DECAY OF THE SOLUTION ENERGY FOR A NONLINEARLY DAMPED WAVE EQUATION

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الخلاصـة :

تستحوذ مسألة استقرار حلول معادلة الموجة غير الخطية على حيزٍ كبيرٍ من هذه الدراسة، كما أن عدة نتائجَ خاصةٍ بتهافت طاقة الحلول قد تم إثباتها.

نـقوم في هذا البحث بدراسة معادلة أمواج ذات حدٍّ مخمـدّ لاخطي ونبرهن أنه مهما كانت القيم الإبتدائية فإن طاقة حل المعادلة تتهافت على شكل دالة أُسيّـة.

ABSTRACT

The issue of stablity of solutions to nonlinear wave equations has been addressed by many authors. Thus, many results concerning energy decay have been established. Here in this paper, we consider the following nonlinearly damped wave equation:

$$u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t + bu|u|^{p-2} = 0,$$

$a, b > 0$, in a bounded domain, and show, for arbitrary initial data, that the energy of the solution decays exponentially if $2 \leq m \leq p$.

Keywords: Damped equation, Decay, Local, Energy, Global, Exponential, Stability.

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1 INTRODUCTION

In [1] Nakao considered the following initial boundary value problem:

$$\begin{gathered} u_{tt} - \Delta u + \rho(u_t) + f(u) = 0, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega\ , \quad t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{gathered} \tag{1.1}$$

where $\rho(v) = |v|^\beta v$, $\beta > -1$, $f(u) = bu|u|^\alpha$,$\alpha, b > 0$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. He showed that (1.1) has a unique global weak solution if $0 \leq \alpha \leq 2/(n-2)$, $n \geq 3$ and a global unique strong solution if $\alpha > 2/(n-2)$, $n \geq 3$; (of course if $n = 1$ or 2 then there is no restriction on α). In addition to global existence the issue of the decay rate was addressed. In both cases, it has been shown that the energy of the solution decays algebraically if $\beta > 0$ and it decays exponentially if $\beta = 0$. This improves an earlier result by the same author [2], where he studied the problem in an abstract setting and established a theorem concerning the decay of the solution energy only for the case $\alpha \leq 2/(n-2)$, $n \geq 3$. Later on, in a joint work with Ono [3], this result has been extended to the Cauchy problem:

$$\begin{gathered} u_{tt} - \Delta u + \lambda^2(x)u + \rho(u_t) + f(u) = 0, \quad x \in \mathbb{R}^n, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.2}$$

where $\rho(u_t)$ behaves like $|u_t|^\beta u_t$ and $f(u)$ behaves like $-bu|u|^\alpha$. In this case the authors required that the initial data be small enough in H^1 x L^2 norm and of compact support.

Pucci and Serrin [4] discussed the stability of the following problem:

$$\begin{gathered} u_{tt} - \Delta u + Q(x,t,u,u_t) + f(x,u) = 0, \qquad x \in \Omega, \qquad t > 0 \\ u(x,t) = 0, \quad x \in \partial\Omega\ , t \geq 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{gathered} \tag{1.3}$$

and proved that the energy of the solution is a Liaponov function. Although they did not discuss the issue of the decay rate, they did show that in general the energy goes to zero as t approaches infinity. They also considered an important special case of (1.3), which occurs when $Q(x,t,u,u_t) = a(t)t^\alpha u_t$ and $f(x,u) = V(x)u$, and showed that the behavior of the solutions depends crucially on the parameter α. If $|\alpha| \leq 1$ then the rest field is asymptotically stable. On the other hand, when $\alpha < -1$ or $\alpha > 1$ there are solutions that do not approach zero or approach nonzero functions $\phi(x)$ as $t \to \infty$.

Concerning nonexistence in (1.1), it is well known that, if $\rho(u_t) \equiv 0$ then the source term $f(u) = -bu|u|^{p-2}$ causes finite time blow up of solutions with negative initial energy (see [5–8]). The interaction between the damping and the source terms has been first considered by Levine [7, 8] in the linear damping case ($\rho(u_t) = au_t$). He showed that solutions with negative initial energy blow up in finite time. Recently Georgiev and Todorova [9] extended Levine's result to the nonlinear damping case. In their work, the authors introduced a different method and determined suitable relations between α and β, for which there is global existence or alternatively finite time blow up. Precisely, they showed that solutions with negative energy continue to exist globally 'in time' if $\beta \geq \alpha$ and blow up in finite time if $\beta < \alpha$ and the initial energy is sufficiently negative.

This result has been lately generalized to an abstract setting and to unbounded domains by Levine and Serrin [10] and Levine, Park, and Serrin [11] (See also [12]). In these papers, the authors showed that no solution with negative energy can be extended on $[0, \infty)$, if $\beta < \alpha$. This generalization allowed them also to apply their result to quasilinear situations. Vitillaro [13] combined the arguments in [9] and [10] to extend these results to situations where the damping is nonlinear and the solution has positive initial energy. It is also worth mentioning that the blow up result of [9] has been improved by Messaoudi [14], where the condition of sufficiently negative has been weakened to negative only.

In this paper we are concerned with the following initial boundary value problem:

$$
\begin{aligned}
&u_{tt} - \Delta u + a(1 + |u_t|^{m-2})u_t + bu|u|^{p-2} = 0, \qquad x \in \Omega, \qquad t > 0 \\
&u(x,t) = 0, \qquad x \in \partial\Omega\,,\ t \geq 0 \\
&u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega,
\end{aligned} \tag{1.4}
$$

where a, $b > 0$, m, $p > 2$, and Ω is a a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. We first state an existence theorem, which is known as a standard one (see [15–19]).

Theorem 1. *Suppose that $m > 2$, $p > 2$, such that*

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3 \tag{1.5}$$

and let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Then the problem (1.4) has a unique global solution

$$u \in C\left([0,\ \infty);\ H_0^1(\Omega)\right), \quad u_t \in C\left([0,\ \infty);\ L^2(\Omega)\right) \cap L^m\left(\Omega \times (0,\ \infty)\right). \tag{1.6}$$

Remark 1.1. This theorem can be also established by repeating the argument of [9]. However we do not need the condition $p \leq m$, imposed by the authors, due to the difference in the two problems.

2 MAIN RESULT

In this section we show that the solution energy, defined by:

$$E(t) := \frac{1}{2}\int_\Omega [u_t^2 + |\nabla u|^2](x,t)dx + \frac{b}{p}\int_\Omega |u(x,t)|^p dx, \tag{2.1}$$

decays exponentially if $2 \leq m \leq p$. For this purpose we prepare some material we need. Precisely we know, from Poincaré's inequality and Sobolev embedding theorems, that there exists a constant β depending on Ω only such that:

$$||u||_2 \leq \beta||\nabla u||_2; \quad ||u||_m \leq \beta||u||_p,\ m \leq p; \quad ||u||_m \leq \beta||\nabla u||_2\ . \tag{2.2}$$

Theorem 2. *Suppose that $2 \leq m \leq p$, with p satisfying (1.5). Then there exist positive constants K and α such that any solution of (1.4) in the class (1.6) satisfies*

$$E(t) \leq Ke^{-\alpha t}, \quad \forall t \geq 0. \tag{2.3}$$

Proof. We multiply Equation (1.4) by u_t and integrate over Ω to get:

$$E'(t) = -a\left[\int_\Omega |u_t(x,t)|^2 dx + \int_\Omega |u_t(x,t)|^m dx\right] \tag{2.4}$$

for any regular solution of (1.4). This identity remains valid for solutions (1.6) by a simple density argument.

Inspired by the idea of [20] and [21], we define:

$$H(t) := E(t) + \varepsilon \int_\Omega uu_t(x,t)dx, \tag{2.5}$$

for ε to be specified later. By using the Schwarz inequality and (2.2), we have:

$$|H(t) - E(t)| \le \varepsilon(1+\beta^2)E(t). \tag{2.6}$$

We differentiate (2.5) and use equation (1.4) and (2.1) to obtain:

$$\begin{aligned} H'(t) &= -a\int_\Omega |u_t(x,t)|^2 dx - a\int_\Omega |u_t(x,t)|^m dx + \varepsilon\int_\Omega [u_t^2 - |\nabla u|^2](x,t)dx \\ &\quad -a\varepsilon\int_\Omega u_t u(x,t)dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx - \varepsilon b\int_\Omega |u(x,t)|^p dx \\ &= -a\int_\Omega |u_t(x,t)|^2 dx - a\int_\Omega |u_t(x,t)|^m dx + \frac{3}{2}\varepsilon\int_\Omega u_t^2(x,t)dx - \frac{1}{2}\varepsilon\int_\Omega |\nabla u(x,t)|^2 dx \\ &\quad -a\varepsilon\int_\Omega u_t u(x,t)dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx - b(1-\frac{1}{p})\varepsilon\int_\Omega |u(x,t)|^p dx - \varepsilon E(t). \\ &\le -a\int_\Omega |u_t(x,t)|^2 dx - a\int_\Omega |u_t(x,t)|^m dx + \frac{3}{2}\varepsilon\int_\Omega u_t^2(x,t)dx \\ &\quad -\frac{1}{2}\varepsilon\int_\Omega |\nabla u(x,t)|^2 dx - a\varepsilon\int_\Omega u_t u(x,t)dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx - \varepsilon E(t). \end{aligned} \tag{2.7}$$

By using:

$$a\left|\int_\Omega u_t u(x,t)dx\right| \le \frac{1}{4}\int_\Omega |\nabla u(x,t)|^2 dx + a^2\beta\int_\Omega |u_t(x,t)|^2 dx,$$

inequality (2.7) takes then the form

$$\begin{aligned} H'(t) &\le -a\int_\Omega |u_t(x,t)|^2 dx - a\int_\Omega |u_t(x,t)|^m dx + (\frac{3}{2} + a^2\beta)\varepsilon\int_\Omega u_t^2(x,t)dx \\ &\quad -\frac{1}{4}\varepsilon\int_\Omega |\nabla u(x,t)|^2 dx - a\varepsilon\int_\Omega |u_t|^{m-2}u_t u(x,t)dx - \varepsilon E(t). \end{aligned} \tag{2.8}$$

We then exploit Young's inequality:

$$XY \le \delta X^r + c(\delta)Y^s,\ X, Y, \ge 0,\ \delta, c(\delta) > 0, 1/r + 1/s = 1,$$

with $r = m$ and $s = m/(m-1)$ to get:

$$\left|\int_\Omega |u_t|^{m-2}u_t u(x,t)dx\right| \le \delta||u||_m^m + c(\delta)||u_t||_m^m, \qquad \forall\delta > 0.$$

Therefore (2.8) becomes:

$$H'(t) \le -a\int_\Omega |u_t(x,t)|^2dx - a\int_\Omega |u_t(x,t)|^m dx + (\frac{3}{2}+a^2\beta)\varepsilon\int_\Omega u_t^2(x,t)dx$$
$$-\frac{1}{4}\varepsilon\int_\Omega |\nabla u(x,t)|^2dx + a\varepsilon\left[\delta||u||_m^m + c(\delta)||u_t||_m^m\right] - \varepsilon E(t), \qquad \forall\delta > 0,$$

which yields:

$$H'(t) \le -\varepsilon E(t) - \frac{1}{4}\varepsilon||\nabla u||_2^2 + a\varepsilon\delta||u||_m^m \tag{2.9}$$
$$-\left[a - (\frac{3}{2}+a^2\beta)\varepsilon\right]\int_\Omega u_t^2(x,t)dx - a\left[1-\varepsilon c(\delta)\right]||u_t||_m^m, \qquad \forall\delta > 0.$$

Up to this point, we distinguish two cases:

(1) $||u||_m \le 1$. In this case we have $||u||_m^m \le ||u||_m^2$; hence:

$$\frac{1}{4}||\nabla u||_2^2 - a\delta||u||_m^m \ge \frac{1}{4}||\nabla u||_2^2 - a\delta||u||_m^2 \ge 0, \tag{2.10}$$

if $\delta \le \delta_1 = 1/4a\beta^2$ (β is the constant appearing in (2.2)).

(2) $||u||_m \ge 1$. In this case we have $||u||_m^m \le ||u||_m^p \le \beta^p||u||_p^p$. This implies that:

$$\frac{1}{2}E(t) - a\delta||u||_m^m \ge \frac{b}{2p}||u||_p^p - a\delta||u||_m^m \ge \left(\frac{b}{2p} - a\delta\beta^p\right)||u||_p^p \ge 0, \tag{2.11}$$

if $\delta \le \delta_2 = b/2ap\beta^p$. Therefore by taking $\delta \le \min\{\delta_1, \delta_2\}$ and combining (2.9)–(2.11), we arrive at:

$$H'(t) \le -\frac{\varepsilon}{2}E(t) - a\left[1-\varepsilon c(\delta)\right]||u_t||_m^m - \left[a - (\frac{3}{2}+a^2\beta)\varepsilon\right]\int_\Omega u_t^2(x,t)dx. \tag{2.12}$$

By selecting $\varepsilon \le \min\{1/\ c(\delta), a/(\frac{3}{2}+a^2\beta)\}$, estimate (2.12) yields:

$$H'(t) \le -\frac{\varepsilon}{2}E(t). \tag{2.13}$$

We then combine (2.6) and (2.13) to get:

$$H'(t) \le -\frac{\varepsilon\left(1+\varepsilon(1+\beta^2)\right)}{2}H(t). \tag{2.14}$$

A simple integration of (2.14) then gives:

$$H(t) \le H(0)e^{-\alpha t}, \qquad \forall t \ge 0, \tag{2.15}$$

where $\alpha = \varepsilon\left(1+\varepsilon(1+\beta^2)\right)/2$. Again by combining (2.6), (2.15) and choosing ε so small that $1-\varepsilon(1+\beta^2) > 0$, we obtain:

$$E(t) \le \frac{1}{1-\varepsilon(1+\beta^2)}H(t) \le \frac{H(0)}{1-\varepsilon(1+\beta^2)}e^{-\alpha t}, \qquad \forall t \ge 0. \tag{2.16}$$

Therefore (2.3) is established.

Remark 3.1 The case $m = 2$ is a direct result of Nakao [2]. It can also be established by either repeating the same argument and using Schwarz' inequality instead of Young's inequality or using the argument in [22] based on a lemma by Komornik [23].

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