HARTREE INTEGRAL IN CLOSED FORMS

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The Hartree integral in the form

$$I_n(r,g) = \int dq \int d^{d-1}p \left\{ r + p^2 + q^2 + \frac{g^2 q^2}{p^2} \right\}^{-n}, \tag{1}$$

where r and g are arbitrary parameters, has a great influence in studying the crossover behavior in uniaxial dipolar ferromagnets [1]. The integral is usually evaluated in the dimensional regularization scheme using the Feynman identities. Such techniques are tricky, and require special skills to reach the desired results. In this paper, we propose another procedure to solve Equation (1) in a closed and analytic form, which is also valid in both limits $r \to 0$ and $g \to 0$, using the properties of the hypergeometric function ${}_2F_1$.

Starting with the integral

$$I_1(r,g) = \int dq \int d^{d-1}p \left\{ r + p^2 + q^2 + \frac{g^2 q^2}{p^2} \right\}^{-1}, \tag{2}$$

after performing the q-integration using the formula (GR P.292,3.241.4)[2], one obtains

$$I_1(r,g) = \pi S_{d-1,1} I_c(r,g), \tag{3}$$

where $S_{d-1,1} = \dfrac{4\pi\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}$ $(d > 1)$ is the geometrical factor of integration (2), and

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$$I_c(r,g) = \int \frac{d^d p}{\sqrt{r+p^2}\sqrt{g^2+p^2}}. \tag{4}$$

By means of the substitution $p^2 = x$ in (4), and using the formula (GR P.286,3.197.1)[2], we obtain

$$I_c(r,g) = \tfrac{1}{2}\, g^{-1} r^{\left(\frac{d-1}{2}\right)} \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) {}_2F_1\left(\frac{d}{2},\tfrac{1}{2};1;1-\frac{r}{g^2}\right). \tag{5}$$

where $\Gamma(\alpha)$ is the usual gamma function. The singularity that appears in $\Gamma\left(1-\frac{d}{2}\right)$ at $d \geq 2$ is important, and has been used as a tool (ε-expansion) for solving many difficult problems that occur in the area of phase transition and others. In our procedure, Equation (5) is considered the first closed and analytic form. However, due to the inapplicability of this form at $r=0$ or $g=0$, it is not suitable for practical use. Therefore, in the following steps, the multiplication factors of g and r in the prefactor will be transformed into a more suitable form using the well known properties of the hypergeometric function.

Elimination of r in Equation (5) could be done by using the formula (AS P.559,15.3.3)[3], then (5) can take the form:

$$I_c(r,g) = \tfrac{1}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(g^2\right)^{\frac{d}{2}-1} {}_2F_1\left(1-\frac{d}{2},\tfrac{1}{2};1;1-\frac{r}{g^2}\right), \tag{6}$$

Equation (6) is another closed and analytic form, which could be useful to apply for actual calculations even at $r=0$. Another important aspect of Equation (6) is that, since the r- dependent appears explicitly in the last argument of ${}_2F_1$, one can use such an equation to calculate the higher order I_n using the relation: $I_{n+1} = (-1)^n n! \left(\dfrac{\partial^n I_1}{\partial r^n}\right)$.

The inapplicability of Equation (6) at $g=0$ could be resolved by means of the formula (AS P.560,15.3.17)[3] to have the final form:

$$I_c(r,g) = \tfrac{1}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(\frac{g+\sqrt{r}}{2}\right)^{d-2} {}_2F_1\left(1-\frac{d}{2},1-\frac{d}{2};1;Z\right), \tag{7}$$

where $Z = \left(\dfrac{g-\sqrt{r}}{g+\sqrt{r}}\right)^2$. Equation (7) is considered, in our procedure, the most suitable analytic form for Hartree-integral, since it gives the correct expression in both limits $r \equiv 0$ and $g \equiv 0$, *i.e.*:

$$I_1(r=0,g) = \frac{\sqrt{\pi}}{4} S_{d-1,1}\left(g^2\right)^{\frac{d}{2}-1}\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}\right), \tag{8}$$

$$I_1(r,g=0) = \frac{\sqrt{\pi}}{4} S_{d-1,1}(r)^{\frac{d}{2}-1}\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}\right). \tag{9}$$

Equations (8) and (9) followed directly from Equation (7) using the expansions (AS P.556,15.1.20, and P.256,6.1.18) [13]:

$${}_2F_1\left(1-\frac{d}{2},1-\frac{d}{2};1;1\right) = \frac{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)}. \tag{10}$$

The ε-expression could be implemented in our proposed analytical form through the singularity of the gamma function and using the relation $d = 4 - \varepsilon$. With a simple expansion for the hypergeometric function in Equation (7) one can get

$$ {}_2F_1\left(\frac{\varepsilon}{2} - 1, \frac{\varepsilon}{2} - 1; 1; Z\right) = (1 + Z) - Z\varepsilon + O\left(\varepsilon^2\right). \tag{11} $$

In comparison with reference [1], Equation (11) is simple, compact, and could be extended to higher order in ε. Consequently, Equation (7) could be very useful in studying the crossover behavior, and in calculating many different physical quantities, such as: effective exponents for susceptibility and the specific heat, *etc.*, in the crossover regime [1].

In summary, the procedure developed here for the evaluation of Hartree-integral is easy to apply and would have a wide application especially in calculating the physical constants in the critical phenomena. In addition to the closed analytic forms mentioned above; several others could also be derived using the properties of the hypergeometric function.

ACKNOWLEDGEMENT

The author is grateful to Dr. R. Folk for many helpful discussions.

REFERENCES

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Paper Received 28 December 1994; Revised 22 May 1995; Accepted 24 June 1995.