ON DIRECT SUMS OF UNIFORM MODULES

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الخلاصـة :

تُقـدِّمُ هذا البحث دراسة للمجاميع المباشرة للنماذج المنتظمة ، وذلك باعطاء بعض الشروط الضرورية والكافية للمجموع المباشر لكي يكمل المجاميع الجزئية المباشرة لنماذج منتظمة ، أو لكي يحقق خاصية « كل مجموع جُـزئي مباشر وموضعي لنماذج منتظمة يكون مجموع جُـزئي مباشر » . كما نقوم في هذا البحث دراسة المجاميع المباشرة للنماذج المنتظمة غير الشاذة والمُـعَـرَّقة على حلقات جولدى الشبه أولية ونبرهن أنه إذا كانَ كل مجموع جُـزئي لعدد منتهي من الموديلات يكون من النوع CS فإنَّ كـلَّ نموذج جُـزئي مغلق يكون مجموع جُـزئي مباشر . ونتيجة لذلك نُـبيِّـن أن خاصية الحقن النسبي لكـلِّ زوج من النماذج والتي تظهر في ذلك المجموع تكافئ خاصيَّـة شِـبه الاتصال لذلك المجموع المباشر .

ABSTRACT

This paper studies direct sums of uniform modules. We give a number of necessary and sufficient conditions for such a sum to complement uniform summands, or to have that local summands, of uniform submodules, are summands. We also investigate direct sums of nonsingular uniform modules over semiprime right Goldie rings; and prove that if every finite subsum is a CS-module, then every closed submodule is a direct summand. As a result, we show that the relative injectivity of every two distinct submodules appear in such a sum, is equivalent to the quasi-continuity.

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1. INTRODUCTION

Direct sums of indecomposable modules have been investigated in great detail, in a long series of papers, by Harada [1], Harada and Oshiro [2], Müller and Rizvi [3, 4], and by Kamal [5]. The present paper owes a great deal to the work in [4], and some of the arguments here are taken from that source with little modification.

A module M is a CS-module if every complement submodule of M is a direct summand of M, or equivalently, every closed submodule of M is a direct summand of M. The notion of CS-modules is due to Osofsky and Smith [6]. Chatter and Hajarnavis [7], in one of the first papers to study this concept, have investigated rings in which every complement right ideal is a direct summand. Later, other terminology, such as extending, has been used in place of CS [8–10].

A module M is quasi-continuous if it is a CS-module, and the following condition holds: (C_3) For all X, $Y \subseteq^{\oplus} M$ with $X \cap Y = 0$, one has $X \oplus Y \subseteq^{\oplus} M$. M is continuous if it is a CS-module, and the following condition holds : (C_2) if a submodule N of M is isomorphic to a direct summand of M, then N is a direct summand of M. It is clear that CS-modules are generalizations of (quasi) continuous modules, which, in turn, are generalizations of (quasi) injective modules.

All modules here are right-modules over a ring R. m^o denotes the annihilator in R of the element $m \in M$. $X \subseteq^e M$ and $Y \subseteq^{\oplus} M$ signify that X is an essential submodule, and Y is a direct summand, of M. A submodule A is closed in M if it has no proper essential extensions in M.

A module M is a 1-CS-module if every closed uniform submodule is a direct summand. A special case of the condition (C_3) is that: $(1 - C_3)$ For all X, $Y \subseteq^{\oplus} M$ with $X \cap Y = 0$, and with X, Y uniform; one has $X \oplus Y \subseteq^{\oplus} M$. A direct sum $\oplus_{i\in I} N_i$ of submodules of M is called a local direct summand (or for short a local summand) if $\oplus_{i\in F} N_i \subseteq^{\oplus} M$, for all finite $F \subseteq I$.

An element s in a ring R is called a right regular element in R if $sr \neq 0$ for all non-zero $r \in R$. An element of R is called a regular element of R if it is right and left regular. M and N are called relatively injective modules if M is N-injective and N is M-injective (see [11]).

For a given decomposition $M = \oplus_{i\in I} M_i$ and a subset K of the index set I, we denote $\oplus_{i\in K} M_i$ by $M(K)$. The decomposition $M = \oplus_{i\in I} M_i$ is said to have $(1 - C_3^*)$ if for every uniform direct summand X of M and $J \subseteq I$ with $X \cap M(J) = 0$; one has $X \oplus M(J) \subseteq^{\oplus} M$. $M = \oplus_{i\in I} M_i$ is said to satisfy the condition (A_3) if for any choice of distinct $i_j \in I (j \in \mathbb{N})$ and $m_j \in M_{i_j}$, if the sequence m_j^o is ascending, then it becomes stationary.

Lemma 1. ([9], Lemma 17) Let $M = X \oplus Y$ be a module over an arbitrary ring, where Y is X-injective. Let N be a submodule of M, with $N \cap Y = 0$. Then there exists a homomorphism $f : X \to Y$ such that $X^* = \{x + f(x) : x \in X\}$ contains N, and that $M = X^* \oplus Y$.

Corollary 2. Let $M = \oplus_{i\in I} M_i$, where the M_i are M_j-injective for all $i \neq j \in I$. Then for every direct summand N of M and every finite subset F of I, with $N \cap M(F) = 0$, one has $N \oplus M(F) \subseteq^{\oplus} M$.

Proof. Let $N \subseteq^{\oplus} M$ and $N \cap M(F) = 0$, with F finite subset of I. By Lemma 1, and since $M(F)$ is $M(I - F)$-injective, we have $M = M(F) \oplus M^*(I - F)$, and $N \subseteq^{\oplus} M^*(I - F)$; where $M^*(I - F)$ is defined as X^* in Lemma 1.

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Theorem 3. Let $M = \oplus_{i\in I} M_i$, where M_i are uniform and M_j -injective for all $i \neq j \in I$. Then the following are equivalent:

(1) M has $(1 - C_3^*)$,

(2) $\oplus_{i\in I} M_i$ complements uniform direct summands.

Proof. (1)$\Rightarrow$(2). Let U be a uniform summand of M. Let $0 \neq x \in U$. It follows that $x \in M(F)$ for some finite subset F of I. Since the submodule xR of $M(F)$ is uniform, we have $xR \cap ker\pi_i = 0$ for some natural projection $\pi_i : M(F) \to M_i$; $i \in F$ (otherwise $0 = xR \cap \left[\bigcap_{i\in I} ker\pi_i\right] = \bigcap_{i\in I}[xR \cap ker\pi_i] \neq 0$, which is a contradiction). Thus $xR \cap M(F - i) = 0$, and hence $xR \oplus M(F - i) \subseteq^e M(F)$. It follows that $xR \cap M(I - i) = 0$, and that $xR \oplus M(I - i) \subseteq^e M$. Since $xR \subseteq^e U$, we have $U \cap M(I - i) = 0$. By $(1 - C_3^*)$, $U \oplus M(I - i) \subseteq^{\oplus} M$. Therefore $U \oplus M(I - i) = M$ (due to $U \oplus M(I - i) \subseteq^e M$).

(2)$\Rightarrow$(1). Let X be a uniform summand of M, with $X \cap M(J) = 0$ for some $J \subseteq I$. Since $\oplus_{i\in I} M_i$ complements uniform summands, we have $M = X \oplus M(I - \beta)$ for some $\beta \in I$. Hence $X \cong M_\beta$ is $M(I - \beta)$-injective. By Corollary 2, and since $M(J) \cap X = 0$; we have that $M(J) \oplus X \subseteq^{\oplus} M$.

Proposition 4. Let $M = \oplus_{i\in I} M_i$, where the M_i are uniform and not imbeddable in $M(I - i)$; for all $i \in I$. Then the following are equivalent:

(1) M has $(1 - C_3^*)$,

(2) $\oplus_{i\in I} M_i$ complements uniform direct summands.

Proof. (1)$\Rightarrow$(2). Follows from Theorem 3.

(2)$\Rightarrow$(1). Let X be a uniform summand of M with $X \cap M(J) = 0$, $J \subseteq I$. Since $\oplus_{i\in I} M_i$ complements uniform summands, we have $X \oplus M(I - \alpha) = M$ for some $\alpha \in I$. Now if $\alpha \in J$, then $M_\alpha \cap X = 0$; and thus π imbeds M_α in $M(I - \alpha)$, where π is the projection of $X \oplus M(I - \alpha)$ onto $M(I - \alpha)$; which contradicts our assumption. Therefore $\alpha \notin J$; *i.e.* $X \oplus M(J) \subseteq^{\oplus} X \oplus M(I - \alpha) = M$.

Lemma 5. If a decomposition $\oplus_{i\in I} M_i$, with M_i uniform for all $i \in I$, complements uniform direct summands; then it complements direct summands of the form $\oplus_{i=1}^n U_i$, with all U_i uniform, for all $n \in N$.

Proof. By induction on n. Assume that the claim holds true for n, and let $A = \oplus_{i=0}^n U_i$, with all U_i uniform, be a direct summand of M. By induction $M = A^* \oplus M(K_1)$ for some $K_1 \subseteq I$, where $A^* := \oplus_{i=1}^n U_i$. By the modular law, $A = A^* \oplus (A \cap M(K_1))$. It follows that $U_0^* := A \cap M(K_1) \cong U_0$ is a uniform summand of A, hence a summand of $M(K_1)$. Since $M(K_1)$ inherits the same property, there exists $K_2 \subseteq K_1$ such that $M(K_1) = U_0^* \oplus M(K_2)$; and therefore $M = A \oplus M(K_2)$.

Lemma 6. Let $M = \oplus_{i\in I} M_i$, where the M_i are uniform for all $i \in I$. If M has $(1 - C_3^*)$, then every direct summand of M of the form $X = \oplus_{i=1}^n X_i$, with all X_i uniform ($n \in \mathbb{N}$), and every subset J of I with $X \cap M(J) = 0$; one has $X \oplus M(J) \subseteq^{\oplus} M$.

Proof. By induction on n. Assume that the claim holds true for n, and let $X = \oplus_{i=0}^n X_i \subseteq^{\oplus} M$, with $X \cap M(J) = 0$; where all X_i are uniform, and $J \subseteq I$. Let $X =: \oplus_{i=1} X_i$, and consider $\mathcal{S} = \{J \subseteq K \subseteq I : X \cap M(K) = 0\}$ ordered by inclusion. Zorn's Lemma yields a maximal member K of $\mathcal{S}$. By the maximality of K, we have

$[\bar{X} \oplus M(K)] \cap M_i \neq 0$ for all $i \in I$. Since the M_i, s are uniform, we get that $[\bar{X} \oplus M(K)] \cap M_i \subseteq^e M_i$; $i \in I$. It follows that $\oplus_{i \in I}([\bar{X} \oplus M(K)] \cap M_i) \subseteq^e M$; and therefore $\bar{X} \oplus M(K) \subseteq^e M$. By induction $\bar{X} \oplus M(K) \subseteq^{\oplus} M$; and hence $M = \bar{X} \oplus M(K)$. Then $X = \bar{X} \oplus (X \cap M(K))$; and thus $X_0 \cong X \cap M(K) =: X_0^* \subseteq^{\oplus} X \subseteq^{\oplus} M$. Now we have X_0^* is a uniform summand of $M(K)$, with $X_0^* \cap M(J) = 0$. Since $M(K)$ inherits $(1 - C_3^*)$, $X_0^* \oplus M(J) \subseteq^{\oplus} M(K)$; and therefore $X \subseteq^{\oplus} M$.

Lemma 7. Let $M = \oplus_{i \in I} M_i$, where the M_i are uniform for all $i \in I$. If M has $(1 - C_3)$, then for every uniform summand X of M and every finite subset F of I with $X \cap M(F) = 0$, one has $X \oplus M(F) \subseteq^{\oplus} M$.

Proof. By induction over the cardinality of F. Assume that it holds true for cardinality $< n$, and let X be a uniform summand of M, and $F \subseteq I$ where $X \cap M(F) = 0$, and $|F| = n$. By induction, $X \oplus M(F - i) \subseteq^{\oplus} M$ for some $i \in F$. Write $M = X \oplus M(F - i) \oplus N$. By the modular law, we have that $M(F) = M(F - i) \oplus M(F) \cap [X \oplus N]$. It follows that $M_i^* := M(F) \cap [X \oplus N] \cong M_i$ is a uniform summand of $M(F)$. Hence M_i^* is a uniform summand of $X \oplus N$, with $M_i^* \cap X = 0$. Since $(1 - C_3)$ is inherited by direct summands, $M_i^* \oplus X \subseteq^{\oplus} X \oplus N$. Therefore $X \oplus M(F) = M_i^* \oplus X \oplus M(F - i) \subseteq^{\oplus} X \oplus M(F - i) \oplus N = M$.

Theorem 8. Let $M = \oplus_{i \in I} M_i$, where the M_i are uniform for all $i \in I$. Let (A_3) hold. Then the following are equivalent:

(1) M has $(1 - C_3^*)$,

(2) M has $(1 - C_3)$, and every local summand of M of the form $\oplus_{j \in J} L_j$ with all L_j uniform, is a summand.

Proof. (1)$\Rightarrow$(2). Let $L = \oplus_{j \in J} L_j$ be a local summand of M, with all L_j uniform. By the same argument as in Lemma 6, there exists $K \subseteq I$ such that $L \cap M(K) = 0$ and that $L \oplus M(K) \subseteq^e M$. By Lemma 6, and since $\oplus_{j \in J} L_j$ is a local summand of M, we get that $L(F) \oplus M(K) \subseteq^{\oplus} M$ for all finite subsets F of J; *i.e.* $L(J) \oplus M(K)$ is a local summand of M. Hence, without loss of generality, we may consider $L \subseteq^e M$. Now we show that $L = M$. Assume, on the contrary, that $L \neq M$; then there exists $x_o \in M_{i_0} \backslash L$, for some $i_0 \in I$. Since $L \subseteq^e M$, there exists $r \in R$ with $0 \neq x_0 r \in L$; and thus $x_0 r \in L(F)$ for a finite subset F of J. Since M has $(1 - C_3^*)$, by Theorem 3 and Lemma 5 we have that the decomposition complements $L(F)$. Write $M = L(F) \oplus M(T)$; $T \subseteq I$. Then $x_0 = b_0 + y$; where $b_0 \in L(F)$, and $y = \Sigma_{t \in T} y_t \in M(T)$. It is clear that $x_0^o \subseteq y_t^o$, and since $0 \neq x_0 r \in L(F)$, we obtain $x_0^o \subset y_t^o$; for all $t \in T$. Observe (due to $x_0 \notin L$) that there exists $t_* \in T$, with $y_{t_*} \in M_{t_*} \backslash L$.

Denote $x_1 := y_{t_*}$ and $i_1 := t_*$, we have $x_j \in M_{i_j} \backslash L (j = 0, 1)$; such that $x_0^o \subset x_1^o$. By repeating the same argument, we eventually obtain a sequence $\{x_j\}_{j \in N}$; where $x_j \in M_{i_j} \backslash L (j \in N)$ for distinct i_j with $x_0^o \subset x_1^o \subset \ldots \subset x_n^o \subset \ldots$; which contradicts (A_3). Therefore $L = M$.

To show that M has $(1 - C_3)$. Let $X, Y \subseteq^{\oplus} M$, X and Y are uniform with $X \cap Y = 0$. Then, by $(1 - C_3^*)$, $M = X \oplus M(I - i)$ for some $i \in I$. Hence $X \cong M_i$; and thus the docomposition $X \oplus M(I - i)$ inherits $(1 - C_3^*)$. Since $Y \cap X = 0$, we conclude $X \oplus Y \subseteq^{\oplus} M$.

(2)$\Rightarrow$(1). Let X be a uniform summand of M, with $X \cap M(J) = 0$, for some $J \subseteq I$. Since M has $(1 - C_3)$, by Lemma 7, we have that $X \oplus M(F)$ is a direct summand of M for all finite subsets F of J. Therefore the decomposition $X \oplus M(J)$ is a local summand of M; and hence a summand of M, by (2).

Corollary 9. Let $M = \oplus_{i \in I} M_i$, with M_i uniform and M_j-injective for all $i \neq j \in I$. Let (A_3) be hold. Then the following are equivalent:

(1) M has $(1 - C_3^*)$,

(2) every local summand of M, of the form $\oplus_{j \in J} L_j$ with all L_j uniform, is a summand.

Proof. Theorem 8, and Corollary 2.

Lemma 10. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. If $M(F)$ is a CS-module, for all finite subsets F of I, then every non-zero summand of M contains a uniform summand.

Proof. Let A be a non-zero summand of M. Let $0 \neq a \in A$, it follows that $a \in M(F)$; and hence $aR \subseteq M(F)$, for some finite subset F of I. Let U be a uniform submodule of aR, and consider a maximal essential extension U^* of U in $M(F)$. Since $M(F)$ is CS-module, with a closed submodule U^*, we have $U^* \subseteq^{\oplus} M(F) \subseteq^{\oplus} M$. Write $M = A \oplus M^*$, and let $x \in U^*$ be arbitrary. Then $x = y + z$; where $y \in A$, $z \in M^*$. Since $U \subseteq^e U^*$, and R is a semiprime right Goldie ring, there exists a regular element $s \in R$ such that $xs \in U$. Thus $xs - ys = zs \in A \cap M^* = 0$, and hence $z = 0$ (due to M^* non-singular). Hence $x = y \in A$; and therefore U^* is the required uniform summand of A.

Lemma 11. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Let $M(F)$ be a CS-module for all finite subsets F of I. Then the following are equivalent:

(1) every local summand of M is a summand;

(2) every local summand of M, of the form $\oplus_{j\in J} L_j$ with all L_j uniform, is a summand.

Proof. (1)$\Rightarrow$(2) is obvious. To show (2)$\Rightarrow$(1), it suffices to show that every direct summand of M is a direct sum of uniform submodules. To this end, let X be a direct summand of M. Let $\mathcal{S} =: \{X_j : j \in J\}$ be the family of all uniform direct summands of X. Observe, by Lemma 10, that $\mathcal{S}$ is non empty. We call a subset K of J local direct if $\Sigma_{k\in K} X_k$ is a direct sum and is a local summand of M. Consider the collection of all local direct subsets of J; ordered by inclusion. An application of Zorn's Lemma yields a maximal local direct subset K of J. By (2), $\oplus_{k\in K} X_k$ is a direct summand of M; hence of X. Write $X = \oplus_{k\in K} X_k \oplus X^*$. If $X^* \neq 0$, then by Lemma 10 X^* contains a uniform direct summand; which contradicts the maximality of K. Therefore $X = \oplus_{k\in K} X_k$ is a direct sum of uniforms.

Lemma 12. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Let $M(F)$ be CS-module, for all finite subsets F of I. Then M is a 1-CS-module.

Proof. Let A be a closed uniform submodule of M. Let $0 \neq x \in A$, it follows that $xR \subseteq M(F)$; with F a finite subset of I. Let U be a maximal essential extension of xR in $M(F)$. By assumption, $M(F)$ is a CS-module; hence $M(F) = U \oplus N$. Thus $M = U \oplus N \oplus M(I - F)$. Now for each $a \in A$, we have that $a = u + y$; $u \in U$ and $y \in N \oplus M(I - F)$. Since $xR \subseteq^e U$, there exists a regular element $s \in R$ such that $us \in xR$. It follows that $(a - u)s = ys \in A \cap [N \oplus M(I - F)] = 0$ (due to A uniform, and $xR \cap [N \oplus M(I - F)] = 0$); and hence $y = 0$ (due to all M_i being non-singular). It follows that $A \subseteq^e U$; and therefore $A = U \subseteq^{\oplus} M$, since A is closed in M.

Lemma 13. ([5], Corollary 14) Let M be a non-singular module over a semiprime right Goldie ring R. Then the following are equivalent:

(1) M is a CS-module,

(2) M is a 1-CS-module, and local direct summands of M are summands.

Corollary 14. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Then the following are equivalent:

(1) M is a CS-module,

(2) $M(F)$ is a CS-module for all finite subsets F of I, and every local summand of M is a summand.

Lemma 15. ([12], Corollary 5) Every direct summand of a CS-module (1-CS-module) is a CS-module (1-CS-module).

Lemma 16. ([13], Theorem 2.13) Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of quasi-continuous modules. Then the following are equivalent:

(1) $M = \oplus_{\alpha\in\Lambda} M_\alpha$ is quasi-continuous,

(2) $M(\Lambda - \alpha)$ is M_α-injective for all $\alpha \in \Lambda$.

Lemma 17. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Let $M(F)$ be a CS-module for all finite $F \subseteq I$. If M has $(1 - C_3)$, then every submodule $A = \oplus_{i=1}^n A_i$, with all A_i uniform summands of M, is a summand of M.

Proof. By induction on n. Let the claim hold true for n, and let $A = \oplus_{i=0}^n A_i$, with all A_i uniform summands of M. By induction, $\oplus_{i=1}^n A_i \subseteq^\oplus M$. Write $M = \oplus_{i=1}^n A_i \oplus N$; hence $A = \oplus_{i=1}^n A_i \oplus (A \cap N)$. Let B_0 be a maximal essential extension of $A \cap N$ in N. Since $A_0 \cong A \cap N \subseteq^e B_0$, we have that B_0 is a uniform closed submodule of N. By Lemma 12 and Lemma 15, N is a 1-CS-module; and hence $B_0 \subseteq^\oplus N$. It follows that $A \subseteq \oplus_{i=1}^n A_i \oplus B_0 \subseteq^\oplus M$. Now $\oplus_{i=1}^n A_i \oplus B_0$ inherits $(1-C_3)$, and is a 1-CS-module; hence Lemma 16 yields that A_1 is B_0-injective for all $i = 1, 2, \ldots, n$. By Lemma 1, and since $\oplus_{i=1}^n A_i$ is B_0-injective with $\oplus_{i=1}^n A_i \cap A_0 = 0$, there exists B_0^* such that $A_0 \subseteq B_0^*$ and that $\oplus_{i=1}^n A_i \oplus B_0 = \oplus_{i=1}^n A_i \oplus B_0^*$. Since $A_0 \subseteq^\oplus M$ and $B_0^* \cong B_0$ is uniform, we obtain $A_0 = B_0^*$; and therefore $A = \oplus_{i=1}^n A_i \oplus B_0^* \subseteq^\oplus M$.

Theorem 18. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Let $M(F)$ be CS-module for all finite $F \subseteq I$. Then $(1 - C_3)$ is equivalent to $(1 - C_3^*)$.

Proof. Let M have $(1-C_3)$, and let X be a uniform summand of M with $X \cap M(J) = 0$; $J \subseteq I$. By Zorn's Lemma, there exists $K \subseteq I$ a maximal with respect to $J \subseteq K$ and $X \cap M(K) = 0$. One can check that $X \oplus M(K) \subseteq^e M$. We show that $M = X \oplus M(K)$. To this end, let $m \in M$ be arbitrary. Then, by the essentiality of M over $X \oplus M(K)$, there exists a regular element s in R, such that $ms \in X \oplus M(K)$; and thus $ms \in X \oplus M(F)$ for some finite $F \subseteq I$. Then $X \oplus M(F) \subseteq^\oplus M$, by Lemma 17. Write $M = X \oplus M(F) \oplus N$; thus $m = b + n$, where $b \in X \oplus M(F)$, $n \in N$. We deduce $(m - b)s = ns \in (X \oplus M(F)) \cap N = 0$. Since N is non-singular, we have that $n = 0$; and hence $m = b \in X \oplus M(F) \subseteq X \oplus M(K)$. Therefore $X \oplus M(J) \subseteq^\oplus X \oplus M(K) = M$.

The converse is obvious, by Theorem 8.

Lemma 19. ([4], Theorem 9). Let $M = \oplus_{i\in I} M_i$, with all M_i uniform. Then M is quasi-continuous if and only if $(1 - C_1)$ and $(1 - C_3^*)$ hold.

Theorem 20. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Let $M(F)$ be CS-module for all finite $F \subseteq I$. Then M is quasi-continuous if and only if $(1 - C_3)$ holds.

Proof. Lemma 12, Theorem 18, and Lemma 19.

Lemma 21. ([12],Theorem 18) Let $M = \oplus_{i=1}^n M_i$, where M_i is M_j-injective for all $i \neq j$. Then M is CS-module if and only if all M_i are CS-modules.

Theorem 22. Let $M = \oplus_{i\in I} M_i$, with all M_i non-singular uniform, be a module over a semiprime right Goldie ring R. Then the following are equivalent:

(1) M_i is M_j-injective for all $i \neq j \in I$,

(2) M is quasi-continuous,

(3) $M(I-\alpha)$ is M-injective for all $\alpha \in I$.

Proof. (1)$\Rightarrow$(2): By Lemma 21, $M(F)$ is CS-module for all finite $F \subseteq I$. In view of Theorem 20, it remains to show that M has $(1-C_3)$. Let X and Y be uniform summands of M, with $X \cap Y = 0$. Let $0 \neq x \in X$, by the same argument as in Theorem 3, we have $xR \subseteq M(F)$ with $xR \cap M(F-\alpha) = 0$ for a finite $F \subseteq I$ and $\alpha \in F$. Let B be a maximal essential extension of xR in $M(F)$; hence $B \cap M(F-\alpha) = 0$. By Lemma 1 and since $M(F-\alpha)$ is M_α-injective, we have that $M(F) = M_\alpha^* \oplus M(F-\alpha)$ with $B \subseteq M_\alpha^*$; and hence $M = M_\alpha^* \oplus M(I-\alpha)$. Now for each $a \in X$, we get $a = c + d$ where $c \in M_\alpha^*$ and $d \in M(I-\alpha)$. Since $xR \subseteq^e B \subseteq^e M_\alpha^*$ (due to $M_\alpha^* \cong M_\alpha$ uniform), there is a regular element $s \in R$ such that $cs \in xR$. It follows that $(a-c)s = ds \in M_\alpha^* \cap M(I-\alpha) = 0$; and thus $d = 0$ (due to all being M_i non-singular). Then $X \subseteq^{\oplus} M_\alpha^*$; and hence $X = M_\alpha^*$. Now we have that $M = X \oplus M(I-\alpha)$, where $Y \cap X = 0$ and X is $M(I-\alpha)$-injective. Thus, again by Lemma 1, we have that $M = X \oplus M^*(I-\alpha)$ with $Y \subseteq^{\oplus} M^*(I-\alpha)$. Therefore $X \oplus Y \subseteq^{\oplus} M$.

(2)$\Rightarrow$(3) follows from Lemma 16, and (3)$\Rightarrow$(1) is obvious.

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