SOME FORMS OF COMPACTIFICATIONS

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الخلاصـة :

يَهدف هذا البحث إلى تقديم أنواع قوية من التصميتات (مثل شبه تصميتات وأقوى تصميتات) وذلك باستخدام أنواع من المجموعات قريبة الفتح مثل شبه مفتوحة ، والمفتوحة مبدئياً. ودراسة خواص هذه الأنواع الجديدة من التصميتات وعلاقاتها ببعض المسلمات المنفصلة.

ABSTRACT

Using some sorts of nearly-open sets (semi-open [1], pre-open [2]), we introduce and study stronger forms of compactifications; semi-compactifications (strongly-compactifications). We discuss properties for these type, also their relations with separation axioms are investigated.

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INTRODUCTION

Throughout the present paper, (X,τ) and (Y,σ) (or simply X and Y) always mean topological space and $f: X \to Y$ represents a function. The closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively (resp.). First recall that a set $A \subseteq X$ is semi-open [1] (resp. pre-open [2]), if $A \subseteq Cl\text{-}Int(A)$ (resp. $A \subseteq Int\text{-}Cl(A)$). The complement of a semi-open (pre-open) set is called semi-closed [3] (pre-closed [2]). The class of all semi-open (pre-open) sets in X is denoted by $SO(X,\tau)$ $(PO(X,\tau))$. A space X is called semi-compact [4]) (strongly compact [5]) if every semi-open (pre-open) cover of X has a finite subcover. A space X is called semi-normal [6] (pre-normal [7]), if for any two disjoint closed sets F_1 and F_2 in X, there exist two disjoint semi-open (pre-open) sets containing F_1 and F_2, respectively. A space X is called semi-T_2 [8] (pre-T_2 [9]), if for every two distinct points x, y in X there exist two disjoint semi-open (pre-open) sets in X containing x, y respectively. The filter of all sets in a topological space (X,τ) containing a non empty subset A of X is called the main filter belonging to A [10] and is denoted by $\rho(A)$ (if $A = \{x\}$, we write $\rho(A)$).

A filter $\mathcal{F}$ in a topological space (X,τ) is called a closed filter [10] iff $\mathcal{F}$ has a filter base consisting only of closed sets. We call $F_1 > F_2$ iff $F_1 \subseteq F_2$, for two different filters F_1, F_2 [10]. A minimal proper closed filter on a topological space (X,τ) is called an ultraclosed filter [10]. A bijective mapping $f: (X,\tau) \to (Y,\mu)$ is called a semi-homeomorphism [11] (pre-homeomorphism [12]), if $f^{-1}(V) \in SO(X,\tau)$ ((if $f^{-1}(V) \in PO(X,\tau)$) for each $V \in SO(Y,\mu)$ $(V \in PO(Y,\mu))$ and $f(U) \in SO(Y,\mu)$ $(f(U) \in PO(Y,\mu)$ for each $U \in SO(X,\tau)$ $(U \in PO(X,\tau))$. If A is a subset of a topological space (X,τ) such that $A \notin \tau$, then the topology $\tau(A) = \{O \cup (O' \cap A), O, O' \in \tau, A \notin \tau\}$ is called the simple extension of τ by A [13]. The topology $\tau[A] = \{U \backslash B, U \in \tau, B \subseteq A \subseteq X\}$ is called the local discrete extension of τ by A [14] (where $\tau(A)$, $\tau[A]$ are independent [15]). If (X,τ) is a topological space and $\mathcal{F}$ is a filter on X, then the topology $\tau(\mathcal{F}) = \{U \cap F: U \in \tau, F \in \mathcal{F}\}$ is called a filter extension of τ by the filter $\mathcal{F}$ on X [15]. A space (X,τ) is submaximal [16] if all dense subsets are open.

2. SOME FORMS OF COMPACTIFICATIONS

Definition 2.1. Let (X,τ), (Y,μ) be two topological spaces. Let S be a dense subset of Y. If there exists a semi-homeomorphism (pre-homeomorphism) $f: (X,\tau) \to (S,\mu_s)$ then the pair $(f,(Y,\mu))$ is called a semi-extension (pre-extension) of X.

Definition 2.2. A semi-extension (pre-extension) $(f,(Y,\mu))$ is called a semi-compactification (strongly compactification) if the space (Y,μ) is semi-compact (strongly compact).

Theorem 2.1. Let (X,τ) be a topological space which is neither semi-compact nor strongly compact. Set $X^* = X \cup \{\omega\}$, $\omega \notin X$, let $\tau^* = \{\tau(\rho(\omega)) \cup \{H: H \in \rho(\omega), X^* \backslash H$ is closed and semi-compact (strongly compact) in $X\}\} = \tau(\rho(\omega)) \cup K$. Then τ^* is a topology on X^* and $(i,(X^*,\tau^*))$ where i is the identity map on X, is a semi-compactification (strongly-compactification).

Proof. It is not difficult to verify that τ^* is a topology on X^*. We show that X^* is semi-compact. Let $\{G_1^*\}$ be a semi-open cover of X^*, then there must be at least one element say $G^*_{1(\omega)}$ in the cover which contains ω. Then $A = X^* \backslash G^*_{1(\omega)}$ is a closed and a semi-compact set in X and is covered by the family $\{G_i^* \cap X\}$ of semi-open sets in X (because X is an open subspace of X^*). Hence there exists a finite subfamily $G^*_{i(1)} \cap X, \ldots, G^*_{i(n)} \cap X$ which covers A. The finite family $G^*_{1(\omega)}, G^*_{i(1)}, \ldots, G^*_{1(n)}$ is a semi-open cover of X^* and is a subcover of the original cover. Thus X^* is semi-compact. In the same way, we can prove X^* is strongly-compact. We show that $(i,(X^*,\tau^*))$

is a semi-extension (strongly extension). Since $\tau_x^* = \tau$ (because $G^* \cap X \in \tau = \tau_x^*$, $G^* \in \tau^*$. Hence $i = (X,\tau) \to (X,\tau_x^*)$ is a semi-homeomorphism. We show that X is dense in X^*, since $G^* \cap X \neq \varnothing$ for every $G^* \in \tau^*$ (because (*a*) If $G^* \in \tau\,(\rho(\omega))$, then $G^* \cap X \neq \varnothing$; (*b*) If $G^* \in K$, then $X^* \setminus G^*$ is closed and semi-compact in X), $(X \setminus X^* \setminus G^*) \cap X \neq \varnothing$, which implies $X \cap G^* \cap X \neq \varnothing$, hence $G^* \cap X \neq \varnothing$, and therefore $(i,(X^*,\tau^*))$ is a semi-compactification. In the same way we can prove the other part of the theorem.

Remark 2.1. In this type of a stronger form of compactification, we could take the space (X,τ) to be compact but neither semi-compact nor strongly compact and we know that semi-compact $\Rightarrow$ compact and strongly compact $\Rightarrow$ compact.

Lemma 2.1 [16]. If A is τ-dense and τ-semi-open, then $SO(X,\tau) = SO(X,\tau\,(A))$.

Lemma 2.2 [16]. If (X,τ) is submaximal and $A \in PO(X,\tau)$, then $PO(X,\tau) = PO(X,\tau\,(A))$.

Theorem 2.2. Let (X,τ) be a topological space which is not semi-compact. Set $X^* = X \cup \{\omega\}$, $\omega \notin X$. Then $\tau^* = \tau(A) \cup \{H : H \in \rho(\omega),\ X^* \setminus H \text{ is closed and semi-compact in } X\} = \tau(A) \cup K$ is a topology on X^* provided that A is τ-dense and τ-semi-open, and $(i,(X^*,\tau^*))$, where i is the identity map on X, is a semi-compactification.

Proof. Obvious by Lemma 2.1 and Theorem 2.1.

Theorem 2.3. Let (X,τ) be a submaximal space which is not strongly compact. Set $X^* = X \cup \{\omega\}$, $\omega \notin X$. Then $\tau^* = \tau(A) \cup \{H : H \in \rho(\omega),\ X^* \setminus H \text{ is closed and strongly compact in } X\}\ \tau(A) \cup K$ is a topology on X^* provided that $A \in PO(X,\tau)$ and $(i,(X^*,\tau^*))$ where i is the identity map on X, is a strongly compactification.

Proof. Obvious by Lemma 2.2 and Theorem 2.1.

Lemma 2.3 [16]. Let (X,τ) be a topological space and A be a subset of X such that $A \cap U = \varnothing$, for every $U \in \tau, U \neq X$. Then $PO(X,\tau) = PO(X,\tau[A])$.

Theorem 2.4. Let (X,τ) be a topological space which is not strongly-compact and A be a subset of X such that $A \cap U = \varnothing$, for every $U \in \tau, U \neq X$. Set $X^* = X \cup \{\omega\}$, $\omega \notin X$. Then $\tau^* = \tau[A] \cup \{H : H \in \rho(\omega),\ X^* \setminus H \text{ is closed and strongly-compact in } X\} = \tau(A) \cup K$ is a topology on X^*, and $(i,(X^*,\tau^*))$, where i is the identity map on X, is a strongly compactification.

Proof. Obvious by Lemma 2.3 and Theorem 2.1. •

The previous compactifications are called 1-point semi-compactification (1-point strongly compactification).

Clearly:

$$
\begin{array}{ccc}
 & \text{1-point-compactification} & \\
\Uparrow & & \Uparrow \\
\text{1-point-semi-compactification} & & \text{1-point strongly compactification}
\end{array}
$$

If the semi-compactification (strongly compactification) has any property P then the compactification has P.

Theorem 2.5. The 1-point semi-compactification (1-point strongly compactification) is semi-T_i [resp. pre-T_i, $i = 0, 1, 2$] iff (X,τ) is semi-T_i (pre-T_i).

Proof. Let (X^*, τ^*) be semi-T_2, then for every two distinct points x, y in X, there are two disjoint semi-open sets U, V in X^* containing x, y respectively, since $x \in U \cap X = N$ and $y \in V \cap X = M$, where M, N are disjoint semi-open sets in X containing x, y respectively [because X is an open subspace of X^* and $M \cap N = U \cap X \cap V \cap X = U \cap V \cap X = \varnothing$]. Hence (X, τ) is semi-T_2. Conversely, let (X, τ) be semi-T_2 and let x, y be two distinct points in X also in X^*. If $y = \omega$, then there is a semi-compact-neighborhood U of x which is semi-closed (because X is semi-T_2), so its complement is a semi-open set containing ω which is disjoint from some semi-open sets containing x. Hence X^* is semi-T_2 and also semi-normal (because X^* is semi-compact). In the same way, we prove the other parts of the theorem.

Lemma 2.4 [10].

(*a*) Given a closed filter F in a space (X, τ), there exists an ultraclosed filter F' containing F.

(*b*) In a T_1-space (X, τ), every $\rho(x)$ is an ultraclosed filter

(*c*) If $U, V \in \tau$, F is an ultraclosed filter, then $U \cup V \in F$ implies $U \in F$ or $V \in F$.

(*d*) If F, F' are ultraclosed filter on X, $F \neq F'$, there exist closed sets C, C' such that $C \in F$, $C' \in F'$, $C \cap C' = \varnothing$. •

Theorem 2.6. Let $W(X)$ be the collection of all ultraclosed filters on a T_1 which is not semi-compact space (X, τ). Let $\mathcal{X}: X \to W(X)$ be the mapping, $\mathcal{X}(x) = \rho(x)$. Finally for every $U \in SO(X, \tau)$ (resp. $U \in PO(X, \tau)$), define $U^* = \{F : F \in W(X), U \in F\}$ and the family $\beta = \{U^* : U \in SO(X, \tau)(U \in PO(X, \tau))\}$ is a base for a topology ω on $W(X)$, then $(\mathcal{X}(W(X)\omega))$ is a T_1-semi-compactification (T_1-strongly compactification).

Proof. We must prove that:

(*a*) β is a base for a topology ω on $W(X)$. Let $\beta = \{U^* : U \in SO(X, \tau)\}$, then

$$U^* \cap V^* = \{F : F \in W(X), U \in F\} \cap \{F : F \in W(X), V \in F\}$$

$$= \{F : F \in W(X), U \cap V \in F\} = (U = V)^*.$$

Thus β is a base for a topology ω on $W(X)$.

(*b*) $U^* = V^*$ implies $U = V$, because if $U \neq V$ there exists a point $X \in U$, $x \notin V$, this implies $U \in \rho(x)$ and $V \notin \rho(x)$, hence $U^* \neq V^*$.

(*c*) We observe that $\mathcal{X}(X) \cap U^* = [\rho(x) : U \in \rho(x)] = [\rho(x) : x \in U] = \mathcal{X}(U)$.

This ensures that $\mathcal{X}: (X, \tau) \to (\mathcal{X}(X), \omega_{\mathcal{X}(X)})$ is a semi-homeomorphism (pre-homeomorphism), because

(*i*) $\mathcal{X}(X) \cap U^* \neq \varnothing$, this implies $\mathcal{X}(X)$ is a dense subset of $W(X)$.

(*ii*) If $G \in \tau$ then $\mathcal{X}(G) = \mathcal{X}(X) \cap G^* \in \omega_{\mathcal{X}(X)}$.

(*iii*) If $H^* \in \omega_{\mathcal{X}(X)}$, then $\mathcal{X}^{-1}(H^*) = \mathcal{X}^{-1}(\mathcal{X}(X) \cap G^*) = G$ where $G^* \in \omega$ and $G \in SO(X, \tau)$. Hence $(\mathcal{X}, (W(X), \omega)$ is a semi-extension of the space (X, τ).

(*d*) We show that $W(X)$ is semi-compact.

Let $\{C^*_{ik}\}$ be a semi-closed family of subsets of $W(X)$ which has the finite intersection property $\cap \{C^*_{ik}\} \neq \varnothing$, which implies $\cap \{Cl_\omega C^*_{ik}\} \neq \varnothing$, so $X^* \setminus \cap \{Cl_\omega C^*_{ik}\} \neq X^*$, which implies $\cup \{X^* \setminus Cl_\omega C^*_{ik}\} \neq X^*$, so $\cup \{U^*_{ik}\} \neq X^*$, but $\cup \{U^*_{ik}\} = \{\cup U_{ik}\}^*$, hence $\{\cup U_{ik}\}^* \neq X^*$ this implies $X^* \setminus \cup U^*_{ik} \neq \varnothing$, so $\cap \{X \setminus U_{ik}\} \neq \varnothing$, which implies $\cap \{C_{ik}\} \neq \varnothing$, hence the family $\{C_i\}$ forms a subbase for a closed filter in X and by Lemma 2.4 (a), there exists an ultraclosed filter containing $\{C_i\}$, so $\cap\{C_i^*\} \neq \varnothing$. Hence $W(X)$ is semi-compact. Therefore $(X(X), \omega_{X(X)})$ is a semi-compactification. In the same way we can prove $(X(X), \omega_{X(X)})$ is strongly compactification.

(e) We show that $W(X)$ is T_1-space. Let $F, F' \in W(X)$, $F \neq F'$, then from Lemma 2.4(d) there exists two disjoint closed sets C, C' such that $C \in F$, $C' \in F'$, which implies $X \backslash C \supseteq C'$. Take $U = X\backslash C \supseteq C'$, so that $U \in F'$ and hence $F' \in U^*$, since $X\backslash U = C \in F$, this implies $U \notin F$ and $F \notin U^*$, similarly, one shows that $(X\backslash C')^*$ is an open-neighborhood of F which does not contain F'. Thus $W(X)$ is T_1.

Theorem 2.7. $W(X)$ is semi-normal space, (resp. pre-normal) iff (X,τ) is semi-normal (resp. pre-normal).

Proof. Let (X,τ) be a semi-normal space, let $F, F' \in W'(X)$, $F \neq F'$ then there exists two disjoint closed sets C, C' such that $C \in F$, $C' \in F'$, but (X,τ) is semi-normal, so there exists two disjoint semi-open sets U, V in X such that $C \subseteq U$, $C' \subseteq V$, which implies $F \in U^*$, $F' \in V^*$ and $(U^* \cap V^*) = (U \cap V)^* = \varnothing$. Hence $W(X)$ is semi- T_2 space, also semi-normal (because $W(X)$ is semi-compact). Conversely, let $W(X)$ be semi-T_2 (semi-normal), let A, B be two disjoint closed sets in X, define.

$$A^* = [\{[A]^c\}^*]^c, \quad B^* = [\{[B]^c\}^*]^c,$$

$$\begin{aligned} A^* \cap B^* &= [\{[A]^c\}^*]^c \cap [\{[B]^c\}^*]^c \\ &= [\{[A]^c\}^* \cup \{[B]^c\}^*]^c \\ &= [\{[A]^c \cup [B]^c\}^*]^c = [\{[A \cap B]^c\}^*]^c = [\{X^*\}]^c = \varnothing. \end{aligned}$$

Since $W(X)$ is semi-normal, there exists two disjoint semi-open sets W, W' in $W(X)$ such that $A^* \subseteq W$, $B^* \subseteq W'$, we have $W = \cup U_i^*$, $W' = \cup V_i^*$,where $\{U_i^*\}$ is a semi-open cover of A^* which is a semi-closed in a semi-compact space, hence itself semi-compact, there exist a finite number of $\{U_i^*\}$ such that

$$A^* \subseteq \cup U_{ik}^* = \{\cup U_{ik}^*\} = U^*$$

$$B^* \subseteq \cup V_{ik}^* = \{\cup V_{ik}^*\} = U^*,$$

since $U^* \subseteq W$, $V^* \subseteq W'$, and $W \cap W' = \varnothing$,

this implies $U^* \cap V^* = \varnothing$, $(U \cap V) = \varnothing$, and $A \subseteq U$, $B \subseteq V$.

Hence (X,τ) is semi-normal. In the same way we can prove the other part of the theorem.

Example 2.1. Let X be an infinite set, τ be the indiscrete topology, then (X,τ) is compact and semi-compact but not strongly compact, if we take $W(X)$ to be the collection of all ultraclosed filters in X and $X: X \to W(X)$ is the mapping $X(x) = \rho(x)$ and for every $U \in PO(X,\tau)$, define $U^* = \{F : F \in W(X), U \in F\}$ and the family $\beta = \{U \in PO(X,\tau)\}$ is a base for the topology ω on $W(X)$. Then $(X,(W(X),\omega))$ is a T_1-strongly compactification of (X,τ).

Example 2.2. Let (X,τ) be an infinite cofinite space. This is T_1 and compact but not strongly compact and by Lemma 2.4(b), $\rho(x)$ is an ultraclosed filter, hence the collection of all ultraclosed filters $W(X) = \{\rho(x): x \in X\}$, for every $U \in PO(X,\tau)$, take $\beta = \{\rho(x): x \in U, U \in \rho(x)\}$ this is a base for a topology ω on $W(X)$. Let $X: X \to W(X)$ be the mapping $X(x) = \rho(x)$. Then $(X,(W(X),\omega))$ is a T_1-strongly-compactification of (X,τ).

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Paper Received 16 September 1992; Revised 19 April 1995; Accepted 24 June 1995.