ON A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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الخلاصــة :

ليكن $S_s^*(\alpha,\beta)$ فصل الدوال $f(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0)$ التحليلية على قرص الوحدة $U = \{z : |z| < 1\}$ والتي تحقق الشرط

$$\left| \frac{\dfrac{2zf'(z)}{f(z)-f(-z)} - 1}{\dfrac{2zf'(z)}{f(z)-f(-z)} + (1-2\alpha)} \right| < \beta$$

وذلك لقيم $\alpha\,(0 \leq \alpha < 1)$ و $\beta\,(0 < \beta \leq 1)$ ولكل نقطة z في قرص الوحدة. ولنعتبر فصل الدوال $C_S(\alpha,\beta)$ والذي يرتبط ارتباطاً قوياً بالفصل $S_s^*(\alpha,\beta)$ بالعلاقة التالية $f \in C_S(\alpha,\beta) \Leftrightarrow zf' \in S_S^*(\alpha,\beta)$.

نهدف في هذا البحث إلى دراسة أهم خصائص هذين الفصلين مثل خاصية التشوه وتحديد المعاملات وبعض نظريات الإغلاق. كما نتناول بالدراسة خصائص المؤثر التكاملي $\frac{c+I}{z^c}\int_0^z t^{c-1} f(t)\,dt,\ c > -1$ والمؤثر التكاملي الكسري للدوال في الفصلين $C_S(\alpha,\beta)$ و $S_s^*(\alpha,\beta)$.

ABSTRACT

Let $S_s^*(\alpha,\beta)$ denote the class of functions $f(z) = z - \sum_{k=2}^{\infty} a_k z^k (a_k \geq 0)$ which are analytic in the unit disc $U = \{z : |z| < 1\}$ and which satisfy the inequality

$$\left| \frac{\frac{2zf'(z)}{f(z)-f(-z)} - 1}{\frac{2zf'(z)}{f(z)-f(-z)} + (1-2\alpha)} \right| < \beta$$

for $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$ and for all $z \in U$. Further $f(z)$ is said to belong to the class $C_S(\alpha,\beta)(0 \leq \alpha < 1$ and $0 < \beta \leq 1)$ if and only if $zf'(z) \in S_S^*(\alpha,\beta)$. The object of the present paper is to obtain distortion theorems, coefficient estimates, closure theorems for functions in the classes $S_s^*(\alpha,\beta)$ and $C_S(\alpha,\beta)$. Properties of the integral operator $\frac{c+I}{z^c}\int_0^z t^{c-1} f(t)dt$, $c > -1$ are also studied. Finally, we present a result about a fractional integral operator.

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1. INTRODUCTION

Let S_- denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, (a_k \geq 0) \tag{1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in S_-$ is said to be in the class $S^*(\alpha, \beta)$ if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + (1 - 2\alpha)} \right| < \beta \tag{2}$$

for some α $(0 \leq \alpha < 1)$, β $(0 < \beta \leq 1)$ and for all $z \in U$. Further, $f(z) \in S_-$ is said to be in the class $C^*(\alpha, \beta)$ if and only if $zf'(z) \in S^*(\alpha, \beta)$. The classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ were studied by Gupta and Jain [1], Owa [2], and Kumar and Shukla [3]. In [4] Gupta introduced the class of close-to-convex functions of order α and type β, $0 \leq \alpha < 1$, defined as follows. A function $f \in S_-$ is in $K(\alpha, \beta)$, the class of close-to-convex functions of order α and type β, if there exists a function $\phi(z) \in S_-$ such that

$$\left| \frac{\frac{zf'(z)}{\phi(z)} - 1}{\frac{zf'(z)}{\phi(z)} + (1 - 2\alpha)} \right| < \beta, \qquad z \in U.$$

A subclass $B(\alpha, \beta)$ of $K(\alpha, \beta)$ was defined [4] as follows.

A function $f \in S_-$ is in $B(\alpha, \beta)$, if there exists a function $\phi(z) = z - \sum_{k=2}^{\infty} b_k z^k$ in S_- such that

(*i*) $\sum_{k=2}^{\infty} \{(1 + \beta)ka_k - (1 - \beta + 2\alpha\beta)b_k\} \leq 2\beta(1 - \alpha)$

and

(*ii*) $ka_k - b_k \geq 0$ for every k.

In 1959, Sakaguchi [5] defined the class of starlike functions with respect to symmetrical points as follows:

Let f be analytic in U and suppose that for every $r < 1$ $(r \to 1)$ and every ξ on $|z| = r$, the angular velocity of $f(z)$ about the point $f(-\xi)$ is positive at $z = \xi$ as z traverses the circle $|z| = r$ in the positive direction, that is

$$Re \frac{zf'(z)}{f(z) - f(\xi)} > 0, \; z = \xi, \; |\xi| = r.$$

Then f is said to be starlike with respect to symmetrical points.

In [6] Das and Singh defined the class C_S of univalent convex functions with respect to symmetrical points as follows:

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $\quad z \in U$. Then $f \in C_S$ if and only if

$$\mathrm{Re}\frac{(zf')'}{(f(z) - f(z))'} > 0, \quad \text{for} \quad z \in U.$$

In [7] Das and Singh defined the order of such functions as follows:

A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $\quad z \in U$ is said to belong to the class $S_S^*(\alpha)$, $\; 0 \leq \alpha \leq \frac{1}{2}$ if and only if, for $z \in U$,

$$\mathrm{Re}\frac{zf'(z)}{f(z) - f(-z)} > \alpha$$

and similarly the class $C_S(\alpha)$.

The aim of this paper is to introduce the concept of type for these families of functions.

Definition 1. A function f of the form (1) is in $S_S^*(\alpha, \beta)$ if and only if the inequality

$$\left| \frac{\dfrac{2zf'(z)}{f(z) - f(-z)} - 1}{\dfrac{2zf'(z)}{f(z) - f(-z)} + (1 - 2\alpha)} \right| < \beta,$$

holds for some α $(0 \leq \alpha < 1)$, $\;\beta$ $(0 < \beta \leq 1)$ and for all $z \in U$.

Definition 2. A function f of the form (1) is in the class $C_S(\alpha, \beta)$ if and only if the inequality

$$\left| \frac{\dfrac{2(zf'(z))'}{(f(z) - f(-z))'} - 1}{\dfrac{2(zf'(z))'}{(f(z) - f(-z))'} + (1 - 2\alpha)} \right| < \beta,$$

holds for some α $(0 \leq \alpha < 1)$, $\;\beta(0 < \beta \leq 1)$ and for all $z \in U$.

It follows immediately from definitions 1 and 2 that

$$f(z) \in C_S(\alpha, \beta) \text{ if and only if } zf'(z) \in S_S^*(\alpha, \beta).$$

2. MAIN RESULTS

Theorem 1. A function $f = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0$ is in $S_S^*(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \{(k-1)(1+\beta)a_{2k-2} + [k(\beta+1) - (1+\alpha\beta)]\, a_{2k-1}\} \leq \beta(1-\alpha). \tag{3}$$

The result is sharp for the functions:

$$f_\mu(z) = z - \frac{\mu\beta(1-\alpha)}{(k-1)(1+\beta)} z^{2k-2} - \frac{(1-\mu)\beta(1-\alpha)}{[k(1+\beta) - (1+\alpha\beta)]} z^{2k-1} \qquad (0 \leq \mu \leq 1, \quad k \geq 2). \tag{4}$$

Proof. Let $|z| = r < 1$. Noting that

$$|2zf'(z) - f(z) + f(-z)| < 4\sum_{k=2}^{\infty} \{(k-1)\, [a_{2k-2} + a_{2k-1}]\}\, r$$

and

$$|2zf'(z) + (1-2\alpha)(f(z) - f(-z))| > 4r$$

$$\left\{(1-\alpha) - \sum_{k=2}^{\infty} [(k-1)a_{2k-2} + (k-\alpha)a_{2k-1}]\right\},$$

we see that

$$|2zf'(z) - (f(z) - f(-z))| - \beta|2zf'(z) + (1-2\alpha)(f(z) - f(-z))|$$

$$< 4r\left[\sum_{k=2}^{\infty} \{(k-1)(1+\beta)a_{2k-2} + (k(1+\beta) - (1+\alpha\beta))a_{2k-1}\} - \beta(1-\alpha)\right]. \tag{5}$$

The right-hand side of (5) is non-positive by (3), so that $f(z) \in S_S^*(\alpha, \beta)$ by Definition (1).

For the second part, we assume that $f(z) \in S_S^*(\alpha, \beta)$, then

$$\left|\frac{2zf'(z) - (f(z) - f(-z))}{2zf'(z) + (1-2\alpha)(f(z) - f(-z))}\right| < \beta, \quad z \in U.$$

This gives that

$$\operatorname{Re}\left\{\frac{-4\sum_{k=2}^{\infty} \{(k-1)a_{2k-2}z^{2k-2} + (k-1)a_{2k-1}z^{2k-1}\}}{4z(1-\alpha) - 4\sum_{k=2}^{\infty}[(k-1)a_{2k-2}z^{2k-2} + (k-\alpha)a_{2k-1}z^{2k-1}]}\right\} < \beta. \tag{6}$$

Since $|\mathrm{Re}\, z| \le |z|$ for all z. Choose values of z on the real axis so that $\left[\frac{2zf'(z)}{f(z)-f(-z)}\right]$ is real. Upon clearing the denominator in (6) and letting $z \to 1$ through real values, we obtain the inequality

$$\sum_{k=2}^{\infty}(k-1)\,(a_{2k-2}+a_{2k-1}) \le \beta\left[(1-\alpha)-\sum_{k=2}^{\infty}\{(k-1)a_{2k-2}+(k-\alpha)a_{2k-1}\}\right].$$

This on simplification gives the required coefficient inequality (3).

Corollary 1. If $f \in S_S^*(\alpha,\beta)$, then $\frac{1}{2}[f(z)-f(-z)] \in S^*(\alpha,\beta)$.

Proof. It can be verified by applying

$$\sum_{k=2}^{\infty}\{(k-1)+\beta\,(k+1-2\alpha)\}\,a_k \le 2\beta(1-\alpha)$$

which is a necessary and sufficient condition for a function to be in $S^*(\alpha,\beta)$ [8].

Corollary 2. Let the function $f(z)$ defined by (1) be analytic in U. Then $f(z)$ is in $C_S(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}\left\{2(k-1)^2(1+\beta)a_{2k-2}+[k(\beta+1)-(1+\alpha\beta)]\,(2k-1)a_{2k-1}\right\} \le \beta(1-\alpha). \tag{7}$$

The result is sharp.

Proof. Since

$$zf'(z) = z-\sum_{k=2}^{\infty}ka_kz^k$$

by replacing a_k by ka_k in Theorem 1, we immediately have Corollary 2.

Corollary 3. $S_S^*(\alpha,\beta) \subseteq B(\alpha,\beta) \subseteq K(\alpha,\beta)$.

Theorem 2 (Distortion Theorem). If $f \in S_S^*(\alpha,\beta)$, then for $|z| \le r < 1$

$$r-\frac{\beta(1-\alpha)}{1+\beta}r^2 \;\le\; |f(z)| \le r+\frac{\beta(1-\alpha)}{1+\beta}r^2 \tag{8}$$

$$1-\frac{2\beta(1-\alpha)}{1+\beta}r \;\le\; |f'(z)| \le 1+\frac{2\beta(1-\alpha)}{1+\beta}r. \tag{9}$$

The bounds in (8) and (9) are sharp, since the equalities are attained by the function

$$f(z) = z-\frac{\beta(1-\alpha)}{1+\beta}z^2, \quad (z=\pm r). \tag{10}$$

Proof. In view of Theorem 1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta(1-\alpha)}{1+\beta}.$$

Hence (8) follows from

$$r - r^2 \sum_{k=2}^{\infty} a_k \leq |f(z)| \leq r + r^2 \sum_{k=2}^{\infty} a_k.$$

Further, since

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2\beta(1-\alpha)}{(1+\beta)}$$

(9) follows from

$$1 - r \sum_{k=2}^{\infty} k a_k \leq |f'(z)| \leq 1 + r \sum_{k=2}^{\infty} k a_k.$$

Corollary 4. If a function $f(z)$ defined by (1) is in the class $C_S(\alpha, \beta)$, then

$$1 - \frac{\beta(1-\alpha)}{1+\beta} r \leq |f'(z)| \leq 1 + \frac{\beta(1-\alpha)}{1+\beta} r.$$

The result is sharp.

Corollary 5. Let $f \in S_S^*(\alpha, \beta)$. Then the disc $|z| < 1$ is mapped onto a domain that contains the disc

$$|\omega| < \frac{1+\alpha\beta}{1+\beta}.$$

Let the function $f_j(z)$ be defined, for $j = 1, 2, \ldots, m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k, \quad (a_{k,j} \geq 0) \tag{11}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the classes $S_S^*(\alpha, \beta)$ and $C_S(\alpha, \beta)$.

Theorem 3. Let the functions $f_j(z)$ $(j = 1, 2, \ldots, m)$ defined by (11) be in the class $S_S^*(\alpha, \beta)$. Then the functions

$$h(z) = z - \sum_{k=2}^{\infty} b_k z^k, \quad \left(b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \tag{12}$$

and

$$g(z) = \sum_{j=1}^{m} d_j f_j(z), \left(d_j \geq 0, \sum_{j=1}^{m} d_j = 1 \right) \tag{13}$$

also belong to the class $S_S^*(\alpha, \beta)$.

Proof. Since $f_j(z) \in S_S^*(\alpha, \beta)$, it follows from Theorem 1 that

$$\sum_{k=2}^{\infty} \{(1+\beta)(k-1)a_{2k-2,j} + [k(1+\beta) - (1+\alpha\beta)]a_{2k-1,j}\} \leq \beta(1-\alpha), \quad j = 1, 2, \ldots, m$$

Therefore

$$\sum_{k=2}^{\infty} \{(1+\beta)(k-1)b_{2k-2} + [k(1+\beta) - (1+\alpha\beta)]\, b_{2k-1}\}$$

$$= \sum_{k=2}^{\infty} \left\{ (1+\beta)(k-1) \left[\frac{1}{m} \sum_{j=1}^{m} a_{2k-2,j} \right] + [k(1+\beta) - (1+\alpha\beta)] \times \left[\frac{1}{m} \sum_{j=1}^{m} a_{2k-1,j} \right] \right\} \leq \beta(1-\alpha)$$

hence by Theorem 1, $h(z) \in S_S^*(\alpha, \beta)$.

Also

$$\sum_{k=2}^{\infty} \left\{ (1+\beta)(k-1) \left[\sum_{j=1}^{m} d_j a_{2k-2,j} \right] + [k(1+\beta) - (1+\alpha\beta)] \times \left[\sum_{j=1}^{m} d_j a_{2k-1,j} \right] \right\} = \sum_{j=1}^{m} d_j [\sum_{k=2}^{\infty} \{(1+\beta)(k-1)a_{2k-2,j}$$

$$+(k(1+\beta) - (1+\alpha\beta))a_{2k-1,j}\}]$$

$$\leq \left[\sum_{j=1}^{m} d_j \right] \beta(1-\alpha) = \beta(1-\alpha)$$

which implies that $g(z) \in S_S^*(\alpha, \beta)$. Thus we have the theorem.

By using Corollary 2, we have

Theorem 4. Let the functions $f_j(z)$ defined by (11) be in the class $C_S(\alpha,\beta)$ for every $j = 1,2,\ldots,m$. Then the functions $h(z)$ and $g(z)$ defined by (12) and (13) also belong to the same class $C_s(\alpha,\beta)$.

Theorem 5. Let the function $f_1(z)$ defined by (11) be in the class $S_S^*(\alpha,\beta)$ and the function $f_2(z)$ defined by (11) be in the class $C_S(\alpha,\beta)$. Then the function $k(z)$ defined by

$$k(z) = z - \frac{2}{3}\sum_{k=2}^{\infty}(a_{k,1} + a_{k,2})\,z^k \tag{14}$$

is in $S_S^*(\alpha,\beta)$.

Proof. Since $f_1(z) \in S_S^*(\alpha,\beta)$ and $f_2(z) \in C_S(\alpha,\beta)$, by using Theorem 1 and Corollary 2, we get, respectively,

$$\sum_{k=2}^{\infty}\{(1+\beta)(k-1)a_{2k-2,1} + [k(1+\beta) - (1+\alpha\beta)]\,a_{2k-1,1}\} \le \beta(1-\alpha)$$

and

$$\sum_{k=2}^{\infty}\{(k-1)(1+\beta)a_{2k-2,2} + [k(1+\beta) - (1+\alpha\beta)]\,a_{2k-1,2}\} \le \frac{1}{2}\beta(1-\alpha).$$

Therefore, we have

$$\frac{2}{3}\sum\{(k-1)(1+\beta)\,(a_{2k-2,1} + a_{2k-2,2}) + [k(1+\beta) - (1+\alpha\beta)] \times$$
$$(a_{2k-1,1} + a_{2k-1,2})\} \le \beta(1-\alpha)$$

which implies that $k(z) \in S_S^*(\alpha,\beta)$, and proof of Theorem 5 is thus complete.

3. INTEGRAL OPERATORS

Theorem 6. Let the function $f(z)$ defined by (1) be in the class $S_S^*(\alpha,\beta)$ and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{15}$$

also belongs to the class $S_S^*(\alpha,\beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{16}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{17}$$

Therefore

$$\sum_{k=2}^{\infty}\{(k-1)(1+\beta)b_{2k-2} + [k(1+\beta) - (1+\alpha\beta)]\, b_{2k-1}\}$$

$$= \sum_{k=2}^{\infty}\left\{(k-1)(1+\beta)\left(\frac{c+1}{c+k}\right) a_{2k-2} + [k(1+\beta) - (1+\alpha\beta)]\right.$$

$$\left.\left(\frac{c+1}{c+k}\right) a_{2k-1}\right\} \le \sum_{k=2}^{\infty}\{(k-1)(1+\beta)a_{2k-2}$$

$$+ [k(1+\beta) - (1+\alpha\beta)]\, a_{2k-1}\} \le \beta(1-\alpha). \tag{18}$$

Since $f(z) \in S_S^*(\alpha,\beta)$. Hence by Theorem 1, $F(z) \in S_S^*(\alpha,\beta)$.

Theorem 7. Let c be a real number such that $c > -1$. If $F(z) \in S_S^*(\alpha,\beta)$, then the function $f(z)$ defined by (15) is univalent in $|z| < R^*$, where

$$\begin{aligned} R^* &= \min(r_1, r_2), \\ r_1 &= \inf_k \left[\frac{(1+\beta)(c+1)}{2\beta(1-\alpha)(c+2k-2)}\right]^{\frac{1}{(2k-3)}}, \quad (k \ge 2) \\ r_2 &= \inf_k \left[\frac{\{k(1+\beta) - (1+\alpha\beta)\}(c+1)}{\beta(1-\alpha)(c+2k-1)(2k-1)}\right]^{\frac{1}{2k-2)}} \end{aligned} \tag{19}$$

The result is sharp.

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $(a_k \ge 0)$. It follows from (15) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}\,[z^c F(z)]'}{(c+1)} \quad (c > -1) \\ &= z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k z^k \\ &= z - \sum_{k=2}^{\infty}\frac{c+2k-1}{c+1} a_{k-1} z^{2k-1} \\ &\quad - \sum_{k=2}^{\infty}\frac{c+2k-2}{c+1} a_{2k-2} z^{2k-2}. \end{aligned} \tag{20}$$

In order to obtain the required result it suffices to show that $|f'(z)-1|<1$ in $|z|<R^*$.

Now $|f'(z)-1|<1$ if

$$\sum_{k=2}^{\infty}\left|\frac{c+2k-1}{c+1}(2k-1)a_{2k-1}z^{2k-2}+\frac{c+2k-2}{c+1}\ (2k-2)\times\right.$$

$$\left.a_{k-2}z^{2k-3}\right|<1. \tag{21}$$

According to Theorem 1, we have

$$\sum_{k=2}^{\infty}\left\{\frac{(k-1)(1+\beta)a_{2k-2}+[k(1+\beta)-(1+\alpha\beta)]\,a_{2k-1}}{\beta(1-\alpha)}\right\}\leq 1 \tag{22}$$

Hence (21) will be true if

$$\left(\frac{c+2k-1}{c+1}\right)(2k-1)a_{2k-1}|z|^{2k-2}+\left(\frac{c+2k-2}{c+1}\right)(2k-2)\times$$

$$a_{2k-2}|z|^{2k-3}<\frac{(k-1)(1+\beta)}{\beta(1-\alpha)}a_{2k-2}+$$

$$\frac{[k(1+\beta)-(1+\alpha\beta)]}{\beta(1-\alpha)}a_{2k-1}$$

$$\text{or if}\quad |z|<R^*=\min(r_1,r_2), \tag{23}$$

where

$$r_1=\inf_k\left[\frac{(1+\beta)(c+1)}{2\beta(1-\alpha)(c+2k-2)}\right]^{\frac{1}{2k-3}}$$

and

$$r_2=\inf_k\left[\frac{\{k(1+\beta)-(1+\alpha\beta)\}(c+1)}{\beta(1-\alpha)(c+2k-1)(2k-1)}\right]^{\frac{1}{2k-2}},\quad k\geq 2.$$

Therefore $f(z)$ is univalent in $|z|<R^*$. Sharpness follows if we take

$$f_\mu(z)\ =\ z-\frac{\mu\beta(1-\alpha)(c+2k-2)}{(1+\beta)(c+1)(k-1)}z^{2k-2}$$

$$-\frac{(1-\mu)\beta(1-\alpha)(c+2k-1)}{\{k(\beta+1)-(1+\alpha\beta)\}(c+1)}z^{2k-1},\quad (k\geq 2,\mu=0\text{ or }1) \tag{24}$$

Theorem 8. Let c be a real number such that $c > -1$. If $F(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0)$ belongs to the class $S_S^*(\alpha, \beta)$, then the function $f(z)$ defined by (15) is starlike of order $\rho (0 \leq \rho < 1)$ in $|z| < r^*(\rho, \alpha, \beta)$, where

$$\begin{aligned} r^*(\rho, \alpha, \beta) &= \min(R_1, R_2), \\ R_1 &= \inf_k \left[\frac{(k-1)(1+\beta)(c+1)(1-\rho)}{\beta(1-\alpha)(2k-2-\rho)(c+2k-2)} \right]^{\frac{1}{2k-3}} \\ R_2 &= \inf_k \left[\frac{\{k(1+\beta) - (1+\alpha\beta)\}(1-\rho)(c+1)}{\beta(1-\alpha)(2k-1-\rho)(c+2k-1)} \right]^{\frac{1}{2k-2}}. \end{aligned}$$

The result is sharp.

Proof. In order to establish the required result, it suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \rho \text{ in } |z| < r^*(\rho, \alpha, \beta).$$

Now

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{-\sum_{k=2}^{\infty}(k-1)\left(\frac{c+k}{c+1}\right) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k z^{k-1}} \right| \\ &\leq \frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{c+k}{c+1}\right) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k |z|^{k-1}} \\ &< (1-\rho) \end{aligned} \tag{25}$$

provided

$$\sum_{k=2}^{\infty} \left(\frac{k-\rho}{1-\rho}\right) \left(\frac{c+k}{c+1}\right) a_k |z|^{k-1} < 1. \tag{26}$$

By using (22), the inequality (26) holds if

$$\left(\frac{2k-2-\rho}{1-\rho}\right) \left(\frac{c+2k-2}{c+1}\right) |z|^{2k-3} < \frac{(k-1)(1+\beta)}{\beta(1-\alpha)}$$

and

$$\left(\frac{2k-1-\rho}{1-\rho}\right)\left(\frac{c+2k-1}{c+1}\right)|z|^{2k-2} < \frac{k(1+\beta)-(1+\alpha\beta)}{\beta(1-\alpha)},$$

or if $|z| < r^*(\rho,\alpha,\beta)$, where

$$\begin{aligned} r^*(\rho,\alpha,\beta) &= \min(R_1, R_2) \\ R_1 &= \inf_k \left[\frac{(k-1)(1+\beta)(c+1)(1-\rho)}{\beta(1-\alpha)(2k-2-\rho)(c+2k-2)}\right]^{\frac{1}{2k-3}} \\ R_2 &= \inf_k \left[\frac{\{k(1+\beta)-(1+\alpha\beta)\}(1-\rho)(c+1)}{\beta(1-\alpha)(2k-1-\rho)(c+2k-2)}\right]^{\frac{1}{2k-2}}. \end{aligned}$$

Hence, $f(z) \in S_\rho^*$ in $|z| < r^*(\rho,\alpha,\beta)$. Sharpness follows if we take the functions $F_\mu(z)$ given by

$$F_\mu(z) = z - \frac{\mu\beta(1-\alpha)}{(k-1)(1+\beta)} z^{2k-2} - \frac{(1-\mu)\beta(1-\alpha)}{k(1+\beta)-(1+\alpha\beta)} z^{2k-1}, \quad k \geq 2, \ \mu = 0 \text{ or } 1. \tag{27}$$

Theorem 9. Let the function $f(z)$ be defined by (1). If $f(z) \in S_S^*(\alpha,\beta)$, then the function $F(z)$ defined by (15) belongs to $S_S^*(\rho)$ where

$$\rho = \frac{(c+2)+\beta[2\alpha(c+1)-c]}{(c+2)(1+\beta)}. \tag{28}$$

The result is sharp. Further, the converse need not be true.

Proof. Let $F(z) = z - \sum_{k=2}^{\infty} b_k z^k \in S_S^*(\sigma)$, where b_k is given by (17), then by Theorem 1, it holds if and only if

$$\sum_{k=2}^{\infty}\left\{\frac{2(k-1)}{(1-\sigma)} b_{2k-2} + \frac{(2k-1-\sigma)}{(1-\sigma)} b_{2k-1}\right\} \leq 1. \tag{29}$$

Thus we have to find the largest value of σ so that the inequality (29) holds.

Now, by using (22), (29) holds if

$$\frac{2(k-1)}{1-\sigma} b_{2k-2} \leq \frac{(k-1)(1+\beta)}{\beta(1-\alpha)} a_{2k-2},$$

and

$$\frac{(2k-1-\sigma)}{1-\sigma} b_{2k-1} \leq \frac{k(1+\beta)-(1+\alpha\beta)}{\beta(1-\alpha)} a_{2k-1}, \quad (k \geq 2). \tag{30}$$

Or if

$$\frac{2(c+1)}{(1-\sigma)(c+2k-2)} \leq \frac{1+\beta}{\beta(1-\alpha)}$$

and

$$\frac{(2k-1-\sigma)(c+1)}{(1-\sigma)(c+2k-1)} \leq \frac{k(1+\beta)-(1+\alpha\beta)}{\beta(1-\alpha)}, \quad (k \geq 2) \tag{31}$$

which is equivalent to

$$\sigma \leq \rho_k, \quad (k \geq 2)$$

and

$$\sigma \leq \delta_k, \quad (k \geq 2)$$

where

$$\rho_k = \frac{(1+\beta)(c+2k-2)-2(c+1)\beta(1-\alpha)}{(1+\beta)(c+2k-2)}$$

$$\delta_k = \frac{(c+2k-1)[k(1+\beta)-(1+\alpha\beta)]-(2k-1)(c+1)\beta(1-\alpha)}{(c+2k-1)[k(1+\beta)-(1+\alpha\beta)]-(c+1)\beta(1-\alpha)}. \tag{32}$$

It is easy to verify that ρ_k, δ_k ar increasing functions of k $(k \geq 2)$. Therefore, $\rho_2 = \inf_{k\geq 2} \rho_k$ and $\delta_2 = \inf_{k \geq 2} \delta_k$, where

$$\rho_2 = \frac{(c+2)+\beta[2\alpha(c+1)-c]}{(1+\beta)(c+2)},$$

$$\delta_2 = \frac{(c+3)+\beta[2\alpha c+3-c\}}{(c+3)+\beta[c-2\alpha+5\}},$$

and hence

$$\rho = \min(\rho_2, \delta_2) = \rho_2.$$

To show the sharpness we take the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1-\alpha)}{1+\beta} z^2. \tag{33}$$

Then

$$\frac{zF'(z)}{F(z)-F(-z)} = -\frac{2\beta(1-\alpha)(c+1)}{(c+2)(1+\beta)}z$$

$$= \frac{(c+2)+\beta[2\alpha(c+1)-c]}{(1+\beta)(c+2)}, \quad \text{for } z=1.$$

Hence the result is sharp.

We now show that the converse of the theorem need not be true. To this end we consider the function

$$F(z) = z - \left(\frac{(1-\rho)}{3-\rho}\right) z^3. \tag{34}$$

Theorem 1 guarantees that $F(z) \in S_S^*(\rho)$. But the corresponding function

$$f(z) = z - \frac{(c+3)}{(c+1)}\frac{(1-\rho)}{(3-\rho)} z^3 \tag{35}$$

does not belong to $S_S^*(\alpha,\beta)$, since, for this $f(z)$ the coefficient inequality of Theorem 1 is not satisfied.

4. FRACTIONAL INTEGRAL OPERATOR

We need the following definition of fractional integral operator given by Srivastava, Saigo, and Owa [9].

Definition. Let

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k, \tag{36}$$

where

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} \begin{cases} 1 & , \quad k=0 \\ \nu(\nu+1)\cdots(\nu+k-1) & , \quad k \in N = \{1,2,3,\ldots\} \end{cases} \tag{37}$$

For real number $\rho > 0$, δ, η, and $\epsilon > \max\{0, \delta-\eta\} - 1$, the fractional integral operator $I_{0,z}^{\rho,\delta,\eta}$ is defined by

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{z^{-\rho-\delta}}{\Gamma(\rho)} \int_0^z (z-t)^{\rho-1} F\left(\rho+\delta, -\eta; \rho; 1-\frac{t}{z}\right) f(t)dt. \tag{38}$$

where $f(z)$ is a function analytic in a simply connected region of the z -plane containing the origin, satisfying

$$f(z) = O(|z|)^{\epsilon}, \quad z \to 0,$$

and the multiplicity of $(z-t)^{\rho-1} = \exp((\rho-1)\log(z-t))$ where $\log(z-t)$ is supposed to be real when $z-t>0$.

Remark. For $\delta=-\rho$, we note that

$$I_{0,z}^{\rho,-\rho,-\eta} f(z) = D_z^{-\rho} f(z),$$

where $D_z^{-\rho} f(z)$ is the fractional integral of order ρ of $f(z)$ which was introduced by Owa [10, 11].

In order to prove our results for the fractional operator, we have to recall here the following lemma due to Srivastava, Saigo, and Owa [9].

Lemma 1. If $\rho>0$ and $k>\delta-\eta-1$, then

$$I_{0,z}^{\rho,\delta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\rho+\eta+1)} z^{k-\delta}. \tag{39}$$

With the aid of Lemma 1, we have

Theorem 10. Let $\rho>0$, $\delta<2$, $\rho+\eta>-2, \delta-\eta<2$, and $\delta(\rho+\eta)\le 3\rho$.

If $f(z)\in S_S^*(\alpha,\beta)$, then

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \ge \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{1-\frac{\beta(1-\alpha)(2-\delta+\eta)}{(1+\beta)(2-\delta)(2+\rho+\eta)}|z|\right\} \tag{40}$$

and

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \le \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{1+\frac{\beta(1-\alpha)(2-\delta+\eta)}{(1+\beta)(2-\delta)(2+\rho+\eta)}|z|\right\} \tag{41}$$

for $z\in U_0$, where

$$U_0 = \begin{cases} U & (\delta\le 1) \\ U-\{0\} & (\delta>1). \end{cases}$$

The result is sharp.

Proof. By using Lemma 1, we have

$$\begin{aligned} I_{0,z}^{\rho,\delta,\eta} f(z) &= \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} z^{1-\delta} \\ &\quad -\sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\rho+\eta+1)} a_k z^{k-\delta} \end{aligned} \tag{42}$$

letting

$$H(z) = \frac{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}{\Gamma(2-\delta+\eta)} z^{\delta} I_{0,z}^{\rho,\delta,\eta} f(z)$$

$$= z - \sum_{k=2}^{\infty} h(k) a_k z^k, \tag{43}$$

where

$$h(k) = \frac{(2-\delta+\eta)_{k-1}(1)_k}{(2-\delta)_{k-1}(2+\rho+\eta)_{k-1}}, \qquad (k \geq 2). \tag{44}$$

We can see that $h(k)$ is non-increasing for integers $k \geq 2$, and we have

$$0 < h(k) \leq h(2) = \frac{2(2-\delta+\eta)}{(2-\delta)(2+\rho+\eta)}. \tag{45}$$

Since $f(z) \in S_S^*(\alpha, \beta)$, Theorem 1 implies that

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta(1-\alpha)}{1+\beta}. \tag{46}$$

Therefore, by using (46) and (45), we have

$$|H(z)| = |z| - h(2)|z|^2 \sum_{k=2}^{\infty} a_k$$

$$\geq |z| - \frac{\beta(1-\alpha)(2-\delta+\eta)}{(1+\beta)(2-\delta)(2+\rho+\eta)} |z|^2 \tag{47}$$

and

$$|H(z)| = |z| + h(2)|z^2| \sum_{k=2}^{\infty} a_k$$

$$\leq |z| + \frac{\beta(1-\alpha)(2-\delta+\eta)}{(1+\beta)(2-\delta)(2+\rho+\eta)} |z|^2. \tag{48}$$

Sharpness follows if we take the function

$$f(z) = z - \frac{\beta(1-\alpha)}{(1+\beta)} z^2. \tag{49}$$

Similarly, by applying Corollary 2, to the function $f(z)$ belonging to the class $C_S(\alpha, \beta)$, we can derive

Theorem 11. Let $\rho > 0$, $\delta < 2$, $\rho + \eta > -2$, $\delta - \eta < 2$ and $\delta(\rho + \eta) \leq 3\rho$.

If $f(z) \in C_S(\alpha, \beta)$, then

$$\left| I_{0,z}^{\rho,\delta,\eta} f(z) \right| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{ 1 - \frac{\beta(1-\alpha)(2-\delta+\eta)}{2(1+\beta)(2-\delta)(2+\rho+\eta)} |z| \right\} \tag{50}$$

and

$$\left| I_{0,z}^{\rho,\delta,\eta} f(z) \right| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} \left\{ 1 + \frac{\beta(1-\alpha)(2-\delta+\eta)}{2(1+\beta)(2-\delta)(2+\rho+\eta)} |z| \right\} \tag{51}$$

for $z \in U_0$, where U_0 is defined in Theorem 10. The equalities in (50) and (51) are attained by the function

$$f(z) = z - \frac{\beta(1-\alpha)}{2(1+\beta)} z^2. \tag{52}$$

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