ON SOME WEAKER FORMS OF ALEXANDROFF SPACES

Francisco G. Arenas
Area of Geometry and Topology, Faculty of Science
Universidad de Almería, 04071 Almería, Spain
e-mail: fgarenas@obelix.cica.es

Julian Dontchev
Department of Mathematics, University of Helsinki
PL 4, Yliopistonkatu 15, 00014 Helsinki, Finland
e-mail: dontchev@cc.helsinki.fi

and

Maximilian Ganster*
Department of Mathematics, Graz University of Technology
Steyrergasse 30, A-8010 Graz, Austria
e-mail: ganster@weyl.math.tu-graz.ac.at

Dedicated to Professor P. Alexandroff's 100th birthday

الخلاصة :

تُعَرَّف فضاءات الكساندروف بأنها فضاءات توبولوجية عناصرها مُحتواة في مجموعات صُغرى مفتوحة، أو التي يكون فيها أي تقاطع لمجموعاتها المفتوحة مفتوح. نطرح في هذا البحث مفهومين من مفاهيم الكساندروف أضعف من فضاءاته هما : فضاء الكساندروف المُعمَم، وشبيه فضاء الكساندروف. وفي فضاء $T_{1/2}$ تتطابق مفاهيم الكساندروف وفضاء (g–الكساندروف) بينما يقع شبيه فضاء الكساندروف بين فضائي الكساندروف وفضاء P'.

ABSTRACT

Alexandroff spaces are the topological spaces in which each element is contained in a smallest open set or equivalently the spaces where arbitrary intersections of open sets are open. In this paper we introduce and study two weaker concepts of Alexandroff spaces, namely generalized Alexandroff and semi-Alexandroff spaces. In $T_{1/2}$-spaces the concepts of Alexandroff and g-Alexandroff spaces coincide, while the class of semi-Alexandroff spaces is properly placed between the classes of Alexandroff and P'-spaces.

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* To whom correspondence should be addressed.

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1. INTRODUCTION

In digital topology, many problems are directly or indirectly connected with the classes of locally finite and Alexandroff spaces. A topological space (X, τ) is called *locally finite* if each element x of X is contained in a finite open set and a finite closed set. It is called *Alexandroff* [1] if arbitrary intersections of open sets are open. Clearly all finite topological spaces are locally finite and all locally finite topological spaces are Alexandroff.

The major building block of the digital n-space is the *digital line* or the so called *Khalimsky line* [2]–[4]. This Alexandroff space is the set of the integers, $\mathbf{Z}$, equipped with the topology $\mathcal{K}$, generated by $\mathcal{G}_{\mathcal{K}} = \{\{2n-1, 2n, 2n+1\}: n \in \mathbf{Z}\}$. The Khalimsky line has higher separation than T_0, in fact it is $T_{\frac{3}{4}}$ [5]. Spaces in which non-closed singletons are regular open are called $T_{\frac{3}{4}}$ in [5].

In this paper, we consider a weaker form of Alexandroff spaces called generalized Alexandroff, which in $T_{\frac{3}{4}}$-spaces, even in $T_{\frac{1}{2}}$-spaces, coincides with Alexandroff. We define a topological space (X, τ) to be *generalized Alexandroff* if arbitrary intersections of open sets are generalized open. A subset A of a topological space (X, τ) is called *generalized open* [6] (= g-open) if its interior contains every closed subset of A. It is called *generalized closed* (= g-closed) if its closure is included in every open superset of A [6]. Clearly, complements of g-closed sets are g-open. Also in [6], Levine introduced the concept of $T_{\frac{1}{2}}$*-spaces* as the spaces where every generalized closed (= g-closed) set is closed or equivalently these are the spaces whose non-closed singletons are isolated.

We also investigate a class of spaces called semi-Alexandroff, which is properly placed between the classes of Alexandroff and P'-spaces. Recall that a P'*-space* [7] is a topological space whose non-empty G_δ-sets have non-empty interiors. As mentioned in [8], it is not difficult to see that a topological space (X, τ) is a P'-space if and only if arbitrary countable intersections of open sets are semi-open. A *semi-open* set is a set which lies between an open set and its closure. Complements of semi-open sets are called *semi-closed.*

2. GENERALIZED ALEXANDROFF SPACES

Definition 1 A topological space (X, τ) is called *generalized Alexandroff* (= g-Alexandroff) if any intersection of open sets is g-open. A point $x \in X$ is an *A-point* (= Alexandroff point) if x has a minimal (necessarily open) neighborhood. Let us also define a g-neighborhood of x to be any g-open set containing x.

Remark 2.1 The spaces where every point has a minimal generalized open neighborhood are precisely the Alexandroff spaces (since if a space has a minimal g-open neighborhood then this neighborhood has to be open, because the interior of each neighborhood is clearly g-open).

Theorem 2.2 For a topological space (X, τ) the following conditions are equivalent:

(1) X is g-Alexandroff.

(2) Intersections of g-open sets are g-open.

(3) Unions of closed sets are g-closed.

(4) Unions of g-closed sets are g-closed.

Proof. It is clear that (2) implies (1). Now suppose that (1) holds. Let $\{A_i : i \in I\}$ be a collection of g-open sets and let A be its intersection. We have to show that A is g-open. So let $F \subseteq X$ be closed and $F \subseteq A$. Then $F \subseteq A_i$ and so $F \subseteq \text{int}A_i$ for each $i \in I$. If $B = \cap_{i \in I}\text{int}A_i$, then B is g-open by (1) and so $F \subseteq \text{int}B$. Since $\text{int}B \subseteq \text{int}A$ we are done, *i.e.* A is g-open.

The equivalences (1) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are obvious. □

Theorem 2.3 (1) The property g-Alexandroff is a topological property.

(2) Every Alexandroff space is g-Alexandroff. □

Proof. (1) Let $f: (X, \tau) \to (Y, \sigma)$ be a homeomorphism, let X be g-Alexandroff and let $\{V_i : i \in I\}$ be a collection of closed sets in Y. By assumption $U = \cup_{i \in I} f^{-1}(V_i)$ is g-closed in X. Set $V = \cup_{i \in I}(V_i)$. By Theorem 3.6 from [6] (f is both closed and continuous), $V = f(U)$ is g-closed. Hence Y is g-Alexandroff.

(2) is obvious, since open sets are g-open. □

We have the following lemma which gives us many g-Alexandroff spaces.

Lemma 2.4 If a topological space X has a point p whose only neighborhood is X itself, then X is g-Alexandroff.

Proof. First observe that p has to be in every non-empty closed set. Let $\{O_i : i \in I\}$ be a collection of open sets and let O be its intersection. Let $F \subseteq X$ be closed with $F \subseteq O$. If $F = \emptyset$, then $F \subseteq \text{int}O$. If $F \neq \emptyset$, then $p \in F$ and $p \in O_i$ for each $i \in I$. Hence $O_i = X$ for each $i \in I$ and so $O = X$ and $F \subseteq \text{int}O$. □

The following example shows that not every g-Alexandroff space is Alexandroff.

Example 2.5 Let X be the real line with the following topology: the non-trivial open sets are the intervals $(-\frac{1}{k}, \frac{1}{k})$, where $k \in \mathbb{N}$, $\mathbb{N}$ being the set of all positive integers. By Lemma 2.4, it follows that X is g-Alexandroff (*i.e.* the only neighborhood of 2 is X itself). Clearly $\{0\}$ is an intersection of open sets but not open. Hence X is not Alexandroff.

Theorem 2.6 For a g-Alexandroff space (X, τ) the following conditions are valid and equivalent to each other:

(1) Each point $x \in X$ has a minimal g-neighborhood.

(2) Each closed singleton is an A-point.

Proof. It is easy to see that in g-Alexandroff spaces condition (1) is valid, since in g-Alexandroff spaces any intersection of g-open sets is g-open according to Theorem 2.2.

Now suppose that (1) holds and let $\{x\}$ be closed. Let U be the minimal g-neighborhood of x. Since U is g-open, then we have $x \in \text{int}U$, therefore U is a minimal neighborhood of x, *i.e.* x is an A-point. Conversely, suppose that (2) holds and let $x \in X$. If $\{x\}$ is not closed, then $\{x\}$ is a minimal g-neighborhood of x. If $\{x\}$ is closed, then x is an A-point and therefore has a minimal neighborhood, which is necessarily a minimal g-neighborhood. □

There are, however, spaces in which each point has a minimal g-neighborhood but which fail to be g-Alexandroff as the following example shows.

Example 2.7 Let X be an infinite set and let $\{A_n : n \in \mathbf{N}\}$ be a partition of X where each A_n is infinite. For each cofinite subset $M \subseteq \mathbf{N}$ let $O_M = \cup_{n \in M} A_n$. Then $\{\emptyset, X\} \cup \{O_M : M \subseteq \mathbf{N} \text{ cofinite}\}$ is a topology on X. Since $A_n = X \setminus O_{\mathbf{N} \setminus \{n\}}$, then each A_n is closed. On the other hand, $A_n = \cap_{m \neq n} O_{\mathbf{N} \setminus \{m\}}$, so each A_n is also an intersection of open sets. If $x \in X$, then $\{x\}$ is clearly not closed, hence $\{x\}$ is a minimal g-neighborhood of x. However, X is not g-Alexandroff, since each A_n is closed but not g-open.

Theorem 2.8 A $T_{\frac{1}{2}}$-space is g-Alexandroff if and only if it is Alexandroff. □

Proof. A topological space (X, τ) is $T_{\frac{1}{2}}$ if and only if the collection of all g-open sets in X coincides with τ [6]. □

Thus as a corollary of Theorem 2.8 above, in the notation of Theorem 2.7 and Theorem 2.8 from [9], we have that all g-Alexandroff $T_{\frac{1}{2}}$-spaces are locally path-connected, first countable, orthocompact, and that in g-Alexandroff $T_{\frac{1}{2}}$-spaces the concepts of path-connectedness, connectedness, and chain-connectedness coincide.

Recall that a topological space (X, τ) is called *submaximal* if every dense subset of X is open or equivalently if every subset of X is locally closed (= intersection of an open and a closed set). Note that a submaximal space is hereditarily irresolvable, where a space is said to be *irresolvable* if any two dense subsets have non-empty intersection.

Theorem 2.9 (1) [10] A submaximal space is $T_{\frac{1}{2}}$.

(2) If a space (X, τ) is $T_{\frac{1}{2}}$ and g-Alexandroff, then it is submaximal and Alexandroff.

Proof. To prove (2), first observe that (X, τ) is Alexandroff by Theorem 2.8. Let $D \subseteq X$ be dense. Then $D = X \setminus \cup_{x \notin D} \{x\}$. Since (X, τ) is $T_{\frac{1}{2}}$, then $\{x\}$ is closed for each $x \notin D$ and therefore $\cup_{x \notin D} \{x\}$ is g-closed by Theorem 2.2. Thus D is g-open and therefore open, since (X, τ) is $T_{\frac{1}{2}}$, *i.e.* (X, τ) is submaximal. □

Corollary 2.10 A $T_{\frac{1}{2}}$ g-Alexandroff space is hereditarily irresolvable. □

Recall that a space (X, τ) is called a *partition space* if every open subset is closed. (X, τ) is an *R_0-space* if $\overline{\{x\}} \subseteq U$ for each open neighborhood U of x.

Theorem 2.11 For a space (X, τ) the following conditions are equivalent:

(1) (X, τ) is R_0 and g-Alexandroff.

(2) (X, τ) is a partition space.

Proof. It is clear that (2) $\Rightarrow$ (1).

Now suppose that (1) holds and let $U \subseteq X$ be open. Then $U = \cup_{x \in U} \overline{\{x\}}$ and therefore U is also g-closed, hence closed. Thus (1) $\Rightarrow$ (2). □

Since a T_1 partition space is discrete, then we have:

Corollary 2.12 A space (X, τ) is T_1 and g-Alexandroff if and only if it is discrete. □

Remark 2.13 The T_1 condition in the theorem above cannot be reduced to $T_{\frac{3}{4}}$ as the digital line shows.

Theorem 2.14 Closed subsets of g-Alexandroff spaces are g-Alexandroff.

Proof. Let (X,τ) be g-Alexandroff and let $A \subseteq X$ be closed. Let $\{V_i: i \in I\}$ be a collection of open sets in $(A, \tau|A)$ and let V be its intersection. For each $i \in I$ there is a τ-open set O_i with $V_i = A \cap O_i$. Now let $F \subseteq A$ be closed in $(A,\tau|A)$ with $F \subseteq V$. Then F is also closed in (X,τ) and $F \subseteq \cap\{O_i: i \in I\}$, hence $F \subseteq \text{int}(\cap\{O_i: i \in I\})$. Since $A \cap \text{int}(\cap\{O_i: i \in I\})$ is open in $(A,\tau|A)$ and contained in V, we have $F \subseteq \text{int}_A(V)$, *i.e.* $(A,\tau|A)$ is g-Alexandroff. □

Our next example shows that g-Alexandroff spaces need not be hereditarily g-Alexandroff.

Example 2.15 Now consider the set X of all reals with the topology generated by the empty set, X and all intervals $(-1/k, 1/k)$ as well as $(1-1/k, 1+1/k)$. This space clearly satisfies the requirements of Lemma 2.4 (*e.g.* the only neighborhood of 2 is X) and therefore is g-Alexandroff. Now consider the subspace $Y = [0,1)$. Then $\{0\}$ is an intersection of open sets in Y and also closed in Y! Clearly $\{0\}$ is not open in Y therefore Y is not g-Alexandroff.

Clearly every Alexandroff space is hereditarily g-Alexandroff. It is easily checked that the space from Example 2.5 is hereditarily g-Alexandroff (but not Alexandroff).

A *GO-continuum* is a connected GO-compact space (= every cover by g-open sets has a finite subcover [11]). Sets that can be represented as the intersection of a g-closed and a g-open set are called in [12] *generalized locally closed.* We refer to a *feebly-$T_{\frac{1}{2}}$-space* as to a space where the union of any two disjoint g-open sets is g-open.

Theorem 2.16 Every subset of a g-Alexandroff feebly-$T_{\frac{1}{2}}$ GO-continuum (X,τ) is generalized locally closed.

Proof. Let $A \subseteq X$. Set $A = A_1 \cup A_2$, with $A_1 \cap A_2 = \emptyset$ such that every singleton of A_1 is closed in X and every singleton of A_2 is g-open in X (note that non-closed singletons must be g-open). Clearly A_1 is g-closed, since X is g-Alexandroff. We need to show that A_2 is g-open. Let F be closed (in X) with $F \subseteq A_2$. Since X is GO-compact, then F is GO-compact [11, Proposition 8]. Note that $F = \cup_{x \in F}\{x\}$, where each $\{x\}$ is g-open. By the GO-compactness of F, it follows that F is finite. Since X is feebly-$T_{\frac{1}{2}}$, then F is g-open and moreover clopen, since it is also closed. Due to the connectedness of X it follows that $F = \emptyset$ or $F = X$. If $F = \emptyset$, then A_2 is trivially g-open. If $F = X$, then each point of X is g-open, hence X is finite (due to its GO-compactness) and also each one of its subsets is g-open (X is feebly-$T_{\frac{1}{2}}$). In particular each set is generalized locally closed. □

Recall that a function $f:(X,\tau) \to (Y,\sigma)$ is called *g-closed* (resp. *g-open*) [13] if the image of every closed set of X is g-closed (resp. g-open) in Y.

Theorem 2.17 Let $f:(X,\tau) \to (Y,\sigma)$ be a surjective continuous g-closed function. If X is Alexandroff, then Y is g-Alexandroff.

Proof. Let $\{B_i: i \in I\}$ be a collection of closed subsets of Y. Set $B = \cup_{i \in I} B_i$. Since f is continuous and since X is Alexandroff, then $A = f^{-1}(B) = \cup_{i \in I} f^{-1}(B_i)$ is closed in X. Since f is onto and g-closed, then $B = f(A)$ is g-closed in Y. Thus Y is g-Alexandroff. □

A bijection $f:(X,\tau) \to (Y,\sigma)$ is called a *g-homeomorphism* (= generalized homeomorphism) [14] if f is both g-continuous and g-open. Note that for a bijective and g-continuous function $f:(X,\tau) \to (Y,\sigma)$, the statements of f being g-open and being g-closed are equivalent [14]. The kernel of a set A, denoted by $A^{\wedge}$ [15], is the intersection of all open supersets of A. A subset A of a topological space (X,τ) is called a Λ*-set* [15] if $A = A^{\wedge}$ and a λ*-set* [16] if $A = L \cap F$, where L is a Λ-set and F is closed. A function $f:(X,\tau) \to (Y,\sigma)$ is called *g-continuous* [11] if $f^{-1}(V)$ is g-closed in (X,τ) for every closed set V of (Y,σ) and λ*-continuous* [16] if $f^{-1}(V)$ is λ-closed in (X,τ) for every closed set V of (Y,σ).

The following lemma is from [16]:

Lemma 2.18 [16]. A function $f:(X,\tau) \to (Y,\sigma)$ is continuous if and only if it is g-continuous and λ-continuous. □

As a corollary of Theorem 2.17, in the notion of the above given lemma, we have the following result:

Corollary 2.19 If (X,τ) is Alexandroff, $f:(X,\tau) \to (Y,\sigma)$ is both λ-continuous and g-homeomorphism, then (Y,σ) is g-Alexandroff. □

A function $f:(X,\tau) \to (Y,\sigma)$ is called *gc-irresolute* [11] if for every g-closed set V of Y its inverse image $f^{-1}(V)$ is g-closed in X. Recall that a bijection $f:(X,\tau) \to (Y,\sigma)$ is said to be a *gc-homeomorphism* [14] if f is gc-irresolute and its inverse f^{-1} is also gc-irresolute.

Theorem 2.20 Let $f:(X,\tau) \to (Y,\sigma)$ be gc-homeomorphism. If X is g-Alexandroff, then Y is also g-Alexandroff, *i.e.* the property g-Alexandroff is gc-topological property.

Proof. Let $\{V_i : i \in I\}$ be a collection of g-closed subsets of Y and let $V = \cup_{i\in I} V_i$. Since f is gc-irresolute and since X is g-Alexandroff, then in the notion of Theorem 2.2 $U = f^{-1}(V) = \cup_{i\in I} f^{-1}(V_i)$ is g-closed in X. Since f is onto and since f^{-1} is gc-irresolute, then $V = f(U)$ is g-closed in Y. Therefore, according to Theorem 2.2 Y is g-Alexandroff. □

It is well-known that finite union of g-closed sets is g-closed [6]. The following lemma is an improvement of that result and it enables us to prove a result on topological sums of g-Alexandroff spaces.

Lemma 2.21 Let $(A_i)_{i\in I}$ be a locally finite family of g-closed sets. Then $A = \cup_{i\in I} A_i$ is g-closed as well. □

Theorem 2.22 Let $(X_i)_{i\in I}$ be a family of topological spaces. For the topological sum $X = \sum_{i\in I} X_i$ the following conditions are equivalent:

(1) X is a g-Alexandroff space.

(2) Each X_i is a g-Alexandroff space.

Proof. (1) ⇒ (2) follows from Theorem 2.14.

(2) ⇒ (1) Let $(V_\alpha)_{\alpha\in A}$ be a collection of closed subsets of X. Set $V = \cup_{\alpha\in A} V_\alpha$. For each $i \in I$ and for each $\alpha \in A$, let $V_{(i,\alpha)} = X_i \cap V_\alpha$. Moreover, each $V_{(i,\alpha)}$ is closed in X_i. Now for each $i \in I$, the set $W_i = \cup_{\alpha\in A} V_{(i,\alpha)}$

is g-closed in X_i, since X_i is g-Alexandroff. By Theorem 2.6 from [6], each W_i is g-closed in X. Note that $V = \cup_{i \in I} W_i$ and that $(W_i)_{i \in I}$ is a locally finite (in fact discrete) family of sets. By Lemma 2.21, V is g-closed in X. Thus X is a g-Alexandroff space. □

It is clear that an arbitrary product of g-Alexandroff spaces fails to be g-Alexandroff (take *e.g.* an infinite product of discrete spaces). We have however, not been able to answer the following:

Question 1. Is the product of two g-Alexandroff spaces necessarily g-Alexandroff?

3. SEMI-ALEXANDROFF SPACES

Definition 2 A topological space (X, τ) is called *semi-Alexandroff* if any intersection of open sets is semi-open.

Remark 3.1 An Alexandroff space need not necessarily satisfy the condition "Arbitrary intersections of semi-open sets are semi-open". Topological spaces satisfying that condition must be extremally disconnected (= ED = open sets have open closures) [17, see Theorem 3] but one easily finds an Alexandroff space that fails to be ED, for example the either-or topology [18, Example 17].

Theorem 3.2 (1) The property "semi-Alexandroff" is a topological property.

(2) Every Alexandroff space is semi-Alexandroff.

(3) A topological space is semi-Alexandroff if and only if each Λ-set is semi-open. □

We first state a lemma, which allows us to find semi-Alexandroff spaces:

Lemma 3.3 If a space (X, τ) contains a point p such that $\{p\}$ is open and generic (*i.e.* it is dense as a subset), then (X, τ) is semi-Alexandroff.

Proof. Let $\{O_i : i \in I\}$ be a collection of open sets and let O be its intersection. Suppose that O is not semi-open, *i.e.* $O \not\subseteq \overline{\text{int} O}$. Then there exists $x \in O$ and an open set V containing x with $V \cap \text{int} O = \emptyset$. Since $\{p\}$ is open and dense, then we have $p \in V$ and $p \in O_i$ for each $i \in I$, hence also $p \in \text{int} O$, a contradiction. Thus O is semi-open. □

We now provide an example showing that not every semi-Alexandroff space is Alexandroff.

Example 3.4 Let X be the real line with the topology $\tau = \{\emptyset, X, \{0\}\} \cup \{G \subseteq X : 0 \in G \text{ and } X \setminus G \text{ is finite}\}$. Clearly $\{0\}$ is open and dense in (X, τ) and so (X, τ) is semi-Alexandroff by Lemma 3.3. On the other hand, the intersection of all open sets $X \setminus \{x\}$, where x is irrational, is not open. Thus (X, τ) is not Alexandroff.

Example 3.5 With the help of the example above, we have that even closed subsets of semi-Alexandroff spaces need not be semi-Alexandroff. Consider the closed subspace $X \setminus \{0\}$. What we have is an infinite set with the cofinite topology. Such spaces are not semi-Alexandroff.

Theorem 3.6 For a topological space (X, τ) the following conditions are equivalent:

(1) X is T_1 and semi-Alexandroff.

(2) X is discrete.

Proof. (1) $\Rightarrow$ (2) For each $x \in X$ and each $y \neq x$, there exists an open set U_y containing x such that $y \notin U_y$ (X is T_1). Since X is semi-Alexandroff and since $\{x\} = \cap_{y \neq x} U_y$, then $\{x\}$ is semi-open and hence open, since a singleton is semi-open if and only if it is open. Thus X is discrete.

(2) $\Rightarrow$ (1) is obvious. □

Recall that a topological space (X, τ) is called *open hereditary irresolvable* [19] if each open subset of X is irresolvable and *quasi-maximal* [19] if every dense set with non-empty interior has dense interior. A subset $A \subseteq X$ is said to be *simply-open* [20] if $A = U \cup N$, where U is open and N is nowhere dense (a set S is *nowhere dense* if $\text{int}\overline{S} = \emptyset$). In [19], Chattopadhyay and Roy called A a *δ-set* if $\text{int}\overline{A} \subseteq \overline{\text{int}A}$. It is easily observed that a set A is simply-open if and only if it is a δ-set. The following lemma gives us a characterization of simply-open sets, which will be used in the proof of a result on semi-Alexandroff spaces.

Lemma 3.7 [21]. For a subset A of a space X the following conditions are equivalent:

(1) A is a simply-open set.

(2) A is the intersection of a semi-open and a semi-closed set. □

Lemma 3.8 [19, in Theorem 2.2]. For a topological space X the following conditions are equivalent:

(1) Every subset of X is simply-open.

(2) X is open hereditary irresolvable and quasi-maximal. □

Theorem 3.9 Let (X, τ) be $T_{\frac{1}{2}}$ and semi-Alexandroff. Then X is open hereditary irresolvable and quasi-maximal.

Proof. According to Lemma 3.8 above, we need to show that every subset of X is simply-open. Let $A \subseteq X$. Set $A = A_1 \cup A_2$ (with $A_1 \cap A_2 = \emptyset$), where each point of A_1 is closed in X and each point of A_2 is open in X (this is possible, since X is $T_{\frac{1}{2}}$). Since X is semi-Alexandroff, then A_1 is semi-closed and moreover A_2 is (semi-)open. In the notion of Lemma 3.7 above, each subset of X is the complement of a simply-open set, *i.e.* each subset of X is simply-open. □

A bijection $f: (X, \tau) \to (Y, \sigma)$ is called a *semihomeomorphism* [22] if both f and f^{-1} preserve semi-open sets, *i.e.* if f is presemiopen and irresolute. Any property transmitted by semihomeomorphisms is called *semitopological* [22]. Every homeomorphism a semihomeomorphism but not conversely. On the other hand, not every topological property is semitopological.

Question 2. Is the property semi-Alexandroff semitopological?

Theorem 3.10 Every semi-Alexandroff space is a P'-space. □

The next example shows that not every P'-space is semi-Alexandroff.

Example 3.11 Let X be the real line where the non-trivial open sets are all sets containing the zero point and having countable complements. Clearly this is a P-space and hence a P'-space. The intersection of all open sets of the form $X \setminus \{x\}$, for $x \neq 0$, is not semi-open. This shows that X is not semi-Alexandroff.

Recall that a topological space (X, τ) is called *nodec* [23] if all nowhere dense sets are closed.

Theorem 3.12 For an ED, nodec space the following conditions are equivalent:

(1) X is semi-Alexandroff.

(2) X is Alexandroff.

Proof. In extremally disconnected, nodec spaces the collection of all semi-open sets coincides with the original topology. □

Also, following the proof of Theorem 2.8 in [9] we obtain that every semi-Alexandroff space is semi-orthocompact, semi-first countable, and semi-locally connected.

Recall that a function $f: (X, \tau) \to (Y, \sigma)$ is called *semi-closed* [24] if the image of every closed set of X is semi-closed in Y. The proof of the next theorem is very similar to the one of Theorem 2.17 and hence omitted.

Theorem 3.13 Let $f: (X, \tau) \to (Y, \sigma)$ be a surjective continuous semi-closed function. If X is Alexandroff, then Y is semi-Alexandroff. □

A subset A of a topological space (X, τ) is called *interior-closed* (= ic-set) [25] if int$A$ is closed in A. A function $f: (X, \tau) \to (Y, \sigma)$ is called *ic-continuous* [25] if the inverse image under f of each open set of Y is an ic-set in X.

The following decomposition of continuity is due to Ganster and Reilly:

Lemma 3.14 [25]. A function $f: (X, \tau) \to (Y, \sigma)$ is continuous if and only if it is semi-continuous and ic-continuous. □

As a consequence of Theorem 3.13, due to the Lemma above, we have the following corollary:

Corollary 3.15 If (X, τ) is Alexandroff, $f: (X, \tau) \to (Y, \sigma)$ is ic-continuous and a semihomeomorphism, then (Y, σ) is semi-Alexandroff. □

Theorem 3.16 Every P'-space (X, τ), whose Λ-sets are G_δ is semi-Alexandroff.

Proof. Since the Λ-sets of X are G_δ and since X is P'-space, then G_δ-sets are semi-open. From Theorem 3.2 (3), since each Λ-set is open, it follows that the space is semi-Alexandroff. □

The proof of the following result is easy and left to the reader.

Theorem 3.17 Let X be semi-Alexandroff and let $f: X \to Y$ be open, continuous and onto. Then Y is semi-Alexandroff. □

Corollary 3.18 Let $(X_i, \tau_i)_{i \in I}$ be a collection of topological spaces and let $X = \prod_{i \in I} X_i$. If X is semi-Alexandroff, then each X_i is also semi-Alexandroff. □

An arbitrary product of Alexandroff spaces fails to be semi-Alexandroff in general. If we take an infinite product of discrete spaces, then the resulting space is T_1 and not discrete, hence not semi-Alexandroff by Theorem 3.6. We do, however, have the following result:

Theorem 3.19 If X and Y are semi-Alexandroff, then $X \times Y$ is semi-Alexandroff.

Proof. Let $\{W_i : i \in I\}$ be a collection of open sets in $X \times Y$ and let W be its intersection. Suppose that W is not semi-open. Then there exists $(x, y) \in W$ and open sets $U \subseteq X$, $V \subseteq Y$ with $(x, y) \in U \times V$ and $(U \times V) \cap \text{int} W = \emptyset$. For each $i \in I$ there exist open sets $O_i \subseteq X$, $R_i \subseteq Y$ such that $(x, y) \in O_i \times R_i \subseteq W_i$. Since X and Y are semi-Alexandroff, then there exist non-empty open sets $G \subseteq X$ and $H \subseteq Y$ with $G \subseteq U \cap (\cap_{i \in I} O_i)$ and $H \subseteq V \cap (\cap_{i \in I} R_i)$. Clearly $G \times H \subseteq (U \times V) \cap \text{int} W$, a contradiction. Thus $X \times Y$ is semi-Alexandroff. □

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REFERENCES

[1] P. Alexandroff, "Diskrete Räume", *Mat. Sb.*, **2** (1937), pp. 501–518.

[2] E.D. Khalimsky, "Applications of Connected Ordered Topological Spaces in Topology", *Conference of Math. Departments of Povolsia*, 1970.

[3] E.D. Khalimsky, R. Kopperman, and P.R. Meyer, "Computer Graphics and Connected Topologies on Finite Ordered Sets", *Topology Appl.*, **36** (1990), pp. 1–17.

[4] V. Kovalevsky and R. Kopperman, "Some Topology-Based Image Processing Algorithms", *Annals of the New York Academy of Sciences*, **728** (1994), pp. 174–182.

[5] J. Dontchev and M. Ganster, "On δ-Generalized Closed Sets and $T_{\frac{3}{4}}$-Spaces", *Mem. Fac. Sci. Kochi Univ. Ser. A, Math.*, **17** (1996), pp. 15–31.

[6] N. Levine, "Generalized Closed Sets in Topology", *Rend. Circ. Mat. Palermo*, **19 (2)** (1970), pp. 89–96.

[7] A.I. Veksler, "P'-Points, P'-Sets, P'-Spaces: A New Class of Order-Continuous Measures and Functionals", *Soviet Math. Dokl.*, **14** (1973), pp. 1445–1450.

[8] T.R. Hamlett, D. Janković, and Ch. Konstadilaki, "On Some Properties Weaker than S-Closed", *Math. Japon.*, to appear.

[9] F.G. Arenas, *Alexandroff Spaces*, preprint.

[10] J. Dontchev, "On Submaximal Spaces", *Tamkang J. Math.*, **26 (3)** (1995), pp. 253–260.

[11] K. Balachandran, P. Sundaram, and H. Maki, "On Generalized Continuous Maps in Topological Spaces", *Mem. Fac. Sci. Kochi Univ. Ser. A, Math.*, **12** (1991), pp. 5–13.

[12] K. Balachandran, P. Sundaram, and H. Maki, "Generalized Locally Closed Sets and glc-Continuous Functions", *Indian J. Pure Appl. Math.*, **27** (**3**) (1996), pp. 235–244.

[13] S.R. Malghan, "Generalized Closed Maps", *J. Karnatak Univ. Sci.*, **27** (1982), pp. 82–88.

[14] H. Maki, P. Sundaram, and K. Balachandran, "On Generalized Homeomorphisms in Topological Spaces", *Bull. Fukuoka Univ. Ed. Part* III, **40** (1991), pp. 13–21.

[15] H. Maki, "Generalized Λ-Sets and the Associated Closure Operator", *The Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement.* Fukuoka University, 1986, pp. 139–146.

[16] F.G. Arenas, J. Dontchev, and M. Ganster, "On λ-Sets and the Dual of Generalized Continuity", *Questions Answers Gen. Topology*, **15** (**1**) (1997), pp. 3–13.

[17] M. Ganster and I. Reilly, "A Decomposition of Continuity", *Acta Math. Hungar.*, **56** (1990), pp. 299–301.

[18] L.A. Steen and J.A. Seebach, Jr., *Counterexamples in Topology.* New York – Montreal – London: Holt, Rhinehart, and Winston, Inc., 1970.

[19] C. Chattopadhyay and U.K. Roy, "δ-Sets, Irresolvable and Resolvable Space", *Math. Slovaca*, **42** (**3**) (1992), pp. 371–378.

[20] A. Neubrunnová, "On Transfinite Sequences of Certain Types of Functions", *Acta Fac. Rer. Natur. Univ. Comenianae Math.*, **30** (1975), pp. 121–126.

[21] M. Ganster, I.L. Reilly, and M.K. Vamanamurthy, "Remarks on Locally Closed Sets", *Math. Pannonica*, **3** (**2**) (1992), pp. 107–113.

[22] S.G. Crossley and S.K. Hildebrand, "Semitopological Properties", *Fund. Math.*, **74** (1972), pp. 233–254.

[23] J. van Mill and C.F. Mills, "A Boojum and Other Snarks", *Nederl. Acad. Wetensch. Proc. Ser. A*, **83** (1980), pp. 419–424.

[24] T. Noiri, "A Generalization of Closed Mappings", *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **54** (**8**) (1973), pp. 412–415.

[25] M. Ganster and I. Reilly, "Another Decomposition of Continuity", *Annals of the New York Academy of Sciences*, **704** (1993), pp. 135–141.

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