BLOW UP IN SOLUTIONS OF A LINEAR WAVE EQUATION WITH MIXED NONLINEAR BOUNDARY CONDITIONS

Salim A. Messaoudi*

Mathematical Sciences Department
King Fahd University of Petroleum & Minerals
Dhahran, Saudi Arabia

الخلاصـة :

ندرس في هذا البحث معادلة الأمواج الخطيَّةِ ذات بُعدٍ واحدٍ مُرفقةً بشروط حديَّة غير خطيَّة؛ حيث نثبت نظريةً متعلقةً بوجود حلول محليَّة. كذلك نبيِّن كيف أن اللاخطية في الحدود تسبب حصول انفجار (شذوذ) لهذه الحلول حتى ولو كانت المعطيات الأولية صغيرة وملساء جدا .

ABSTRACT

We consider a one-dimensional linear wave equation associated with mixed nonlinear boundary conditions. We prove a local existence result and we then show that the nonlinearity at the boundary causes a finite time blow up of the solution, even for small and smooth initial data.

Keywords: wave equation, local existence, blow up, solution energy.
Subject classification 35 L45

*Address for correspondence:
KFUPM Box 5046
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia
e-mail: messaoud@kfupm.edu.sa

BLOW UP IN SOLUTIONS OF A LINEAR WAVE EQUATION WITH MIXED NONLINEAR BOUNDARY CONDITIONS

1. INTRODUCTION

In [1], the wave equation with nonlinear damping and source terms, on a bounded domain Ω of $\mathbb{R}^n$, has been studied. Namely, the authors considered the following problem:

$$\begin{aligned} u_{tt} - \Delta u + a u_t |u_t|^{m-1} &= bu|u|^{p-1}, \qquad x \in \Omega, \quad t > 0 \\ u(x,t) &= 0, \qquad x \in \partial\Omega, \quad t > 0 \\ u(x,0) &= u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{aligned} \tag{1.1}$$

where $a, b > 0$, $p, m > 1$, and Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$. They discussed the interaction between the damping and the source terms and established a global existence for $p > m$ and a blow-up result for $p < m$.

Of course, in the case where $a = 0$, it is well known that the source term destabilizes the solution and causes a finite time blow-up (see [2–4]). On the other hand, if $b = 0$, the nonlinear damping term assures global existence for small initial data (see [5]).

In [6] and [7], the linear wave equation together with a nonlinear feedback at the boundary has been investigated. Precisely, the authors looked into the following problem:

$$\begin{aligned} & u_{tt} - \Delta u = 0, && x \in \Omega, \quad t > 0 \\ & \frac{\partial u}{\partial \nu}(x,t) = -m(x).\nu(x) g(u_t), && x \in \Gamma_0, \quad t > 0 \\ & u(x,t) = 0, && x \in \Gamma_1, \quad t > 0 \\ & u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), && x \in \Omega. \end{aligned} \tag{1.2}$$

where $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n$, $\Gamma_0 = \{x \in \partial\Omega \ : \ m(x).\nu(x) > 0\}$, and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, with $\Gamma_1 \neq \phi$.

They showed that, under certain growth conditions on g and for suitable initial data, the energy of the solution decays exponentially. A similar problem has also been studied by Aliev and Khanmamedov [8] in the n-dimensional open unit cube.

For the one-dimensional situation, several results have been established. In [9], for instance, the following problem has been considered:

$$\begin{aligned} & u_{tt} - u_{xx} = f(x,t,u,u_x,u_t), \qquad x \in I = (0, \ 1), && t > 0 \\ & u_x(0,t) - \alpha u(0,t) = u_x(1,t) + \beta u(1,t) = 0, && t > 0 \\ & u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), && x \in I, \end{aligned} \tag{1.3}$$

where α and β are positive constants. The authors proved a local existence result, as well as the asymptotic expansion of the solution. As they pointed out, their result is a relative generalization of [10]. Messaoudi [11] considered the same problem and established a global existence, as well as a blow up result. It is also worth mentioning that (1.3), for different forms of the function f and different types of boundary conditions, has been discussed by Aregba and Hanouzet [12], Nguyen and Alain [13] and many others. (see [9] for more references.)

In this paper, we consider the following problem:

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad x \in I, \qquad t > 0 \tag{1.4}$$

$$u_x(0,t) = |u(0,t)|^\alpha u(0,t), \quad u_x(1,t) = |u(1,t)|^\alpha u(1,t), \qquad t > 0 \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in I, \tag{1.6}$$

where $\alpha > 0$. We will establish a local existence theorem and show that, for suitably chosen initial data, the solution blows up in finite time.

2. LOCAL EXISTENCE

This section is devoted to the statement and the proof of our local existence theorem. For this purpose we consider the following semilinear problem:

$$\begin{aligned} &v_{tt} - v_{xx} = f(v, v_x, v_t), \qquad x \in I, \qquad t > 0 \\ &v_x(0,t) = \gamma, \quad v_x(1,t) = \gamma, \qquad t > 0 \\ &v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in I, \end{aligned} \tag{2.1}$$

where f is a C^1 function, $\gamma \in \mathbb{R}$, and

$$v_0 \in H^2(I), \ v_1 \in H^1(I) \tag{2.2}$$

Lemma 2.1. *Let v_0 and v_1 be given, satisfying (2.2). Then the problem (2.1) has a unique local solution v defined on a maximal time interval $[0,T)$ and satisfying*

$$v \in C^k\left([0,T); \ H^{2-k}(I)\right) \qquad k = 0,1,2. \tag{2.3}$$

Remark 2.1. The proof of this lemma can be established by either using a classical energy argument [14] or the nonlinear semigroup theory [15].

Theorem 2.2. *Let $u_0 \in H^2(I)$ and $u_1 \in H^1(I)$ be given, such that $u_0(x) > 0, \forall x \in I$. Then the problem (1.4) - (1.6) has a unique local solution u defined on a maximal time interval $[0,T')$ and satisfying*

$$\begin{aligned} &u \in C^k\left([0,T'); \ H^{2-k}(I)\right), \qquad k = 0,1,2 \\ &u(x,t) > 0 \ \forall x \in I, \ 0 \le t < T'. \end{aligned} \tag{2.4}$$

Remark 2.2. There exists a constant M such that $0 < u_0(x) \le M$, $\forall x \in I$, since $u_0 \in H^2(I)$.

Proof. We set:

$$v(x,t) := |u(x,t)|^{-\alpha}$$

and substitute in (1.4) - (1.6), to get:

$$\begin{aligned} &v_{tt} - v_{xx} = \frac{\alpha+1}{\alpha}(v_t^2 - v_x^2)/v, \qquad x \in I, \qquad t > 0 \\ &v_x(0,t) = u_x(1,t) = -\alpha, \quad t > 0, \\ &v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \qquad x \in I, \end{aligned} \tag{2.5}$$

where:

$$v_0(x) = u_0^{-\alpha}(x), \qquad v_1(x) = -\alpha u_0^{-\alpha-1}(x) u_1(x). \tag{2.6}$$

We then define a C^1 function f such that:

$$f(\lambda,\xi,\eta)=\frac{\alpha+1}{\alpha}\frac{\eta^2-\xi^2}{\lambda},\ \lambda\geq\frac{1}{2}M^{-\alpha}, \tag{2.7}$$

and consider the problem (2.1) with f as defined in (2.7) and v_0 and v_1 as defined in (2.6). Therefore the Lemma guarantees the existence of a solution v defined on $[0,T)$. Since $v_0(x)>M^{-\alpha}$ then the uniform continuity of v allows us to choose $T'\leq T$, for which:

$$v(x,t)>\frac{1}{2}M^{-\alpha},\qquad \forall x\in I,\ 0\leq t<T'\ ;$$

hence v is a solution of (2.5) on the interval $[0,T')$.

It is easy to check that:

$$u(x,t)=v^{-1/\alpha}(x,t),\ \forall x\in I,\ 0\leq t<T'$$

is the unique solution of (1.4) - (1.6).

Remark 2.3. A similar result can also be obtained for $u_0(x)<0$.

3. BLOW UP

Theorem 3.1. *Let $u_0\in H^2(I)$ and $u_1\in H^1(I)$ be given satisfying:*

$$E_0=\int_0^1[u_1^2(x)+u_0'^2(x)]dx-\frac{2}{\alpha+2}\left[|u_0(1)|^{\alpha+2}-|u_0(0)|^{\alpha+2}\right]<0. \tag{3.1}$$

Then any solution of (1.4) - (1.6) blows up in finite time.

Proof. We define the formal energy of the solution by:

$$E(t):=\int_0^1[u_t^2(x,t)+u_x^2(x,t)^2]dx-\frac{2}{\alpha+2}\left[|u(1,t)|^{\alpha+2}-|u(0,t)|^{\alpha+2}\right].$$

By multiplying Equation (1.4) by u_t and integrating over I, we easily see that $E'(t)=0$; hence $E(t)=E_0<0$. We also define:

$$F(t):=\frac{1}{2}\int_0^1 u^2(x,t)\,dx+\frac{1}{2}\beta(t+t_0)^2,\ \forall\ t\geq 0,$$

for $t_0>0$ and $\beta>0$ to be chosen later. By differentiating F twice we get:

$$F'(t)=\int_0^1 u(x,t)u_t(x,t)\,dx+\beta(t+t_0)\quad F''(t)=\int_0^1[uu_{tt}+u_t^2](x,t)\,dx+\beta.$$

Straightforward computations, using Equation (1.4) and integration by parts, yield:

$$\begin{aligned}F''(t)&=\int_0^1[u_t^2-u_x^2](x,t)\,dx+\left[|u(1,t)|^{\alpha+2}-|u(0,t)|^{\alpha+2}\right]+\beta \\ &=-(\frac{\alpha}{2}+1)E(t)+\beta+(2+\frac{\alpha}{2})\int_0^1 u_t^2(x,t)\,dx+\frac{\alpha}{2}\int_0^1 u_x^2(x,t)\,dx.\end{aligned}\tag{3.2}$$

We then define $G(t):=F^{-\gamma}(t)$, for γ to be chosen properly. By differentiating G twice we arrive:

$$G'(t)=-\gamma F^{-(\gamma+1)}(t)F'(t),\qquad G''(t)=-\gamma F^{-(\gamma+2)}(t)Q(t),$$

where:

$$
\begin{aligned}
Q(t) &:= F(t)F''(t) - (\gamma+1)F'^2(t) \\
&= F(t)\left(-(\frac{\alpha}{2}+1)E(t) + \beta + (2+\frac{\alpha}{2})\int_0^1 u_t^2(x,t)\,dx + \frac{\alpha}{2}\int_0^1 u_x^2(x,t)\,dx\right) \qquad (3.3)\\
&\quad -(\gamma+1)\left(\int_0^1 u(x,t)u_t(x,t)\,dx + \beta(t+t_0)\right)^2 .
\end{aligned}
$$

By using Schwartz inequality and Young's inequality, (3.3) yields:

$$
\begin{aligned}
Q(t) &\geq F(t)\left(-(\frac{\alpha}{2}+1)E(t) + \beta + (2+\frac{\alpha}{2})\int_0^1 u_t^2(x,t)\,dx + \frac{\alpha}{2}\int_0^1 u_x^2(x,t)\,dx\right)\\
&\quad -(\gamma+1)\left((1+\frac{\varepsilon}{2})\int_0^1 u_t^2(x,t)\,dx\int_0^1 u^2(x,t)\,dx + (1+\frac{1}{2\varepsilon})\beta^2(t+t_0)^2\right).\\
&\geq F(t)\left(-(\frac{\alpha}{2}+1)E(t) + \beta + (2+\frac{\alpha}{2})\int_0^1 u_t^2(x,t)\,dx + \frac{\alpha}{2}\int_0^1 u_x^2(x,t)\,dx\right)\\
&\quad -2F(t)(\gamma+1)\left((1+\frac{\varepsilon}{2})\int_0^1 u_t^2(x,t)\,dx + (1+\frac{1}{2\varepsilon})\beta\right), \qquad \forall\,\varepsilon>0.\\
&\geq F(t)\left(\{\frac{\alpha}{2}-\varepsilon-(2+\varepsilon)\gamma\}\int_0^1 u_t^2(x,t)\,dx + \frac{\alpha}{2}\int_0^1 u_x^2(x,t)\,dx - (\frac{\alpha}{2}+1)E_0\right. \qquad (3.4)\\
&\quad \left. -\{1+\frac{1}{\varepsilon}-(2+\frac{1}{\varepsilon})\gamma\}\beta\right), \ \forall\,\varepsilon>0.
\end{aligned}
$$

By choosing $\varepsilon < \alpha/2$, γ so that $0<\gamma<(\alpha/2-\varepsilon)/(2+\varepsilon)$ and β so small that:

$$
-(\frac{\alpha}{2}+1)E_0 - \{1+\frac{1}{\varepsilon}-(2+\frac{1}{\varepsilon})\gamma\}\beta \geq 0,
$$

we conclude, from (3.4), that $Q(t)\geq 0$, $\forall\, t\geq 0$. Therefore $G''(t)\leq 0\ \forall\, t\geq 0$; hence G' is decreasing. By choosing t_0 large enough we get:

$$
F(0) = \int_0^1 u_0(x)u_1(x)\,dx + \beta t_0 > 0,
$$

hence $G'(0)\leq 0$. Finally Taylor expansion of G yields:

$$
G(t) \leq G(0) + t\ G'(0),\ \forall t,
$$

which shows that $G(t)$ vanishes at a time $t_m \leq -G(0)/G'(0)$. Consequently $F(t)$ must become infinite at time t_m.

Remark 3.1. Note that no assumption has been made on the size of the initial data. In fact the blow up takes place even for small data provided that (3.1) is satisfied.

Example. We consider the following problem:

$$
\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t),\ x\in(0,\ 1),\quad t>0\\
&u_x(x,t) = |u(x,t)|^{1\backslash 3}u(x,t), \qquad x=0,\ 1,\ t>0 \qquad (3.5)\\
&u_0(x) = 27(-x+2)^{-3}, \qquad u_1(x) = 81(-x+2)^{-4}.
\end{aligned}
$$

Direct computations show that the hypothesis (3.1) is satisfied and:

$$
u(x,t) = 27(-x+2-t)^{-3}
$$

is a solution of (3.5). Also by computing:

$$F(t) = \frac{729}{5}\left[\frac{1}{(1-t)^5} - \frac{1}{(2-t)^5}\right], \tag{3.6}$$

we easily see that the blow up occurs at $t = 1$.

ACKNOWLEDGEMENT

The author would like to thank an anonymous referee for a useful suggestion related to the proof of one of the theorems in this paper.

REFERENCES

[1] V. Georgiev and G. Todorova, "Existence of Solutions of the Wave Equation with Nonlinear Damping and Source Terms", *J. Diff. Eq.*, **109(2)** (1994), pp. 295–308.

[2] J. Ball, "Remarks on Blow Up and Nonexistence Theorems for Nonlinear Evolution Equations", *Quart. J. Math., Oxford*, **28(2)** (1977), pp. 473–483.

[3] H.A. Levine, "Instability and Nonexistence of Global Solutions of Nonlinear Wave Equation of the Form $Pu_{tt} = Au + F(u)$", *Trans. Amer. Math. Soc.*, **192** (1974), pp. 1–21.

[4] H.A. Levine, "Some Additional Remarks on the Nonexistence of Global Solutions to Nonlinear Wave Equation", *SIAM J. Math. Anal.*, **5** (1974), pp. 138–146.

[5] A. Haraux and E. Zuazua, "Decay Estimates for Some Semilinear Damped Hyperbolic Problems", *Arch. Rational Mech. Anal.*, **150** (1988), pp. 191–206.

[6] E. Zuazua, "Uniform Stabilization of the Wave Equation by Nonlinear Boundary Feedback", *SIAM J. Control and Opt.*, **28** (1990), pp. 466–477.

[7] A. Komornik and E. Zuazua, "A Direct Method for the Boundary Stabilization of the Wave Equation", *J. Math. Pure and Appl.*, **69** (1990), pp. 33–54.

[8] A.B. Aliev and Kh. Khanmamedov, "Energy Estimates for Solutions of the Mixed Problem for Linear Second-Order Hyperbolic Equations", *Math. Notes*, **59(4)** (1996), pp. 345–349.

[9] Nguyen Thanh Long and Tran Ngoc Diem, "On the Nonlinear Wave Equation $u_{tt} - u_{xx} = f(x,t,u,u_x,u_t)$ Associated with the Mixed Homogeneous Conditions", *Nonlinear Analysis: T.M.A.*, **29(11)** (1997), pp. 1217–1230.

[10] Alain Pham Ngoc Dinh, "Linear Approximation and Asymptotic Expansion Associated with the Nonlinear Wave Equation in One Dimension", *Demonstratio Math.*, **19** (1986), pp. 45–63.

[11] S.A. Messaoudi, "Global Existence and Blow Up in Solutions of a Wave Equation with Mixed Boundary Conditions", *Technical Report # **242**, Math Dept., KFUPM, Saudi Arabia*, April 1999.

[12] D.D. Aregba and B. Hanouzet, "Cauchy Problem for One-Dimensional Semilinear Hyperbolic Systems: Global Existence, Blow-Up", *J. Diff. Eq.*, **125** (1996), pp. 1–26.

[13] Nguyen Thanh Long and Alain Pham Ngoc Dinh, "A Semilinear Wave Equation Associated with a Linear Differential Equation with Cauchy Data", *Nonlinear Analysis: T.M.A.*, **24** (1995), pp. 1261–1279.

[14] C.M. Dafermos and W.J. Hrusa, "Energy Methods for Quasilinear Hyperbolic Initial Value Problems. Applications to Elastodynamics", *Arch. Rational Mech. Anal.*, **87** (1985), pp. 267–292.

[15] T.J.R. Hughes, T. Kato, and J.E. Marsden, "Well-Posed Quasilinear Second Order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity", *Arch. Rational Mech. Anal.*, **63** (1977), pp. 273–294.

Paper Received 6 April 1999; Accepted 11 October 1999.