\,\frac{z}{h/2} \tag{3}$$

and on the subsequent use of Equations (1) through (3) in the differential equations of equilibrium obtained for the remaining stress components:

$$\tau_{xz}=\frac{Q_x}{2h/3}\left[1-\left(\frac{z}{h/2}\right)^2\right] \tag{4}$$

$$\tau_{yz}=\frac{Q_y}{2h/3}\left[1-\left(\frac{z}{h/2}\right)^2\right] \tag{5}$$

$$\sigma_z=\frac{-3q}{4}\left[-\frac{z}{h/2}+\tfrac{1}{3}\left(\frac{z}{h/2}\right)^3+\tfrac{2}{3}\right] \tag{6}$$

Invoking the use of a variational theorem, and making use of Equations (1) through (6), Reissner arrived at the following relationships between the intrinsic variables:

$$D\Delta^2 w=q-\frac{2-v}{10(1-v)}h^2\Delta q \tag{7}$$

$$Q_x-\frac{h^2}{10}\,\Delta Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-\frac{h^2}{10(1-v)}\,\frac{\partial q}{\partial x} \tag{8}$$

$$Q_y-\frac{h^2}{10}\,\Delta Q_y=-D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)-\frac{h^2}{10(1-v)}\,\frac{\partial q}{\partial y} \tag{9}$$

$$M_x=D\left(\frac{\partial\phi_x}{\partial x}+\frac{v\partial\phi_y}{\partial y}\right)+\frac{6v(1+v)}{5Eh}\,qD \tag{10}$$

$$M_y=D\left(\frac{\partial\phi_y}{\partial y}+\frac{v\partial\phi_x}{\partial x}\right)+\frac{6v(1+v)}{5Eh}\,qD \tag{11}$$

$$M_{xy}=-\frac{D(1-v)}{2}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right); \tag{12}$$

where:

$$\phi_x=-\frac{\partial w}{\partial x}+\frac{12(1+v)}{5Eh}\,Q_x \tag{13}$$

$$\phi_y=-\frac{\partial w}{\partial y}+\frac{12(1+v)}{5Eh}\,Q_y. \tag{14}$$

Reissner then proceeded to demonstrate that the system as described by Equations (7) through (14) is of sixth order, as opposed to the fourth-order system of classical plate theory. The solution to the above system was formulated by Reissner in terms of two variables—the transverse displacement w and a stress function ψ.

As remarked by Speare and Kemp [3], the two-variable formulation leads to considerable mathematical complexity, thereby restricting the use of Reissner's theory.

In the next section, Reissner's theory is approximated in terms of a single variable w and the Speare and Kemp formulation of the Reissner problem shown to be a special case of the general solution.

3. APPROXIMATE SOLUTION TO REISSNER'S EQUATIONS

Consider the following classical thin plate equations equivalent to the Reissner set as given by (7) through (14):

$$q = D\Delta^2 w \tag{15}$$

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) \tag{16}$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right) \tag{17}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\nu \partial^2 w}{\partial y^2}\right) \tag{18}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \frac{\nu \partial^2 w}{\partial x^2}\right) \tag{19}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}. \tag{20}$$

Inasmuch as it is the presence of the terms ΔQ_x and ΔQ_y in Equations (8) and (9) that serve to increase the order of Reissner's system, it is postulated that one may use Equations (15) through (17) to approximate ΔQ_x, ΔQ_y, and q (together with its derivatives) as functions of w.

Operating on Equation (15) by Δ, one obtains

$$\Delta q = D\Delta^3 w \tag{21}$$

Similarly, operating on Equations (16) and (17) by Δ, and after simplification yields:

$$\Delta Q_x = -D\left(\frac{\partial^5 w}{\partial x^5} + \frac{2\partial^5 w}{\partial x^3 \partial y^2} + \frac{\partial^5 w}{\partial x \partial y^4}\right) \tag{22}$$

$$\Delta Q_y = -D\left(\frac{\partial^5 w}{\partial x^4 \partial y} + \frac{2\partial^5 w}{\partial x^2 \partial y^3} + \frac{\partial^5 w}{\partial y^5}\right). \tag{23}$$

On substituting Equations (21) through (23) in (7) through (9), one obtains:

$$\Delta^2 w + \frac{(2-\nu)}{10(1-\nu)} h^2 \Delta^3 w = \frac{q}{D} \tag{24}$$

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) - \frac{h^2(2-\nu)}{10(1-\nu)} D\left(\frac{\partial^5 w}{\partial x^5} + \frac{2\partial^5 w}{\partial x^3 \partial y^2} + \frac{\partial^5 w}{\partial x \partial y^4}\right) \tag{25}$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial y \partial x^2}\right) - \frac{h^2(2-\nu)}{10(1-\nu)} D\left(\frac{\partial^5 w}{\partial y^5} + \frac{2\partial^5 w}{\partial x^2 \partial y^3} + \frac{\partial^5 w}{\partial x^4 \partial y}\right). \tag{26}$$

The moment expressions may be obtained by using Equations (15) through (17) to approximate q, Q_x and Q_y in (10) through (12). This yields on simplification:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\nu \partial^2 w}{\partial y^2}\right) - \frac{h^2 D}{10(1-\nu)} \left[(2-\nu)\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\nu \partial^4 w}{\partial y^4}\right] \tag{27}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \frac{\nu \partial^2 w}{\partial x^2}\right) - \frac{h^2 D}{10(1-\nu)} \left[(2-\nu)\frac{\partial^4 w}{\partial y^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\nu \partial^4 w}{\partial x^4}\right] \tag{28}$$

$$M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} + \frac{h^2 D}{5}\left(\frac{\partial^4 w}{\partial x^3 \partial y} + \frac{\partial^4 w}{\partial x \partial y^3}\right). \tag{29}$$

It is interesting to note that Equations (24) through (29) have been derived by Speare and Kemp [3] in an alternative fashion. Equations (24) through (29) represent a sixth-order approximation (in terms of w alone) to the Reissner set (7) through (12).

To obtain the next higher order theory, one uses Equations (24) through (26) to approximate Δq, $\partial q/\partial x$, $\partial q/\partial y$, ΔQ_x and ΔQ_y in Equations (7) through (9). This yields new relationships between q, Q_x and Q_y on one hand and the transverse displacement, w, on the other. The moment—displacement equations are obtained by substitution of Equations (24) through (26) in (10) through (12). The resulting set of equations may be expressed as:

$$\Delta^2 w + \frac{(2-\nu)}{10(1-\nu)} h^2 \Delta^3 w + \left[\frac{(2-\nu)h^2}{10(1-\nu)}\right]^2 \Delta^4 w = q/D \tag{30}$$

$$Q_x = -D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}\, D\left(\frac{\partial^5 w}{\partial x^5}+\frac{2\partial^5 w}{\partial x^3 \partial y^2}+\frac{\partial^5 w}{\partial x \partial y^4}\right)-\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^2 D\left(\frac{\partial^7 w}{\partial x^7}+\frac{3\partial^7 w}{\partial x^5 \partial y^2}+\frac{3\partial^7 w}{\partial x^3 \partial y^4}+\frac{\partial^7 w}{\partial x \partial y^6}\right) \quad (31)$$

$$Q_y = -D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2 \partial y}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}\, D\left(\frac{\partial^5 w}{\partial y^5}+\frac{2\partial^5 w}{\partial x^2 \partial y^3}+\frac{\partial^5 w}{\partial x^4 \partial y}\right)-\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^2 D\left(\frac{\partial^7 w}{\partial y^7}+\frac{3\partial^7 w}{\partial x^2 \partial y^5}+\frac{3\partial^7 w}{\partial x^4 \partial y^3}+\frac{\partial^7 w}{\partial x^6 \partial y}\right) \quad (32)$$

$$\begin{aligned} M_x = & -D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\nu\partial^2 w}{\partial y^2}\right)-\frac{h^2 D}{10(1-\nu)}\left[(2-\nu)\,\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\nu\partial^4 w}{\partial y^4}\right] \\ & -\left[\frac{h^2}{10(1-\nu)}\right]^2 (2-\nu)\, D\left[(2-\nu)\,\frac{\partial^6 w}{\partial x^6}+(4-\nu)\,\frac{\partial^6 w}{\partial x^4 \partial y^2}+(2+\nu)\,\frac{\partial^6 w}{\partial x^2 \partial y^4}+\frac{\nu\partial^6 w}{\partial y^6}\right] \end{aligned} \quad (33)$$

$$\begin{aligned} M_y = & -D\left(\frac{\partial^2 w}{\partial y^2}+\frac{\nu\partial^2 w}{\partial x^2}\right)-\frac{h^2 D}{10(1-\nu)}\left[(2-\nu)\frac{\partial^4 w}{\partial y^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\nu\partial^4 w}{\partial x^4}\right]-\left[\frac{h^2}{10(1-\nu)}\right]^2 (2-\nu)\, D\left[(2-\nu)\,\frac{\partial^6 w}{\partial y^6}\right. \\ & \left.+(4-\nu)\,\frac{\partial^6 w}{\partial x^2 \partial y^4}+(2+\nu)\,\frac{\partial^6 w}{\partial x^4 \partial y^2}+\frac{\nu\partial^6 w}{\partial x^6}\right] \end{aligned} \quad (34)$$

$$M_{xy} = D(1-\nu)\,\frac{\partial^2 w}{\partial x \partial y}+\frac{h^2 D}{5}\left(\frac{\partial^4 w}{\partial x^3 \partial y}+\frac{\partial^4 w}{\partial x \partial y^3}\right)+\frac{h^2 D}{5}\,\frac{h^2(2-\nu)}{10(1-\nu)}\left(\frac{\partial^6 w}{\partial x^5 \partial y}+\frac{2\partial^6 w}{\partial x^3 \partial y^3}+\frac{\partial^6 w}{\partial x \partial y^5}\right) \quad (35)$$

The set of Equations (30) through (35) represents an eighth-order approximation to the original system of equations as given by (7) through (14). This approximation contains terms of the order of h^4.

Operating in a similar fashion, one obtains the h^6 approximation to Reissner's set and given by:

$$\Delta^2 w+\frac{(2-\nu)}{10(1-\nu)}\, h^2 \Delta^3 w+\left[\frac{(2-\nu)h^2}{10(1-\nu)}\right]^2 \Delta^4 w+\left[\frac{(2-\nu)h^2}{10(1-\nu)}\right]^3 \Delta^5 w=\frac{q}{D} \quad (36)$$

$$\begin{aligned} Q_x = & -D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}\, D\left(\frac{\partial^5 w}{\partial x^5}+\frac{2\partial^5 w}{\partial x^3 \partial y^2}+\frac{\partial^5 w}{\partial x \partial y^4}\right) \\ & -\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^2 D\left(\frac{\partial^7 w}{\partial x^7}+\frac{3\partial^7 w}{\partial x^5 \partial y^2}+\frac{3\partial^7 w}{\partial x^3 \partial y^4}+\frac{\partial^7 w}{\partial x \partial y^6}\right) \\ & -\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^3 D\left(\frac{\partial^9 w}{\partial x^9}+\frac{4\partial^9 w}{\partial x^7 \partial y^2}+\frac{6\partial^9 w}{\partial x^5 \partial y^4}+\frac{4\partial^9 w}{\partial x^3 \partial y^6}+\frac{\partial^9 w}{\partial x \partial y^8}\right) \end{aligned} \quad (37)$$

$$\begin{aligned} Q_y = & -D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2 \partial y}\right)-\frac{h^2(2-\nu)}{10(1-\nu)}\, D\left(\frac{\partial^5 w}{\partial y^5}+\frac{2\partial^5 w}{\partial x^2 \partial y^3}+\frac{\partial^5 w}{\partial x^4 \partial y}\right) \\ & -\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^2 D\left(\frac{\partial^7 w}{\partial y^7}+\frac{3\partial^7 w}{\partial x^2 \partial y^5}+\frac{3\partial^7 w}{\partial x^4 \partial y^3}+\frac{\partial^7 w}{\partial x^6 \partial y}\right) \\ & -\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^3 D\left(\frac{\partial^9 w}{\partial y^9}+\frac{4\partial^9 w}{\partial x^2 \partial y^7}+\frac{6\partial^9 w}{\partial x^4 \partial y^5}+\frac{4\partial^9 w}{\partial x^6 \partial y^3}+\frac{\partial^9 w}{\partial x^8 \partial y}\right) \end{aligned} \quad (38)$$

$$\begin{aligned} M_x = & -D\left(\frac{\partial^2 w}{\partial x^2}+\frac{\nu\partial^2 w}{\partial y^2}\right)-\frac{h^2 D}{10(1-\nu)}\left[(2-\nu)\frac{\partial^4 w}{\partial x^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\nu\partial^4 w}{\partial y^4}\right] \\ & -\left[\frac{h^2}{10(1-\nu)}\right]^2 (2-\nu)\, D\left[(2-\nu)\frac{\partial^6 w}{\partial x^6}+(4-\nu)\frac{\partial^6 w}{\partial x^4 \partial y^2}+(2+\nu)\frac{\partial^6 w}{\partial x^2 \partial y^4}+\frac{\nu\partial^6 w}{\partial y^6}\right] \\ & -\left[\frac{h^2}{10(1-\nu)}\right]^3 (2-\nu)^2\, D\left[(2-\nu)\frac{\partial^8 w}{\partial x^8}+(6-2\nu)\frac{\partial^8 w}{\partial x^6 \partial y^2}+\frac{6\partial^8 w}{\partial x^4 \partial y^4}+(2+2\nu)\frac{\partial^8 w}{\partial x^2 \partial y^6}+\frac{\nu\partial^8 w}{\partial y^8}\right] \end{aligned} \quad (39)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2}+\frac{\nu\partial^2 w}{\partial x^2}\right)-\frac{h^2 D}{10(1-\nu)}\left[(2-\nu)\frac{\partial^4 w}{\partial y^4}+\frac{2\partial^4 w}{\partial x^2 \partial y^2}+\frac{\nu\partial^4 w}{\partial x^4}\right]$$

$$-\left[\frac{h^2}{10(1-\nu)}\right]^2 (2-\nu)\, D\left[(2-\nu)\frac{\partial^6 w}{\partial y^6}+(4-\nu)\frac{\partial^6 w}{\partial x^2\partial y^4}+(2+\nu)\frac{\partial^6 w}{\partial x^4\partial y^2}+\frac{\nu\partial^6 w}{\partial x^6}\right] \tag{40}$$

$$-\left[\frac{h^2}{10(1-\nu)}\right]^3 (2-\nu)^2\, D\left[(2-\nu)\frac{\partial^8 w}{\partial y^8}+(6-2\nu)\frac{\partial^8 w}{\partial x^2\partial y^6}+\frac{6\partial^8 w}{\partial x^4\partial y^4}+(2+2\nu)\frac{\partial^8 w}{\partial x^6\partial y^2}+\frac{\nu\partial^8 w}{\partial x^8}\right]$$

$$M_{xy}=D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}+\frac{h^2 D}{5}\left(\frac{\partial^4 w}{\partial x^3\partial y}+\frac{\partial^4 w}{\partial x\partial y^3}\right)+\frac{h^2 D}{5}\frac{h^2(2-\nu)}{10(1-\nu)}\left(\frac{\partial^6 w}{\partial x^5\partial y}+\frac{2\partial^6 w}{\partial x^3\partial y^3}+\frac{\partial^6 w}{\partial x\partial y^5}\right)$$

$$+\frac{h^2 D}{5}\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^2\left(\frac{\partial^8 w}{\partial x^7\partial y}+\frac{3\partial^8 w}{\partial x^5\partial y^3}+\frac{3\partial^8 w}{\partial x^3\partial y^5}+\frac{\partial^8 w}{\partial x\partial y^7}\right). \tag{41}$$

The set of Equations (36) through (41) represent a tenth-order approximation to the original Reissner system as given by Equations (7) through (14).

Higher-order plate theories may be derived in a similar fashion.

There is a very interesting pattern that emerges in each of the equations relating Q_x, Q_y, M_x, M_y, and M_{xy} to the transverse displacement w as the order of accuracy of the approximation is increased.

Pursuing this in detail for one of the force—displacement relations, say Q_x *versus* w, a general form of Equation (37) may be expressed as:

$$Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)-D\left\{\sum_{n=1,2,3}^{\infty}\left[\frac{h^2(2-\nu)}{10(1-\nu)}\right]^n\right.$$

$$\left(a_{1,n}\frac{\partial^{2n+3}w}{\partial x^{2n+3}}+a_{2,n}\frac{\partial^{2n+3}w}{\partial x^{2n+1}\,\partial y^2}\right.$$

$$\left.\left.+a_{3,n}\frac{\partial^{2n+3}w}{\partial x^{2n-1}\,\partial y^4}+\ldots+a_{n+2,n}\frac{\partial^{2n+3}w}{\partial x\partial y^{2n+2}}\right)\right\}. \tag{42}$$

The coefficients $a_{1,n},\ldots, a_{n+2,n}$ in Equation (42) may be readily generated for each value of n as in Table 1.

Table 1. Coefficients for Various Cases of Equation (42)

Case	Coefficients
$n=1$	$a_{1,1}=1$; $a_{2,1}=2$, $a_{3,1}=1$
$n=2$	$a_{1,2}=a_{1,1}$; $a_{2,2}=a_{1,1}+a_{2,1}$; $a_{3,2}=a_{2,1}+a_{3,1}$; $a_{4,2}=a_{3,1}$
$n=3$	$a_{1,3}=a_{1,2}$; $a_{2,3}=a_{1,2}+a_{2,2}$; $a_{3,3}=a_{2,2}+a_{3,2}$; $a_{4,3}=a_{3,2}+a_{4,2}$; $a_{5,3}=a_{4,2}$

The Table 1 may be readily extended to include higher values of n.

The same type of relationships exist for Equations (38) through (41) which may be obvious now by inspection.

The pattern for the governing differential equation is readily recognized to be

$$K^{n-2}\,\Delta^n w+K^{n-3}\,\Delta^{n-1}w+K^{n-4}\,\Delta^{n-2}w+\ldots=q/D \tag{43}$$

where $K=\dfrac{(2-\nu)h^2}{10(1-\nu)}$ and $n=3,\ 4,\ 5,\ \ldots$ and $K^{n-i}=0$ when $n<i$.

4. ALTERNATIVE DERIVATION

Some insight into the nature of the approximation technique described in Section 3 may be gained by noting that an exact form for Δq may be obtained by operating on Equation (7) by Δ and rearranging to yield

$$\Delta q=D\Delta^3 w+K\Delta^2 q. \tag{44}$$

Inasmuch as Δq is not explicitly a function of w in Equation (44), one cannot make use of it directly in order to obtain the Reissner plate theory in terms of w alone.

An initial value for Δq in terms of w alone may be obtained by assuming in Equation (44) that

$$\Delta^2 q=0. \tag{45}$$

On substituting Equation (45) in Equation (44), one obtains

$$\Delta q=D\Delta^3 w, \tag{46}$$

which when substituted into Equation (7) yields the governing differential equation for the sixth-order system

$$D\Delta^2 w=q-KD\Delta^3 w \tag{47}$$

To obtain a better approximation, one now uses Equation (46) to generate

$$\Delta^2 q=D\Delta^4 w \tag{48}$$

which when substituted in Equation (44) yields a new Δq in terms of w alone as

$$\Delta q = D\Delta^3 w + KD\Delta^4 w. \qquad (49)$$

The substitution of Equation (49) into Equation (7) yields the eighth-order differential equation

$$D\Delta^2 w = q - K(D\Delta^3 w + KD\Delta^4 w). \qquad (50)$$

The tenth-order equation may now be obtained by using Equation (49) to obtain a new $\Delta^2 q$ approximation. Thus each successive higher order approximation may be viewed as corresponding to a more refined value for Δq.

It must be emphasized that the approximate governing differential equations do not violate equilibrium. This may be shown by using equations for transverse shears, say (25) and (26), in the transverse force equilibrium equation

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

to obtain the sixth-order equation (24) or (47). Thus, approximations for Δq on one hand and the transverse shears on the other, must constitute a consistent set.

5. APPLICATION TO PLATE PROBLEMS

Assuming a simply supported rectangular plate of dimensions $a \times b$ subjected to a uniformly distributed load $q = q_0$, and following a Navier [4] type approach, one expands q_0 in a double Fourier series as

$$q_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (51)$$

where

$$a_{mn} = \frac{16 q_0}{\pi^2 mn} \qquad (52)$$

Regardless of the order of theory to be used, a solution for w may be assumed in the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (53)$$

The assumed form of Equation (53) satisfies the following boundary conditions:

$$w = 0, \quad M_x = 0 \quad \text{for } x = 0 \text{ and } x = a; \qquad (54)$$

$$w = 0, \quad M_y = 0 \quad \text{for } y = 0 \text{ and } y = b. \qquad (55)$$

w_{mn} in Equation (53) may be determined by substituting Equations (53) and (51) in any of the governing differential equations. Assuming a square plate $a = b$, and taking only one term of the series expansion, one obtains for the maximum deflection ($x = a/2$, $y = b/2$)

$$w_{\max} = \frac{4 q_0 a^4 / \pi^6 D}{1 - \dfrac{2K\pi^2}{a^2} + \dfrac{4K^2\pi^4}{a^4} - \dfrac{8K^3\pi^6}{a^6}} \qquad (56)$$

where the tenth-order system has been used in evaluating $w_{\max}$. Using $\nu = 0$ and $h/a = 0.3$, the maximum transverse deflection from the tenth-order system becomes

$$w_{\max} = 1.377\ (4 q_0 a^4 / \pi^6 D), \qquad (57)$$

where the term in parenthesis represents the Navier one-term solution for $w_{\max}$ in thin-plate theory. This value is very close to the one reported by Salerno and Goldberg [5], using Reissner's formulation in terms of two variables.

It is of interest to note that the maximum transverse displacment using the sixth-order system is given by

$$w_{\max} = 1.55\ (4 q_0 a^4 / \pi^6 D) \qquad (58)$$

which is very close to the value reported by Speare and Kemp [3] using their finite difference approach to the sixth-order problem. Speare and Kemp [3] attributed the over estimation of w to the numerical analysis technique rather than from simplification of the Reissner system to the sixth-order system in w. However, results derived above clearly indicate that theories of order higher than six in w alone are needed to accurately describe the Reissner equations.

CONCLUSIONS

The higher order plate theories in w are found to lend themselves readily to solutions of the Navier type, enabling one to solve simply supported plate problems in which the effects of transverse shear and transverse normal stress are included.

REFERENCES

[1] E. Reissner, 'On the Theory of Bending of Elastic Plates, *J. Math. Phys*, **23** (1944), p. 184.

[2] E. Reissner, 'The Effect of Transverse Shear Deformation on Bending of Elastic Plates', *J. Appl. Mech.* **12**, A69 (1945).

[3] P. R. S. Speare and K. O. Kemp, 'A Simplified Theory for Plate Bending', *Int. J. Solids Structures* (1977), p. 1073.

[4] S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*. New York, McGraw-Hill (1959).

[5] V. L. Salerno and M. A. Goldberg, 'Effect of Shear Deformations on the Bending of Rectangular Plates', *J. Appl. Mech.* **27** (1960), p. 54.

Paper Received 25 April 1979; Revised 22 October 1979.