COMPACTIFICATION IN BITOPOLOGICAL SPACES

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الخلاصة :

يهدف هذا البحث الى تقديم صفات وخواص لمفهوم جديد يسمى الفراغات التوبولوجية الثنائية المُصْمَطة من النوع $(i, j)\lambda_\gamma$ والتي تعتمد على مفهوم المؤثر $(i, j)\lambda_\gamma$ للمفاهيم التوبولوجية الثنائية، والذي يُعتبر تعميما لفكرة التصميط للفراغات التوبولوجية الثنائية المعروفة باسم شبة مصمطة وأقوى إصماطا، ودراسة مدى تأثير بعض الدوال على هذه الأنواع الجديدة من الفراغات.

ABSTRACT

Using the *(i,j)* λ_γ-bitopological concepts, we construct *(i,j)* λ_γ-compactness which generalizes the notion of pairwise compactness (semi-compactness, strong compactness). The images of these concepts under some types of functions are discussed. Also we introduce *(i,j)* λ_γ-compactification of bispaces and many of its properties and characterization are investigated.

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1. INTRODUCTION

Throughout the present paper, (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces (bispaces) and $f: X \rightarrow Y$ represents a function. A subset A of space (X, τ) is α-open [1] (semi-open [2], preopen [3], β-open [4]) in X if $A \subseteq Int_\tau\, Cl_\tau\, Int_\tau(A)$ ($A \subseteq Cl_\tau\, Int_\tau(A)$, $A \subseteq Int_\tau\, Cl_\tau(A)$, $A \subseteq Cl_\tau\, Int_\tau\, Cl_\tau(A)$) where $Int_\tau\, Cl_\tau$ is the interior (closure) operator with respect to (w.r.t.) the topology τ. A bispace (X, τ_1, τ_2) is called weak pairwise T_1, as given by Swart [5], if for every two distinct points in X, there exists a τ_1-open set containing one but not the other and a τ_2-open set containing the second but not the first. We are concerned mainly with the new idea of compactness in bitopological spaces. Several authors have considered the problem of defining compactness for bispace. Five apparently different definitions of bitopological compactness have appeared in the literature namely those of Kim[6], Fletcher, Hoyle, and Patty [7], Birsan [8], Swart [5], and Saegrove [9]. Cooke and Reilly [10] studied the relationships between these definitions and showed that the two definitions in [6, 7] are equivalent as in Theorem 2[10] in spite of the statements of Singal [11, p. 284] and Swart [5, p. 135]. A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open [5] (Definition 4.1) if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2, it is called pairwise open [12; Definition 3]. If every pairwise open cover of (X, τ_1, τ_2) has a finite subcover then the space is called pairwise compact [6].

According to Birsan's definition a bispace X is pair compact if every τ_1-open cover can be reduced to a finite τ_2-open and if every τ_2-open cover can be reduced to a finite τ_1-open cover. A bispace X is pair compact by Swart's definition, if every cover by sets from $\tau_1 \cup \tau_2$ has a finite subcover. In 1983 Mashhour *et al.* [12, 13] introduced the concept of pairwise strongly compact. A bispace X is pairwise strongly compact (semicompact [14]), if every pairwise pre-open cover $\mathcal{U} \subseteq PO(X, \tau_1) \cup PO(X, \tau_2)$ (pairwise semi-open cover $\mathcal{U} \subseteq SO(X, \tau_1) \cup SO(X, \tau_2)$) of X has a finite subcover. In 1979, Kasahara [15] defined an operation θ on a topology τ on a non-empty set X to be a function of τ into the power set $P(X)$ such that $G \subseteq G^\theta$, for every $G \in \tau$, where G^θ denotes the value of θ at G. The family of all operations θ is denoted by $O_{\tau(X)}$ and function $\theta^*: \tau_A \rightarrow P(A)$ satisfies $\omega^{\theta^*} = (G \cap A)^{\theta^*} = G^\theta \cap A$, for every $\omega = (G \cap A)$, $\omega \in \tau_A$, the operation θ^* is called the relative operation with respect to θ. An operation $\theta \in O_{\tau(X)}$ is said to be monotone, if for every $U, V \in \tau$ and $U \subseteq V$, $U^\theta \subseteq V^\theta$. In 1983, Abd El-Monsef *et al.* [16] generalized Kasahara's operation [15] by introducing an operation on the power set $P(X)$ of a topological space (X, τ). A function $\Delta: P(X) \rightarrow P(X)$ (resp. $\delta: P(X) \rightarrow P(X)$ is said to be an operation on $P(X)$ of type I [16] (resp. of type II [16]) if $Int(A) \subseteq A^\Delta$ (resp. $Cl_\tau(A) \supseteq A^\delta$), for every $A \in P(X)$ where $A^\Delta(A^\delta)$ denotes the value $\Delta(\delta)$ at A. The family of all operations of type I (resp. of type II) is denoted by $O_{\tau(X)}$ (resp. $\tilde{O}_{\tau(X)}$). The two operations $\Delta \in O_{P(X)}$ and $\delta \in \tilde{O}_{P(X)}$ are said to be dual, if $(X - A^\Delta) = (X - A)^\delta$, for every $A \in P(X)$ Equivalently, δ, Δ are dual, if $(X - A^\delta) = (X - A)^\Delta$, for every $A \in P(X)$. An operation $\Delta \in O_{P(X)}$ (resp. $\delta \in \tilde{O}_{P(X)}$) is said to be monotone [16], if $A \subseteq B$ implies $A^\Delta \subseteq b^\Delta$ (resp. $A^\delta \subseteq B^\delta$).

2. (j,i) λ_γ-COMPLETE REGULARITY

Definition 2.1. A function $\lambda_\gamma: P(X) \rightarrow P(X)$ is called a (j,i) operation on $P(X)$ of a bispace (X, τ_1, τ_2) if λ_γ is an operation on $P(X)$ of type I and also of type II with respect to (X, τ_j) and (X, τ_i) respectively, *i.e.* $Int_{\tau_j}(A) \subseteq A^{\lambda_\gamma}$ (and $Cl_{\tau_i}(A) \supseteq A^{\lambda_\gamma}$) for every $A \in P(X)$, where A^{λ_γ} denotes the value of λ_γ at A and $i \neq j$, $i, j = 1, 2$.

If $Y \subseteq X$ then the function $\lambda_\gamma^*: P(Y) \rightarrow P(Y)$ is called a (j,i) relative operation with respect to the (j,i) operation λ_γ. The family of all (j,i) operations on $P(X)$ with respect to a bispace (X, τ_1, τ_2) is denoted by (j,i) $O_{P(X)}$. A (j,i) operation $\lambda_\gamma \in (j,i)$ $O_{P(X)}$ is called bimonotone, if $A \subseteq B$ implies $A^{\lambda_\gamma} \subseteq B^{\lambda_\gamma}$.

Definition 2.2. A subset A of a bispace (X, τ_1, τ_2) is called a (j,i) λ_γ-open set, if $A \subseteq A^{\lambda_\gamma}$.

It is easy to get corresponding statements for (j,i) λ_γ-closed sets in bispaces. In a bispace (X, τ_1, τ_2) the class of (j,i) λ_γ-open ((j,i) λ_γ-closed) sets will be denoted by $(j,i)\ \lambda_\gamma O(X)$ ($(j,i)\ \lambda_\gamma C(X)$).

Definition 2.3. A subset Y of a bispace (X, τ_1, τ_2) is called a $\tau_1\tau_2$-open set if Y is τ_1-open and τ_2-open.

Lemma 2.1. If Y is a $\tau_1\tau_2$-open set and $A \in (j,i)\ \lambda^*_\gamma O(Y)$, then $A \in (j,i)\ \lambda_\gamma O(X)$.

Lemma 2.2. If Y is a $\tau_1\tau_2$-open subspace of a bispace (X, τ_1, τ_2) and $V \in (j,i)\ \lambda_\gamma O(X)$, then $(Y \cap V) \in (j,i)\ \lambda^*_\gamma O(Y)$.

Definition 2.4. Let A be a subset of a bispace (X, τ_1, τ_2), then the intersection of all (j,i) λ_γ-closed sets containing A is called the (j,i) λ_γ-closure of A and is denoted by $(j,i)\ \lambda_\gamma\text{-}Cl(A)$.

Definition 2.5. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (j,i) λ_γ-continuous if the inverse image of each σ_j-open set in Y is a (j,i) λ_γ-open set in X.

Definition 2.6. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) if the image of every τ_j-open (τ_j-closed) set in X is a (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) set in Y.

Definition 2.7. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous, if the inverse image of each (j,i) λ^*_γ-open set in Y is (j,i) λ_γ-open in X.

Definition 2.8. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (j,i) $\lambda_\gamma\lambda^*_\gamma$-open [$(j,i)$ $\lambda_\gamma\lambda^*_\gamma$-closed], if the image of each (j,i) λ_γ-open ((j,i) λ_γ-closed) set in X is a (j,i) λ^*_γ-open ((j,i) λ^*_γ-closed) set in Y.

Definition 2.9. Two bispaces X and Y are called (j,i) $\lambda_\gamma\lambda^*_\gamma$-homeomorphic equivalent if there exists a bijective function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ such that f is (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous and (j,i) $\lambda_\gamma\lambda^*_\gamma$-open; such a function f is called a (j,i) $\lambda_\gamma\lambda^*_\gamma$-homeomorphism.

Definition 2.10. A bispace X is called (j,i) λ_γ-completely regular if for each (j,i) λ_γ-closed set F and for each point $x \notin F$ there exists a (j,i) λ_γ-continuous function $f: (X, \tau_1, \tau_2) \to [0,1]$ such that $f(x) = 0, f(y) = 1, y \in F$.

A (j,i) λ_γ-completely regular space which is weak pair T_1 as given by Swart [5] is called weak (j,i) $\lambda_\gamma\text{-}T_{3\frac{1}{2}}$.

Definition 2.11. Let (X, τ_1, τ_2) be a bispace, $\Im$ be a family of (j,i) λ_γ-closed sets and Φ a family of (i,j) λ_γ-closed sets; then the pair $(\Im, \Phi)$ is called a bi λ_γ-normal pair iff for each $A \in \Im$ and $B \in \Phi$, such that $A \cap B = \varnothing$ there exist $C \in \Phi$, $D \in \Im$ such that $(X-C) \cap (X-D) = \varnothing$ and $A \subseteq (X-C)$, $B \subseteq (X-D)$.

Definition 2.12. Let (X, τ_1, τ_2) be a bispace, $\Im$ be a family of (j,i) λ_γ-closed sets and Φ a family of (i,j) λ_γ-closed sets; then the pair $(\Im,\Phi)$ is called a bi λ_γ-separating pair iff (i) and (ii) hold:

(i) If F is (i,j) λ_γ-closed and $x \notin F$, then there exist $A \in \Phi$ and $B \in \Im$ such that $x \in A$, $F \subseteq B$, and $A \cap B = \varnothing$.

(ii) If F is (j,i) λ_γ-closed and $x \notin F$, then there exist $A \in \Im$ and $B \in \Phi$, such that $x \in A$, $F \subseteq B$, and $A \cap B = \varnothing$.

Definition 2.13. A set E is a (i,j) zero set iff there exists a (i,j) λ_γ-continuous function $f: X \to (\Re, R, L)$ such that $E = \{x: f(x) \le 0\}$ ($E = \{x: f(x) \ge 0\}$), where $\Re$ is the real line, R is the set of open right rays, and L the set of open left rays.

Theorem 2.1. A bispace (X, τ_1, τ_2) is (j,i) λ_γ-completely regular iff it possesses a bi λ_γ-normal bi λ_γ-separating pair.

Proof. It is easy to verify that the families $\Im$ of all (j,i) zero sets and Φ of all (i,j) zero sets form a bi λ_γ-normal, bi λ_γ-separating pair. To prove the converse, assume that $(\Im, \Phi)$ is a bi λ_γ-normal bi λ_γ-separating pair and F is a (j,i) λ_γ-closed set with $x \notin F$, then there are $F_0 \in \Im$ and $G_1 \in \Phi$ such that $x \in F_0$, $F \subseteq G_1$, and $F_0 \cap G_1 = \varnothing$. By the bi λ_γ-normality condition there are $G_{1/2} \in \Phi$ and $F_{1/2} \in \Im$ such that $(X-F_{1/2}) \cap (X-G_{1/2}) = \varnothing$ and $F_0 \subseteq (X-G_{1/2})$, $G_1 \subseteq (X-F_{1/2})$. Thus $X \in F_0 \subseteq (X-G_{1/2}) \subseteq F_{1/2} \subseteq (X-G_1)$. Since $F_0 \cap G_{1/2} = \varnothing$, again by bi λ_γ-normality of $(\Im, \Phi)$ there are $F_{1/4}$, $G_{1/4}$ in $\Im$ and Φ respectively such that $(X-F_{1/4}) \cap (X-G_{1/4}) = \varnothing$ and $F_0 \subseteq (X-G_{1/4})$, and $G_{1/2} \subseteq (X-F_{1/4})$, $F_{1/4} \subseteq (X-G_{1/2}) \subseteq F_{1/2} \subseteq (X-G_1)$. So $F_{1/4} \cap G_1 \in \varnothing$, similarly we get sets $F_{3/4}$, $G_{3/4}$ such that $F_{1/2} \subseteq (X-G_{3/4})$, $G_1 \subseteq (X-F_{3/4})$. Thus we now have $x \in F_0 \subseteq (X-G_{1/4}) \subseteq F_{1/4} \subseteq (X-G_{1/2}) \subseteq F_{1/2} \subseteq (X-G_{3/4}) \subseteq F_{3/4} \subseteq (X-G_1)$. Continuing this process we get the collections $\{F_L\}_{L \in D} \subseteq \Im$ and $\{G_k\}_{k \in D} \subseteq \Phi$ where D is the set of diadic rationales between 0,1 such that $K, L \in D$, $K<L$, $F_0 \subseteq (X-G_k) \subseteq F_k \subseteq (X-G_L) \subseteq F_L \subseteq (X-G_1)$. Now define a function f: $(X, \tau_1, \tau_2) \to [0,1]$ by $f(x) = inf\{t \in D: x \in X-G_t\}$, $f(x) = 1$, for $x \in G_1$. We can show that f is a (j,i) λ_γ-continuous function onto ([0,1]. R. L) which is obviously 0 on $\{x\}$ and 1 on F. Hence (X, τ_1, τ_2) is (j,i) λ_γ-completely regular.

3. (j,i) λ_γ-COMPACT SPACES

Definition 3.1. A bispace (X, τ_1, τ_2) is called (j,i) λ_γ-compact. If every (j,i) λ_γ-open cover of X has a finite subcover of X.

Example 3.1.

(*i*) If λ_γ is the identity operation then (j,i) λ_γ-compact is just pairwise compact as in Swart's definition.

(*ii*) If $\lambda_\gamma = Int_{\tau_j}\, Cl_{\tau_i}\, Int_{\tau_j}$ $(Cl_{\tau_j}\, Int_{\tau_i}$, $Int_{\tau_j}\, Cl_{\tau_i}$, $Cl_{\tau_j}\, Int_{\tau_i}\, Cl_{\tau_j})$ then (j,i) λ_γ-compact is (j,i) α-compact ((j,i) semi-compact, (j,i) strongly-compact, (j,i) β-compact).

(*iii*) If $i = j$, we return to the ordinary cases (pairwise cases).

Example 3.2. A bispace $X = [0,1]$ with two topologies $\tau_1 = \{X, \varnothing, \{0\}\}$, $\tau_2 = \{X, \varnothing, \{1\}\}$, is (2,1) semi-compact but not pairwise semi-compact because $\{\{0,x\}, \{1\}, x \in X\}$ has not a finite subcover.

Example 3.3. A bispace $X = [0, 1]$ with two topologies τ_1 = discrete topology $\tau_2 = \{X, \varnothing, (a,1], a \in X\}$ is pairwise semi-compact but not (2,1) semi-compact because $\{\{1\}, [0,a), a \in X\}$ is a (2,1) semi-open cover which has not a finite subcover.

The implications between the above compactness conditions are as in the following diagram.

$$\begin{array}{ccc} (j,i)\ \beta\text{-compact} & \Rightarrow & (j,i)\ \text{semi-compact} \\ \Downarrow & & \Downarrow \\ (i,j)\ \text{strongly compact} & \Rightarrow & (i,j)\ \alpha\text{-compact} \Rightarrow \tau_j\text{-compact.} \end{array}$$

One can easily get examples to verify that the converses of these implications may not be true in general.

Definition 3.2. A subset S of a bispace (X, τ_1, τ_2) is said to be (j,i) λ^*_γ-compact relative to X if every cover of S by (j,i) λ_γ-open subsets of X has a finite subcover, where λ^*_γ: $P(S) \to P(S)$.

Theorem 3.1. A bispace (X, τ_1, τ_2) is (j,i) λ_γ-compact iff any $\tau_1\tau_2$-open subspace S of X is (j,i) λ^*_γ-compact relative to X.

Proof. Obvious from Lemmas (2.1), (2.2).

Theorem 3.2. A bispace is (j,i) λ_γ-closed subset of a (j,i) λ_γ-compact bispace is (j,i) λ_γ-compact.

Theorem 3.3. The following statements are equivalent for a bispace (X, τ_1, τ_2).

(*i*) X is (j,i) λ_γ-compact.

(*ii*) Any family of (j,i) λ_γ-closed sets satisfying the finite intersection property has a non empty intersection.

(*iii*) Any family of (j,i) λ_γ-closed sets of X with empty intersection has a finite subfamily with empty intersection.

Theorem 3.4.

(*i*) The image of a (j,i) λ_γ-compact bispace under a (j,i) λ_γ-continuous function is (j,i) λ^*_γ-compact.

(*ii*) Let f be (j,i) λ_γ-continuous function from a (j,i) λ_γ-compact bispace X onto a bispace Y. Then Y is (j,i) λ^*_γ-compact.

The product of a family $\{X_\alpha, \tau_{1\alpha}, \tau_{2\alpha}\}$ is defined in the natural way to be $\{\Pi_{\alpha\in A}X, \Pi\tau_{1\alpha}, \Pi\tau_{2\alpha}\}$ where $\Pi\tau_{1\alpha}$, $\Pi\tau_{2\alpha}$ are the usual product topologies in $\Pi_{\alpha\in A}(X_\alpha, \tau_{1\alpha})$ and $\Pi_{\alpha\in A}(X_\alpha, \tau_{2\alpha})$ respectively. The standard theorems regarding continuity of projections *etc.* follow immediately from the definitions (see [5]).

Definition 3.3. [17] The product topology $\Pi_\alpha\tau_\alpha$ on the product space $X = \Pi_\alpha X_\alpha$ may now be defined as the weak topology induced by the family of all projections $P_\alpha : X \to X_\alpha$.

Theorem 3.5. All projections $P_{\alpha\kappa} : (\Pi X_\alpha, \Pi\tau_{1\alpha}, \Pi\tau_{2\alpha}) \to (X_k, \tau_{1k}, \tau_{2k})$ are (j,i) λ_γ-continuous and (j,i) λ^*_γ-open where $\Pi\tau_{1\alpha}$, $\Pi\tau_{2\alpha}$ are two product topologies in $\Pi\{X_\alpha\}$.

Proof. Obvious.

Theorem 3.6. A function $f : (X, \tau_1, \tau_2) \to (\Pi_\alpha X_\alpha, \Pi_\alpha\tau_{1\alpha}, \Pi_\alpha\tau_{2\alpha})$ is (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous iff $P_{\alpha\kappa} \circ f$ are (j,i) $\lambda_\gamma\lambda^*_\gamma$-continuous.

Proof. Obvious.

Definition 3.4. Let F be a family of functions $f : (X, \tau_1, \tau_2) \to (Y_f, \sigma_{1f}, \sigma_{2f})$. Then $e : (X, \tau_1, \tau_2) \to (\Pi\{Y_f\}, \Pi\{\sigma_{1f}\}, \Pi\{\sigma_{2f}\})$ which associates with every $x \in X$, $e(x)$ in $\Pi\{Y_f\}$ whose f^{th} coordinate is $f(x)$ is called the evaluation map with respect to F.

Definition 3.5. A family F of functions $f : (X, \tau_1, \tau_2) \to (Y_f, \sigma_{1f}, \sigma_{2f})$ is said to distinguish points iff for each pair of distinct points x, y, there exists $f \in F$, such that $f(x) \neq f(y)$.

Definition 3.6. A family F of functions $f : (X, \tau_1, \tau_2) \to (Y_f, \sigma_{1f}, \sigma_{2f})$ is said to distinguish points and (j,i) λ_γ-closed sets iff for every (j,i) λ_γ-closed set $B \in X$ and every $x \in X$, $x \notin B$, then there exists $f \in F$ such that $f(x) \neq (j,i)$ λ_γ-$Cl(f(B))$.

Theorem 3.7. Let F be a family of $(j,i)\ \lambda_\gamma$-continuous functions $f:(X,\tau_1,\tau_2)\to(\Pi\{Y_f\},\Pi\{\sigma_{1f}\},\Pi\{\sigma_{2f}\})$. If F distinguishes points, and $(j,i)\ \lambda_\gamma$-closed sets, then the evaluation function e provides a $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homoemorphism function from X onto the subspace $e(X)$ of $(\Pi\{Y_f\},\Pi\{\sigma_{1f}\},\Pi\{\sigma_{2f}\})$, where $\lambda^*_\gamma: P(\Pi\{Y_f\}\to P(\Pi\{Y_f\})$.

Definition 3.7. A bispace (X,τ_1,τ_2) is pseudo $(j,i)\ \lambda_\gamma$-compact iff every $(j,i)\ \lambda_\gamma$-continuous function $f:(X,\tau_1,\tau_2)\to(\Re,R,L)$ is bounded.

Definition 3.8. A weak $(j,i)\ \lambda_\gamma$-$T_{3\frac{1}{2}}$ space is $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-real compact iff it is $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homeomorphic to the intersection of a ΠR-closed subset and ΠL-closed subset of a product of copies of $(\Re,R,L)$.

Definition 3.9. A bispace (X,τ_1,τ_2) is $(j,i)\ \lambda_\gamma$-compact iff it is pseudo $(j,i)\ \lambda_\gamma$-compact and $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-real compact.

Theorem 3.8. $([0,1],R,L)$ is $(j,i)\ \lambda_\gamma$-compact.

Proof. Obvious.

Theorem 3.9. A weak $(j,i)\ \lambda_\gamma$-$T_{3\frac{1}{2}}$ space (X,τ_1,τ_2) is $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homeomorphic to a subspace of a product of copies of $([0,1],R,L)$.

Proof. Consider the family $F=\{f_\mu\}_{\mu\in M}$ of all $(j,i)\ \lambda_\gamma$-continuous $f_\mu:(X,\tau_1,\tau_2)\to([0,1],R,L)$. Define $f:(X,\tau_1,\tau_2)\to\Pi_{\mu\in M}([0,1],R,L)$ by $(f(x))_\mu=f_\mu(x)$. It is not difficult to show that f is a $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homeomorphism from Theorem (3.7).

Theorem 3.10. The product of a family $\{(X_\alpha,\tau_{1\alpha},\tau_{2\alpha})\}$ of $(j,i)\ \lambda_\gamma$-compact spaces is $(j,i)\ \lambda_\gamma$-compact.

Proof. Since the product of weak $(j,i)\ \lambda_\gamma$-$T_{3\frac{1}{2}}$ spaces is weak $(j,i)\ \lambda_\gamma$-$T_{3\frac{1}{2}}$ we need to show:

(*i*) $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-real compactness

(*ii*) pseudo $(j,i)\ \lambda_\gamma$-compactness.

To prove (*i*) since each $\{(X_\alpha,\tau_{1\alpha},\tau_{2\alpha})\}$ is $(j,i)\ \lambda_\gamma$-compact, we can find for each α a $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homeomorphism h_α from $\{X_\alpha,\tau_{1\alpha},\tau_{2\alpha}\}$ onto the intersection of a ΠR-closed subset $C_{\Pi R\alpha}$ and ΠL-closed subset $C_{\Pi L\alpha}$ of a product of copies of $(\Re,R,L)$ which we may assume to be contained in a product of bounded intervals $\Pi_{\lambda\in\Lambda_\alpha}\left(\left[a_\lambda,b_\lambda\right],R,L\right)$, $h=\Pi_{\alpha\in\Lambda}h_\alpha$ is a $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-homeomorphism onto $\Pi_{\alpha\in\Lambda}(C_{\Pi R}\cap C_{\Pi L})=\Pi_{\alpha\in\Lambda}C_{\Pi R}\cap\Pi_{\alpha\in\Lambda}C_{\Pi L}$ which is the intersection of a ΠR-closed subset with a ΠL-closed subset of $(\Pi_{\lambda\in\Lambda}(\Pi_{\lambda\in\Lambda\alpha}(\Re,R,L))$.

(*ii*) If $\Pi_{\alpha\in\Lambda}(X_\alpha,\tau_{1\alpha},\tau_{2\alpha})$ is not pseudo $(j,i)\ \lambda_\gamma$-compact, there would exist a $(j,i)\ \lambda_\gamma\lambda^*_\gamma$-continuous functions on the product which is unbounded (assume on the right ray). Hence a sequence $\{x_n\}_{n\in N}$ could be found such that $f(x_n)>n$ for each n. The induced sequence $\{h(x_n)\}_{n\in N}$ is in set $(\Pi_{\alpha\in\Lambda}C_{\Pi R}\cap\Pi_{\alpha\in\Lambda}C_{\Pi L})\subseteq\Pi_{\alpha\in\Lambda}(\Pi_{\lambda\in\Lambda\alpha}([a_\lambda,b_\lambda),R,L))$ and has a cluster point y_0 with respect to the usual topology, hence with respect to the two topologies ΠR and ΠL. Since $y_0\in\Pi_{\alpha\in\Lambda}C_{\Pi R\alpha}\cap\Pi_{\alpha\in\Lambda}C_{\Pi L\alpha}$ there exists $x_0\in\Pi_{\alpha\in\Lambda}(X_\alpha,\tau_{1\alpha},\tau_{2\alpha})$ such that $h(x_0)=y_0$. Since x_0 is a $\Pi\tau_1$ and $\Pi\tau_2$-cluster point for $[x_n]_{n\in N}$ and $f^{-1}(-\infty,f(x_0)+\delta)$ contains infinitely many x_n; this is a contradiction. Then $(\Pi(X_\alpha,\tau_{1\alpha},\tau_{2\alpha}))$ is pseudo $(j,i)\ \lambda_\gamma$-compact and hence $(j,i)\ \lambda_\gamma$-compact.

Example 3.3. $[-1,0)\cup(0,1]$ with the induced right ray and the induced left ray topologies is $(j,i)\ \lambda_\gamma$-compact but is not under Swart's definition.

Corollary 3.1. $\Pi([0,1],R,L)$ is $(j,i)\ \lambda_\gamma$-compact.

Theorem 3.11. The intersection of ΠR-closed subset with a ΠL-closed subset of a (j,i) λ_γ-compact space is (j,i) λ_γ-compact.

Proof. In the same manner as for the proof of Theorem 3.10, we can prove pseudo (j,i) λ_γ-compactness.

If $A = C_{\Pi R} \cap C_{\Pi L}$, where $C_{\Pi R}$ is Π R-closed and $C_{\Pi L}$ is Π L-closed, then $h(A) = h(C_{\Pi R}) \cap h(C_{\Pi L}) = (h(C_{\Pi R}) \cap (C_{\Pi R})) \cap (h(C_{\Pi L}) \cap (C_{\Pi L}))$ which is the intersection of a ΠR-closed subset with a ΠL-closed subset of a product of copies of $(\Re, R, L)$.

4. (j,i) λ^*_γ-COMPACTIFICATIONS

Definition 4.1. A subset E of a bispace (X, τ_1, τ_2) is called (j,i) λ_γ-dense if the (j,i) λ_γ-closure of $E = X$.

Definition 4.2. A bispace $(X^*, \tau_1^*, \tau_2^*)$ is a (j,i) λ^*_γ-compactification for (X, τ_1, τ_2) iff $(X^*, \tau_1^*, \tau_2^*)$ is (j,i) λ^*_γ-compact and (X, τ_1, τ_2) is (j,i) $\lambda_\gamma \lambda^*_\gamma$-homeomorphic to a subspace S of X^*, where S is a (j,i) λ_γ-dense subset of X^*.

Theorem 4.1. If (X, τ_1, τ_2) is a weak (j,i) λ_γ-$T_{3\frac{1}{2}}$ space, then there exists a (j,i) λ_γ-compactification for it.

Proof. From Theorem 3.9 a bispace (X, τ_1, τ_2) is (j,i) λ^*_γ-homeomorphic to a subspace $A = C_{\Pi R} \cap C_{\Pi L}$ of a product $\Pi_{\alpha \in \Lambda}([0,1], R, L)$ and the evaluation function $e: (X, \tau_1, \tau_2) \to \Pi_{\alpha \in \Lambda}(S_\alpha, R_{\alpha s}, L_{\alpha s})$ where $R_{\alpha s}, L_{\alpha s}$ are two relative topologies with respect to the set $S = (j,i)\ \lambda_\gamma\text{-}Cl\,(e(X)) = (j,i)\ \lambda_\gamma\text{-}Cl\,\Pi(A) = \Pi(A)$. Since S is a (j,i) λ_γ-dense subset of $\Pi(A)$ which is (j,i) λ_γ-compact because it is (j,i) λ_γ-closed set in $\Pi_{\alpha \in \Lambda}([0,1], R, L)$. Thus $((j,i)\ \lambda_\gamma\text{-}Cl(\Pi(A), \Pi R_\alpha, \Pi L_\alpha)$ is (j,i) λ_γ-compactification for (X, τ_1, τ_2).

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