A CHARACTERIZATION OF THE POSITIVE SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION

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الخلاصـة :

لنأخذ المعادلة (١) $x'(t) = \int_\Omega x(t + \tau(w))\, d\mu(w)$ والمعادلة المرفقة لها (٢) $\lambda = \exp \int_\Omega \lambda\tau(w)\, d\mu(w)$. من المعروف ان كل جذر حقيقي للمعادلة (٢) يولّد حلاً غير متذبذب (non-oscillatory) للمعادلة (١) مما يطرح السؤال عن حجم فضاء مجموعة الحلول التي لها هذه الصفة . في هذا البحث يجري البرهان على أنّ مجموعة الحلول غير المتذبذب (non-oscillatory) تولد بواسطة حلين $\exp(\alpha t), \exp(\beta t)$ حيث α, β هما جذرا المعادلة (٢) ؛ أو $\exp(\alpha t), \exp(\alpha t)$ اذا كان $\alpha = \beta$.

ABSTRACT

Consider the functional differential Equation (1) $x'(t) = \int_\Omega x(t + \tau(w))\mathrm{d}\mu(w)$ where $\tau: \Omega \to (0, \infty)$ is bounded measurable and $0 < \mu(\Omega) < \infty$, and its characteristic Equation (2) $\lambda = \int_\Omega e^{\lambda\tau(w)}\mathrm{d}\mu(w)$. Every real solution of (2), when it exists, gives rise to a non-oscillatory solutions of (1), but the size of the space of all non-oscillatory solutions has not been considered. In this paper it is shown that the cone of all positive solutions of (1), when non-empty, is two dimensional: it is generated by exp (αt) and exp (βt) where α and β are the two real roots of (2); or by exp (αt) and texp (αt) when $\alpha = \beta$.

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1. INTRODUCTION

Consider the functional differential equation

$$x'(t) = \int_{\Omega} x(t + \tau(w))\, \mathrm{d}\mu(w)\,, \tag{1}$$

where (Ω, m, μ) is a measure space and $\tau\colon \Omega \to R$ is a real-valued function belonging to $L^1(\mu)$ and satisfying for some $M > 0$:

$$-M \leq \tau(w) \leq M \quad \text{for all} \;\; w \in \Omega. \tag{2}$$

A solution of (1) is a function x of the real variable t, defined and differentiable on (M, ∞), having the property that $x(t + \tau(.)) \in L^1(\Omega, \mu)$ for each $t \geq M$ satisfying (1). A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on (M, ∞). An important special case of (1) occurs when $\Omega = [0, \infty), \tau(w) = w$, and $\mathrm{d}\mu(w) = \mathrm{d}H(w)$ where H is a distribution function on $[0, \infty)$. In this case Equation (1) takes the form

$$x'(t) = \int_0^{\infty} x(t + w)\, \mathrm{d}H(w). \tag{3}$$

Such an equation is of the Wiener–Hopf convolution type and may be considered as a limiting case ($\lambda_1 \to \infty$) of an equation [1, p.1] that arises in the study of stochastic processes.

Another special case of (1) occurs when τ is a simple function. In this case Equation (1) takes the form

$$x'(t) = \sum_{i=1}^{n} p_i\, x(t + \tau_i), \tag{4}$$

where $p_i > 0$ and $\tau_i \geq 0$ for $i = 1, 2, \ldots, n$.

The oscillation theory of (4) and of more general equations containing it as a special case has been extensively developed. See, for example [2–5] and the references cited therein. A typical result is the following.

Theorem A. A necessary and sufficient conditions for each solution of (4) to be oscillatory is that the characteristic equation:

$$\lambda = \sum_{i=1}^{n} p_i \exp(\lambda \tau_i) \tag{5}$$

has no real roots.

It is important to note that Equation (4) always possesses oscillatory solutions, since any non-real root $\lambda = \gamma + i\delta$ of (5) gives rise to the oscillatory solution $\exp(\gamma t)\cos\delta t$ and, of course, Equation (5) always has non-real roots. Non-oscillatory solutions can arise only when (5) has real roots. But what is the nature of a non-oscillatory solution when it exists? With the exception of [1], questions of this kind seem not to have been treated in the literature. The main contribution of this paper is to supply a complete answer to this question for Equation (1). In order to elucidate our result let us consider Equation (4) and assume that x is a solution of it that is positive on some semi-infinite interval (M, ∞). It is clear that such a solution x will belong to the class $C^\infty(M, \infty)$. By Berstein's Theorem, x extends to a function analytic in the entire complex plane. In other words every positive solution of (4) extends to an entire function and it is natural to try to determine its order of growth. Using an important transformation (Equation (38)) we are able to recast our differential equation in a form that makes it possible to adapt to it the method in [2], leading to the conclusion that positive solutions must be of exponential type. This is the most important (and difficult) step and we complete the characterization by using standard techniques from complex function theory. The end result is Theorem 3 of this paper. Theorems 1 and 2 are straightforward and deal with the oscillation of Equation (1) and the associated equation:

$$x'(t) = -\int_\Omega x(t - \tau(w))\,\mathrm{d}\mu(w). \tag{1'}$$

Finally, let us note that our result (Theorem 3) does not follow from the result in [1]. The relevant equation considered there which takes the form:

$$x(t) + \lambda_1 x'(t) = \lambda_1 \int_0^\infty x(t + w)\,\mathrm{d}H(w)\,, \tag{6}$$

can be reduced to (1) only by dividing by λ_1 and then letting $\lambda_1 \to \infty$.

Theorem 1. Let $\tau \in L^1(\Omega, \mu)$ where (Ω, m, μ) is a σ-finite measure space and assume that

$$\int_\Omega \tau(w)\,\mathrm{d}\mu(w) > \frac{1}{e}. \tag{7}$$

(*a*) If τ is bounded below then every solution of (1) is oscillatory;

(*b*) If τ is bounded above then every solution of (1′) is oscillatory.

Theorem 2. Let (Ω, m, μ) be a measure space and suppose that $\mu(\Omega) < \infty$ and $\tau \in L^1(\Omega, \mu)$ and satisfies

$$0 \le \tau(\omega) \le M, \quad \text{for all } \omega \in \Omega. \tag{8}$$

A necessary and sufficient condition for all solutions of (1) to be oscillatory is that the characteristic equation

$$\int_\Omega \exp\{\lambda\tau(\omega)\}\,\mathrm{d}\mu(\omega) = \lambda \tag{9}$$

has no real roots.

A solution x of (1) is said to be positive if there is *a* t_1 in its domain such that $x(t) > 0$ for all $t > t_1$.

Theorem 3. Let (Ω, m, μ) and τ be as in Theorem 2, and let A be the set of all positive solutions of (1). If $A \neq \emptyset$ then (9) has exactly two real roots (counting multiplicity) α and β, and

$$A = \{c_1 \exp(\alpha t) + c_2 \exp(\beta t) : c_2 \geq 0\} \quad \text{if } \alpha < \beta \tag{10}$$

and

$$A = \{c_1 \exp(\beta t) + c_2 t \exp(\beta t) : c_2 \geq 0\} \quad \text{if } \alpha = \beta; \tag{11}$$

where c_1 and c_2 are real constants such that when $c_2 = 0$, $c_1 > 0$.

2. PROOF OF THEOREM 1

Let (Ω, m, μ) be a σ-finite measure space and let $\tau \in L^1(\Omega, \mu)$ satisfy (7). There exists a sequence $\{\Omega_n\}$ of measurable subsets of Ω satisfying $\Omega = U_{n=1}^{\infty} \Omega_n$, $\Omega_n \subseteq \Omega_{n+1}$ and

$$0 < \mu(\Omega_n) < \infty \quad \text{for } n = 1, 2, 3, \ldots. \tag{12}$$

The possibility of choosing the Ω_n to satisfy the left hand inequality in (12) follows from the relation $\mu(\Omega) = \lim_{n\to\infty} \mu(\Omega_n)$ and $\mu(\Omega) > 0$; this last inequality being an immediate consequence of (7).

Proof of part (a). Suppose now that τ is bounded below say $\tau(\omega) \geq -M$ for some $M > 0$ and all $\omega \in \Omega$, and that (1) has a non-oscillatory solution x. Clearly, we may, and do, assume that $x(t) > 0$ for all $t > t_1$ for a sufficiently large t_1. Then (1) implies that $x'(t) > 0$ for all $t > t_1 + 2M$ and so $x(t)$ is an increasing function on $(t_1 + M, \infty)$. But then (1) implies that $x'(t)$ is increasing on $(t_1 + 2M, \infty)$ and hence x is a convex function there. Fix $t > t_1 + 2M$ and note that the function $f : \Omega \to R$ defined by $f(\omega) = t + \tau(\omega)$ belongs to $L^1(\Omega_n, \mu)$ and has its range contained in $(t_1 + M, \infty)$ where x is now convex. Applying Jensen's inequality [5, p.61] in the form

$$x\left(\frac{1}{\mu(\Omega_n)} \int_{\Omega_n} f(\omega)\, \mathrm{d}\mu\right) \leq \frac{1}{\mu(\Omega_n)} \int_{\Omega_n} (xof)(\omega)\, \mathrm{d}\mu(\omega)$$

we obtain from (1):

$$\begin{aligned} x'(t) &= \int_{\Omega} x\,(t + \tau(\omega))\, \mathrm{d}\mu(\omega) \geq \int_{\Omega_n} x\,(t + \tau(\omega))\, \mathrm{d}\mu(\omega) \\ &= \int_{\Omega_n} (xof)(\omega)\, \mathrm{d}\mu(\omega) \geq \mu(\Omega_n) x\left(t + \frac{1}{\mu(\Omega_n)} \int_{\Omega_n} \tau(\omega)\, \mathrm{d}\mu(\omega)\right), \end{aligned}$$

valid for $n = 1, 2, \ldots$ and $t > t_1 + 2M$.

Thus x is a solution of the differential inequality

$$x'(t) \geq \mu_n x\left(t + \frac{1}{\mu_n} \int_{\Omega_n} \tau(\omega)\, \mathrm{d}\mu(\omega)\right) \tag{13}$$

where $\mu_n = \mu(\Omega_n)$, $t > t_1 + 2M$, and $n = 1, 2, \ldots$.

By [4, p.135], (13) implies that

$$\int_{\Omega_n} \tau(\omega)\,\mathrm{d}\mu(\omega) \le 1/e \quad \text{for } n = 1, 2, \ldots. \tag{14}$$

Taking limits in (14) as $n \to \infty$ and using the dominated convergence theorem [5, p.26] we obtain

$$\int_{\Omega} \tau(\omega)\,\mathrm{d}\mu(\omega) \le 1/e \tag{15}$$

which contradicts (7). Hence, under (7), every solution of (1) is oscillatory if τ is bounded below.

Proof of part (b). Suppose now that τ is bounded above; say $\tau(\omega) \le M$ for some $M > 0$ and all $\omega \in \Omega$. If we assume that (1′) has a solution $x(t) > 0$ for all $t > t_1$ for a sufficiently large t_1, we conclude as in the proof of part (*a*) that, successively, $x'(t) < 0$, $x(t)$ is decreasing, $x'(t)$ is increasing and finally $x(t)$ is convex on $(t_1 + M, \infty)$. Instead of (13) we obtain

$$x'(t) \le -\mu_n x\left(t - \frac{1}{\mu_n}\int_{\Omega_n} \tau(\omega)\,\mathrm{d}\mu(\omega)\right) \tag{16}$$

and applying [4, p.135] we arrive at (14) and so (15) thereby obtaining a contradiction to (7). This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

Let (Ω, m, μ) be a measure space. Suppose that $\mu(\Omega) < \infty$ and $\tau\colon \Omega \to R$ is a real-valued measurable function satisfying (8). To avoid trivialities we assume that $\mu = \mu(\Omega) > 0$ and τ is not equivalent to zero. If Equation (9) has a real root λ then, necessarily, $\lambda > 0$ and $x(t) = \exp(\lambda t)$ is a non-oscillatory solution of (1). This proves one part of Theorem 2.

Conversely, suppose that (1) has a solution $x(t) > 0$ for all $t > t_1$ and some sufficiently large t_1. Since τ is bounded and $\mu(\Omega) < \infty$ one shows in a standard way that, successively, x' is continuous, x'' exists, x'' is continuous and so forth. By induction we conclude that x belongs to the class $C^\infty(t_1, \infty)$ and that

$$x^n(t) = \int_\Omega \cdots \int_\Omega x(t + \tau(\omega_1) + \cdots + \tau(\omega_n))\,\mathrm{d}\mu(\omega_1)\cdots\mathrm{d}\mu(\omega_n). \tag{17}$$

Fix $t_2 > t_1$ and extend x to a function defined for all complex z by setting

$$x(z) = \sum_{k=0}^{\infty} \frac{x^{(k)}(t_2)}{k!}\,(z - t_2)^k. \tag{18}$$

Since $x(t) > 0$ for all $t > t_1$ the same is true of all its derivatives by (17). The positivity of the derivatives on (t_1, ∞) makes it possible [6, p.139] to prove that the series in (18) converges to $x(z)$ for every real $z \ge t_2$. Since the coefficients are positive, the series will continue to converge for all complex z thereby providing the definition of $x(z)$ for those values. Writing (18) in the form

$$x(t) = \sum_{k=0}^{\infty} a_k (t - t_2)^k \tag{19}$$

we have $a_k > 0$ for $k = 0, 1, 2, \ldots$ and so $x(t) \geq a_0$ for all $t \geq t_2$. By (18) and (19) we then have

$$k! a_k = x^{(k)}(t_2) \geq a_0 (\mu(\Omega))^k \equiv a_0 \mu^k, \tag{20}$$

which, when used in (19), gives

$$x(t) \geq a_0 \exp\{\mu(t - t_2)\} \quad \text{for } t \geq t_2. \tag{21}$$

Define a sequence $\{c_n\}$ by

$$c_1 = \mu, \; c_{n+1} = \int_\Omega \exp\{c_n \tau(\omega)\} \, \mathrm{d}\mu(\omega). \tag{22}$$

By induction, c_n is a monotone increasing sequence and, for $t \geq t_2$,

$$x(t) \geq a_0 \exp\{c_n(t - t_2)\} \tag{23}$$

for $n = 1, 2, \ldots$. Indeed, for $n = 1$, (23) is true by (21) and if we suppose that we have (23), then, together with (17), it implies

$$k! a_k \geq a_0 \left(\int_\Omega \exp\{c_n \tau(\omega)\} \, \mathrm{d}\mu(\omega) \right)^k = a_0 (c_{n+1})^k \tag{24}$$

which, when used in (19), gives $x(t) \geq a_0 \exp\{c_{n+1}(t - t_2)\}$, $t \geq t_2$. Thus (23) is proved. Now (23) shows that the sequence c_n is bounded above by $\log x(1 + t_2) - \log a_0$ and it follows that $\lambda = \lim_{n \to \infty} c_n$ exists as a finite number. Since $c_n < c_{n+1}$ and τ is non-negative we may use the Lebesgue monotone convergence theorem to obtain

$$\lambda = \lim_{n \to \infty} c_{n+1} = \lim_{n \to \infty} \int_\Omega \exp\{c_n \tau(\omega)\} \, \mathrm{d}\mu(\omega) = \int_\Omega \lim_{n \to \infty} \exp\{c_n \tau(\omega)\} \, \mathrm{d}\mu(\omega) = \int_\Omega \exp\{\lambda \tau(\omega)\} \, \mathrm{d}\mu(\omega). \tag{25}$$

Hence λ is a real root of Equation (9) and the proof of Theorem 2 is completed.

4. STUDY OF A TRANSCENDENTAL EQUATION

In this section we determine the roots of Equation (9) in a suitably chosen disk about the origin in the complex plane. Our results will be presented in three lemmas.

Lemma 1. Let p and τ be two positive real numbers satisfying

$$p\tau \leq 1/e. \tag{26}$$

Then the equation

$$p \exp(\lambda\tau) = \lambda \tag{27}$$

has only two real roots α and β satisfying

$$p < \alpha \leq ep \leq \beta < \infty. \tag{28}$$

Furthermore, α and β are the only roots of (27), real or complex, in the disk $|\lambda| \leq \beta$. Conversely, if (27) has real roots then (26) holds true.

Proof. Let $F(\lambda) = p \exp(\lambda\tau) - \lambda$ then $F'(\lambda) = p\tau \exp(\lambda\tau) - 1$ and $F''(\lambda) = p\tau^2 \exp(\lambda\tau)$. Thus F' is strictly increasing and, since $F'(-\infty) = -1$ and $F'(+\infty) = \infty$, it has a unique real zero $-\tau^{-1} \log p\tau = \lambda_1$. It follows that F has real zeros if and only if $F(\lambda_1) \leq 0$. This last inequality is equivalent to $\lambda_1\tau \geq 1$ which in turn is equivalent to (26). A double root of (27) occurs if and only if equality holds in (26) and this is equivalent to $\lambda_1 = pe$. Suppose now that (26) is satisfied and let $\lambda = r \exp(i\theta)$ be a root of (27) satisfying $0 < r \leq \beta$ and $-\pi < \theta \leq \pi$. Then $r = p \exp(\tau r \cos\theta)$ and $\theta = \tau r \sin\theta + 2k\pi$ where k is an integer. Then $r \leq p \exp(\tau r)$ and so $F(r) \geq 0$ implying that either $r \leq \alpha$ or $r \geq \beta$. If $r \leq \alpha$ then $r \leq ep$ and $|2k\pi| \leq |\theta| + \tau r \leq \pi + \tau ep \leq \pi + 1$ implying that $k = 0$. Then $|\theta| \leq \tau ep|\sin\theta| \leq |\sin\theta|$ which implies that $\theta = 0$. This shows that λ is real and so $\lambda = \alpha$. If $r \geq \beta$ we need consider only the case $r = \beta$ since we are interested in locating the zeros inside the disk $|\lambda| \leq \beta$. In this case we have $\beta = p \exp(\tau\beta\cos\theta) \leq p \exp(\tau\beta) = \beta$ so that $\theta = 0$ and $\lambda = \beta$. Hence the only roots, real or complex, of Equation (27) lying inside the disk $|\lambda| \leq \beta$ are its real roots α and β.

Lemma 2. Let $p_1, p_2, \cdots, p_n$ *and* $\tau_1, \tau_2, \cdots, \tau_n$ be positive numbers and $n \geq 2$. Then the equation

$$F_n(\lambda) \equiv \sum_{i=1}^{n} p_i \exp(\lambda\tau_i) - \lambda = 0 \tag{29}$$

has at most two real roots α_n and β_n satisfying

$$\alpha_1 < \alpha_n \leq \beta_n < \beta_1 \tag{30}$$

where α_1 and β_1 are the real roots of the equation $F_1(\lambda) = 0$. Furthermore, when they exist, α_n and β_n are the only roots of (29), real or complex, in the disk $|\lambda| \leq \beta_n$.

Proof. F_n'' is positive for all real λ and F_n' has exactly one real zero, so F_n has at most two real zeros which, when they exist, are necessarily positive. Suppose now that F_n has a real zero, say λ. Then $\lambda = F_n(\lambda)+\lambda \geq \sum_{i=1}^{n} p_i e\lambda\tau_i$ where we have used the elementary inequality $\exp(x) \geq ex$. It follows that $\sum_{i=1}^{n} p_i\tau_i \leq 1/e$ and, in particular, that $p_1\tau_1 < 1/e$. This last inequality implies, by Lemma 1, that F_1 has two real and distinct zeros α_1 and β_1; and since $F_1(\lambda) < F_n(\lambda)$ for all positive λ, (30) follows.

Now let $C_1 = \{\lambda : |\lambda| = \beta_n, Re\lambda \leq x\}$ and $C_2 = \{\lambda : Re\lambda = x, |\lambda| \leq \beta_n\}$ where x is chosen to lie between β_n and the unique real zero of $F_n^{'}$. If λ belongs to C_1 then we have

$$|F_n(\lambda) - F_1(\lambda)| \leq \sum_{i=2}^{n} p_i \exp(Re\lambda\tau_i) < F_n(\beta_n) - F_1(\beta_n)$$
$$= \beta_n - p_1 \exp(\beta_n\tau_1) < |\lambda| - |p_1 \exp(\lambda\tau_1)| \leq |\lambda - p_1 \exp(\lambda\tau_1)|.$$

If λ belongs to C_2 write $\lambda = x + iy$, then

$$|F_n(\lambda) - F_1(\lambda)| \leq F_n(x) - F_1(x) < F_n(\beta_n) - F_1(x)$$
$$= x - p_1 \exp(x\tau_1) \leq x - p_1 \exp(x\tau_1) \cos y\tau_1$$
$$= Re\,\{\lambda - p_1 \exp(\lambda\tau_1)\} \leq |\lambda - p_1 \exp(\lambda\tau_1)|.$$

It follows that $|F_n(\lambda) - F_1(\lambda)| < |F_1(\lambda)|$ on $C_1 \cup C_2$ and so, by Rouche's Theorem, F_n must have inside the contour $C_1 \cup C_2$ the same number of zeros as F_1. By Lemma 1, F_1 has exactly two zeros α_1 and β_1 in the disk $|\lambda| \leq \beta_1$ and so it has one zero inside the contour $C_1 \cup C_2$ since $\beta_n < \beta_1$ by (30). Therefore, F_n has exactly one zero inside the contour $C_1 \cup C_2$. Since x may be chosen arbitrarily close to β_n it follows that F_n has exactly two zeros in $|\lambda| \leq \beta_n$ namely α_n and β_n.

Lemma 3. Let (Ω, m, μ) be a measure space satisfying $\mu(\Omega) < \infty$ and $\tau\colon \Omega \to R$ a non-negative bounded measurable function satisfying

$$0 < \int_\Omega \tau(\omega)\, \mathrm{d}\mu(\omega) < \infty. \tag{31}$$

Then the equation

$$F(\lambda) \equiv \int_\Omega \exp\{\lambda\tau(\omega)\}\, \mathrm{d}\mu(\omega) - \lambda = 0 \tag{32}$$

has at most two real roots. Furthermore, if real roots exist and, if β is the largest one, then Equation (32) has in the disk $|\lambda| \leq \beta$ exactly two roots counting multiplicity.

Proof. We may assume that τ is not a simple function since, otherwise, the conclusion of Lemma 3 is included in Lemma 2. Clearly, F'' is positive for all real λ and F' has exactly one real zero, so F has at most two real zeros which, when they exist, must be positive. Suppose now that F has real roots and let them be denoted by α and β where $\alpha \leq \beta$. If λ is a complex zero of F satisfying $|\lambda| = \beta$ then we have,

$$\beta \leq \int_\Omega \exp\{Re\lambda\, \tau(\omega)\}\, \mathrm{d}\mu(\omega) \leq \int_\Omega \exp\{\beta\tau(\omega)\}\, \mathrm{d}\mu(\omega) = \beta \tag{33}$$

where the last equality occurs because β is a real zero of F. Clearly, (33) implies that $\exp(Re\lambda\, \tau(\omega)) = \exp(|\lambda|\tau(\omega))$ almost everywhere and this in turn implies, since τ is positive on a set of positive measure, that $Re\lambda = |\lambda|$. Thus, $\lambda = \beta$ and we conclude that the only complex zero of F on the circle $|\lambda| = \beta$ is β.

We now show that F has only real zeros inside the disk $|\lambda| < \beta$. Since τ is non-negative and bounded, there exists [5, p.15] a sequence $\{\tau_n\}$ of simple functions converging to τ uniformly on Ω and satisfying

$$0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau \tag{34}$$

on Ω.

Since τ is assumed not simple we have $\tau - \tau_n > 0$ on a set of positive measure and we may assume that $\tau_n > 0$ on a set of positive measure for all n. In particular, if we put

$$F_n(\lambda) = \int_\Omega \exp\{\lambda \tau_n(\omega)\}\, \mathrm{d}\mu(\omega) - \lambda \tag{35}$$

then $F_n(\lambda) < F(\lambda)$ for all $\lambda \geq 0$. This last inequality and $F(\beta) = 0$, together with $F_n(0) > 0$ and $F_n(\infty) > 0$ imply that F_n has at least two distinct zeros which we denote by α_n and β_n. By Lemma 2 they are the only real roots of F_n and they satisfy

$$0 < \alpha_n < \alpha \leq \beta < \beta_n \tag{36}$$

where α and β are the real zeros of F.

Since τ is bounded, $0 < \mu(\Omega) < \infty$ and τ_n converges to τ uniformly on Ω, it is easy to show that F_n converges to F uniformly on compact subsets of the complex plane. Clearly, F_n is an analytic function of the complex variable λ for all λ. Specializing to the region $D = \{\lambda : |\lambda| < \beta, Im\lambda > 0\}$ we have a sequence $\{F_n\}$ of analytic functions each of which is, by Lemma 2 and (36), never zero in D and converging to F uniformly on compact subsets of D: By Hurwitz's theorem [7, p.178] F is analytic and either identically zero or never zero in D. The first possibility being clearly excluded we conclude that F has no zeros in D. The same reasoning applies to the region $\bar{D} = \{\lambda : |\lambda| < \beta, Im\lambda < 0\}$ and we conclude that, in $|\lambda| < \beta$, F has only real zeros. The conclusion of the Lemma now follows immediately.

5. THE POSITIVE SOLUTIONS OF (1); PROOF OF THEOREM 3

Proof of (10). Let (Ω, m, μ) and τ satisfy the conditions of Theorem 2 and suppose that Equation (1) has a positive solution. By Theorem 2, Equation (9) must have real solutions and Lemma 3 shows that there are only two such, say α and β where $\alpha \leq \beta$. Suppose now that $\alpha < \beta$. Let x be a positive solution of (1). The proof of Theorem 2 shows that x extends to an entire function and our first task is to show that x is of exponential type less than or equal to β. Since x is representable by a power series with positive coefficients about t_2, which without loss of generality may be assumed 0, (see proof of Theorem 2), it suffices to show that x is of exponential type $\leq \beta$ on the positive t axis. This will be achieved by setting

$$g(t) = \exp(-\beta t)x(t) \quad (t \geq 0) \tag{37}$$

and showing that for all positive c the integral

$$\int_0^\infty \exp(-cs)\, g(s)\, \mathrm{d}s$$

is finite. Since g is assumed positive, it will suffice to find a positive sequence t_n increasing to ∞ and such that the integrals $\int_0^{t_n} \exp(-cs)g(s)\,ds \le M$ for some constant M and $n = 1, 2, \dots$. We start by finding a suitable representation for g.

Let

$$y(t) = x(t) - \int_\Omega \exp(\beta t) \int_t^{t+\tau(\omega)} \exp\{\beta(\tau(\omega) - \theta)\}x(\theta)\,d\theta\,d\mu(\omega) \quad (t \ge 0). \tag{38}$$

The existence of the integral in (38) is immediate and the possibility of differentiating inside the integral is easily justified using boundedness of τ, finiteness of $\mu(\Omega)$ and analyticity in t of the integrand. We thus have

$$\begin{aligned} y'(t) &= x'(t) - \beta(x(t)) - \int_\Omega \exp(\beta t)\{\exp(-\beta t)x(t+\tau(\omega)) - \exp\{\beta(\tau(\omega) - t)\}x(t)\}\,d\mu(\omega) \\ &= -\beta x(t) + \beta y(t) + x(t)\int_\Omega \exp\{\beta\tau(\omega)\}\,d\mu(\omega) = \beta y(t) \end{aligned}$$

which implies that $y(t) = y(0)\exp(\beta t)$. Equating the two expressions for $y(t)$ we arrive at

$$g(t) = y(0) + \int_\Omega \int_t^{t+\tau(\omega)} \exp\{\beta\tau(\omega)\}g(\theta)\,d\theta\,d\mu(\omega) \tag{39}$$

which is the needed representation.

To eliminate the constant in (39), set $g(t) = z(t) + k$ where $k = y(0)/\left\{1 - \int_\Omega \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega)\right\}$ then

$$z(t) = \int_\Omega \int_t^{t+\tau(\omega)} \exp\{\beta\tau(\omega)\}z(\theta)\,d\theta\,d\mu(\omega) = \int_\Omega \int_0^{\tau(\omega)} \exp\{\beta\tau(\omega)\}z(t+s)\,ds\,d\mu(\omega). \tag{40}$$

At this point we remark that if F is the function defined in (32), then our assumption that $\alpha < \beta$ implies that $F'(\beta) > 0$ and this translates into $\int_\Omega \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega) - 1 > 0$; thus the constant k above is well-defined. Furthermore, since $g(t) \ge 0$ we have $z(t) \ge -|k|$ for all $t \ge 0$.

Fix a number $c > 0$. There exists a positive ε and a subset $\Omega_\varepsilon \subseteq \Omega$ such that

$$\gamma \equiv \int_{\Omega_\varepsilon} \exp\{\beta\tau(\omega)\} \int_{\varepsilon/2}^{\tau(\omega)} \exp(cs)\,ds\,d\mu(\omega) > 1. \tag{41}$$

To see this suppose first that $\tau(\omega) > 0$ for all $\omega \in \Omega$ and let $\Omega_n = \{\omega \in \Omega : \tau(\omega) > 1/n\}$ where $n = 1, 2, \dots$. Then $\Omega_n \subseteq \Omega_{n+1}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$. Then $\int_{\Omega_n} \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega) \to \int_\Omega \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega) > 1$, and $1/n\int_{\Omega_n} \exp\{\beta\tau(\omega)\}\,d\mu(\omega) \to 0$ as $n \to \infty$. If $\omega \in \Omega_n$ then the mean-value theorem implies $\exp\{c^{\tau(\omega)}\} - \exp(c/n) \ge c(\tau(\omega) - 1/n)\exp(c/n) \ge c(\tau(\omega) - 1/n)$. Then $\lim_{n\to\infty} \int_{\Omega_n} \exp\{\beta\tau(\omega)\} \int_{1/n}^{\tau(\omega)} \exp(cs)\,ds\mu(\omega) \ge \lim_{n\to\infty}\left\{\int_{\Omega_n} \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega) - 1/n\int_{\Omega_n} \exp\{\beta\tau(\omega)\}\,d\mu(\omega)\right\} = \int_\Omega \tau(\omega)\exp\{\beta\tau(\omega)\}\,d\mu(\omega) > 1$.

It follows that we can find n large enough so that, with $\varepsilon = 1/n$, (41) holds true. Since we always assume τ is not equivalent to zero, the above still holds true if $\tau \geq 0$. In the remaining part of this proof c will be a fixed positive constant and ε the corresponding positive number for which (41) holds true. Following the proof in [2], we put, for $t \geq 0$,

$$w(t) = \int_0^t \exp(-cu) z(u)\,\mathrm{d}u \tag{42}$$

and use (40) to obtain

$$w(t) = \int_\Omega \int_\Omega^{\tau(\omega)} \exp\{\beta\tau(\omega)\} \exp(cs) \int_s^{t+s} \exp(-cv) z(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\mu(\omega),$$

which, in view of (42), gives

$$w(t) = \int_\Omega \exp\{\beta\tau(\omega)\} \int_0^{\tau(\omega)} \exp(cs)\omega(t+s)\,\mathrm{d}s\,\mathrm{d}\mu(\omega) - C_1 \tag{43}$$

where $C_1 = \int_\Omega \exp\{\beta\tau(\omega)\} \int_0^{\tau(\omega)} \exp(cs) w(s)\,\mathrm{d}s\,\mathrm{d}\mu(\omega)$.

The inequality $z(t) \geq -|k|$ and (42) give $w(t) \geq -|k|/c$ for $t \geq 0$; this last inequality when used in (43) gives:

$$w(t) = \int_\Omega \exp\{\beta\tau(\omega)\} \int_0^{\tau(\omega)} \exp(cs)(w(t+s) + |k|/c)\,\mathrm{d}s\,\mathrm{d}\mu(\omega) - C_2\,,$$

where $C_2 = C_1 + \frac{|k|}{c} \int_\Omega \exp\{\beta\tau(\omega)\} \int_0^{\tau(\omega)} \exp(cs)\,\mathrm{d}s\,\mathrm{d}\mu(\omega)$. It follows that:

$$w(t) \geq \int_{\Omega\varepsilon} \exp\{\beta\tau(\omega)\} \int_{\varepsilon/2}^{\tau(\omega)} \exp(cs) w(t+s)\,\mathrm{d}s\,\mathrm{d}\mu(\omega) - C_3\,, \tag{44}$$

where

$$C_3 = C_2 - \int_{\Omega\varepsilon} \exp\{\beta\tau(\omega)\} \int_{\varepsilon/2}^{\tau(\omega)} \frac{|k|\exp(cs)}{c}\,\mathrm{d}s\,\mathrm{d}\mu(\omega)\,.$$

Given $t_0 > 0$, (44) implies that there exists $w \in \Omega_\varepsilon$ and $t_1 \in (t_0+\varepsilon/2,\ t_0+\tau(\omega))$ such that $w(t_1) \leq (w(t_0)+C_3)/\gamma$. By induction there is a sequence $\{t_n\}$ such that

$$w(t_{n+1}) \leq \frac{w(t_n) + C_3}{\gamma} \quad \text{and} \quad t_n > t_0 + \frac{n\varepsilon}{2} \quad \text{for } n = 1, 2, 3, \ldots\,. \tag{45}$$

Consider the difference equation $x_{n+1} = x_n/\gamma + C_3/\gamma$ with $x_0 = \omega(t_0)$. An easy induction shows that $w(t_n) \leq x_n$; and since x_n converges to $C_3/(\gamma - 1)$, we conclude that $w(t_n)$ is bounded by a constant, say K. Note also that t_n tends to ∞ by (45) so that, by selecting a subsequence if necessary, we may consider t_n to be increasing. We now have

$$\int_0^{t_n} \exp(-cs)g(s)\,\mathrm{d}s = \int_0^{t_n} \exp(-cs)(z(s)+k)\,\mathrm{d}s = w(t_n) + k\int_0^{t_n} \exp(-cs)\,\mathrm{d}s < K + |k|/c.$$

This is the inequality sought and, since $g(t) \geq 0$, we conclude that $\int_0^\infty \exp(-cs)g(t)\,\mathrm{d}t < \infty$ for every positive c. The Cauchy criterion then implies that $\lim_{t\to\infty} \int_t^{t+M} \exp(-cs)g(s)\,\mathrm{d}s = 0$ for any fixed $M > 0$. Since our function τ satisfies $0 \leq \tau(\omega) \leq M$ for every $\omega \in \Omega$, we conclude that $\int_t^{t+\tau(\omega)} \exp(-cs)g(s)\,\mathrm{d}s \to 0$ as $t \to \infty$ uniformly in $\omega \in \Omega$. Now, if in (39), we multiply by $\exp(-ct)$ and use the inequality $\exp(-cs) \geq \exp(-ct).\exp\{-c\tau(\omega)\}$ which is valid for $t \leq s \leq t+\tau(\omega)$, we obtain

$$\exp(-ct)\,(g(t)-y(0)) \leq \int_\Omega \exp\{(\beta+c)\tau(\omega)\} \int_t^{t+\tau(\omega)} \exp(-cs)g(s)\,\mathrm{d}s\,\mathrm{d}\mu(\omega);$$

and as $g(t)-y(0)$ is non-negative by (39), it follows that $\lim_{t\to\infty} \exp(-ct)(g(t)-y(0)) = 0$. Recalling the definition (37) we have shown that for every $c > 0$, and every $\varepsilon > 0$ there exists $T > 0$ such that $x(t) \leq \varepsilon \exp\{(\beta+c)t\}$ for all $t \geq T$. This shows that x is of exponential type $\leq \beta$ on the positive t-axis and, as remarked at the beginning of this proof, this implies that $x(z)$ is an entire function of exponential type $\leq \beta$. We now determine x. Let $\varepsilon > 0$ be so small that the disk $|z| \leq \beta + \varepsilon$ contains no zeros of Equation (9) except α and β. Such an ε exists by Lemma 3. Let C be the positively oriented circle $|z| = \beta + \varepsilon$ and $\phi(z) = \sum_{n=0}^{\infty} \frac{n!a_n}{z^{n+1}}$ where $x(z) = \sum_{n=0}^{\infty} a_n z^n$.

It is well-known [8, p.113] that, since $x(z)$ is of exponential type $\leq \beta, \phi(z)$ is defined for all $|z| > \beta$ and

$$x(z) = \frac{1}{2\pi i}\int_C \exp(zu)\,\phi(u)\,\mathrm{d}u \tag{46}$$

for all complex z.

Differentiating (46), using (1) and the function F defined in (32) we arrive at

$$\frac{1}{2\pi i}\int_C \exp(zu)F(u)\,\phi(u)\,\mathrm{d}u = 0. \tag{47}$$

The equality (47) being valid for all z implies [9, p.110] that $F(u)\phi(u)$ extends to an analytic function inside C. It follows that $\phi(u)$ extends to a function analytic inside C except for two poles at α and β, the zeros of $F(u)$ there. Since they are simple zeros of F, they must be simple poles of ϕ and so the extension of ϕ inside C takes the form

$$\phi(u) = \frac{c_1}{u-\alpha} + \frac{c_2}{u-\beta} + h(u) \tag{48}$$

where $h(u)$ is analytic in $|u| \leq \beta + \varepsilon$.

Now (46), (48), Cauchy's Theorem and the Cauchy integral formula give

$$x(z) = c_1\exp(\alpha z) + c_2\exp(\beta z). \tag{49}$$

We are assuming that $x(t) > 0$ for all large t; so if in (49) we put $z = t$, divide by $\exp(\beta t)$ and let $t \to \infty$ we get $c_2 \geq 0$. If $c_2 = 0$ then, c_1 must be positive; if $c_2 > 0$ then c_1 is unrestricted. This completes the proof of the first part of Theorem 3.

Proof of (11). Let x be a positive solution of (1) and assume that Equation (9) has two equal roots $\alpha = \beta$. We define $g(t)$ as in (37) and obtain the representation (39) for it. At this point a little modification is needed in the proof of part (10) since now we have $\int_\Omega \tau(\omega) \exp\{\beta\tau(\omega)\}\, d\mu(\omega) = 1$. So to eliminate the constant in (39), set $g(t) = z(t) + kt$ where $k = -2y(0)/\int_\Omega \tau^2(\omega)\exp\{\beta\tau(\omega)\}$ to obtain (40). Fix a number $c > 0$ and note that we can still find $\varepsilon > 0$ such that (41) holds true. The proof now proceeds as before. However, we now use $z(t) \geq |k|t$ to obtain $w(t) \geq -|k|/c^2$ for all $t \geq 0$. A sequence t_n is then found which increases to ∞ and such that $w(t_n) \leq M$ and then it follows that

$$\int_0^{t_n} \exp(-cs)\, g(s)\, ds = w(t_n) + k \int_0^{t_n} s \exp(-cs)\, ds < M + k/c^2.$$

We conclude, as before, that x is of exponential type $\leq \beta$, has a representation (46) in terms of ϕ and ϕ has inside C the extension

$$\phi(u) = \frac{c_2}{(u-\beta)^2} + \frac{c_1}{u-\beta} + h(u).$$

It follows that

$$x(z) = c_1 \exp(\beta z) + c_2 z \exp(\beta z) \tag{50}$$

and $c_2 \geq 0$ and, if $c_2 = 0$ then $c_1 > 0$.

This completes the proof of Theorem 3.

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