EIGENFUNCTION EXPANSION ASSOCIATED WITH ELLIPTIC EIGENVALUE PROBLEM WITH EIGENVALUE PARAMETER IN THE BOUNDARY CONDITIONS

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الخلاصة :

ندرس في هذا البحث مسألة القيمة الذاتية المنتظمة والمكوّنة من معادلة تفاضلية جزئية بالاضافة الى الشروط الحدية التي تحتوي القيمة الذاتية البارامترية ، وسنبنى لهذه المسألة مؤثراً ذا ترافق ذاتي أساسي في فراغ هلبرت المُعَرَّف والمناسب لهذه المسألة ، وكذلك نطوِّر نظرية مفكوك دالة القيمة الذاتية المرتبطة بهذه المسألة .

ABSTRACT

In this paper we shall study a regular right-definite eigenvalue problem for elliptic partial differential equation with eigenvalue parameter in the boundary conditions. We associate with it an essentially self-adjoint operator in suitably defined Hilbert space and develop associated eigenfunction expansion theorem.

Key words and phrases: An expansion theorem, an elliptic operator, right-definite eigenvalue problems, eigenvalue parameter in the boundary conditions, Hilbert space formulation.

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1. INTRODUCTION

Regular eigenvalue problem for the Laplace operator in R^1 with eigenvalue parameter in the boundary conditions have been studied by many authors, see for example, Fulton [1], Hinton [2], Walter [3], Schneider [4], Zayed and Ibrahim [5, 6], and Zayed [7]. But regular eigenvalue problems for the Laplace operator in $R^n, n \geq 2$ with eigenvalue parameter in the boundary conditions have been studied by Canavati and Minzoni [8], Odhnoff [9], Eastham [10, 11], Zayed and Ibrahim [12], and many others. In the present paper we shall study a regular eigenvalue problem for the elliptic operator in $R^n, n \geq 2$ with eigenvalue parameter in the boundary conditions.

The problem to be discussed here can be formulated as follows: Let Ω be a normal domain in $R^n, n \geq 2$ with the smooth boundary $\partial\Omega$. We consider the following elliptic eigenvalue equation

$$\tau u = \frac{1}{r}(-E_n + q)\, u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

with the mixed boundary condition

$$u_\nu + \sigma(\mathbf{x})u = \lambda u, \qquad \text{on } \partial\Omega, \tag{1.2}$$

where we assume throughout that:

(*i*) $E_n u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(p_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_j} \right);$

(*ii*) $u_\nu = \sum_{i,j=1}^{n} p_{ij}(\mathbf{x}) u_{x_j} \nu_i$, where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer unit normal of Ω on its boundary $\partial\Omega$ and $\mathbf{x} = (\mathbf{x}_1, \ldots, x_n)$ is a generic point in R^n, where $u_{x_j} = \frac{\partial u}{\partial x_j}$;

(*iii*) $q(\mathbf{x})$ is a real-valued continuous function, and $r(\mathbf{x}) > 0$ is a real-valued function such that $r(\mathbf{x}) \in C(\bar{\Omega})$, $\bar{\Omega} = \Omega U \partial\Omega$ and $C(\bar{\Omega})$ is the space of all continuous functions which are defined on $\bar{\Omega}$;

(*iv*) $p_{ij}(\mathbf{x})$, $i,j = 1, \ldots, n$ are real-valued functions which are continuously differentiable and $p_{ij} = p_{ji}$;

(*v*) $\sigma(\mathbf{x})$ is a real-valued continuous function for all $\mathbf{x} \in \partial\Omega$;

(*vi*) For an arbitrary complex function $f(\mathbf{x})$ and a positive constant c_0, we have the "ellipticity condition";

$$\sum_{i,j=1}^{n} p_{ij}(\mathbf{x}) f_{x_i} \bar{f}_{x_j} \geq c_o \sum_{i=1}^{n} \left| f_{x_i} \right|^2, \tag{1.3}$$

for all $\mathbf{x} \in \Omega$;

(*vii*) λ is a complex number.

Definition 1.1. The eigenvalue problem (1.1) – (1.2) is said to be "regular" if Ω is bounded and $r(\mathbf{x})$ is defined on $\bar{\Omega}$.

Definition 1.2. The regular eigenvalue problem (1.1) – (1.2) is said to be "right-definite" if $r(\mathbf{x})$ is positive on $\bar{\Omega}$.

In this paper we give an operator-theoretic formulation of problem (1.1) – (1.2), by associating with it an essentially self-adjoint operator A with compact resolvent, and prove that the spectrum of A consists of unbounded sequence of real eigenvalues. Moreover, we show that the eigenfunctions of A form a complete fundamental system in the Hilbert space $H = L^2(\Omega; r) \oplus L^2(\partial\Omega)$, and then we prove an expansion theorem for A.

2. HILBERT SPACE FORMULATION

Let $L^2(\Omega; r)$ and $L^2(\partial\Omega)$ be two complex Hilbert spaces of Lebesgue measurable functions $f(\mathbf{x})$ in Ω and $\partial\Omega$ respectively satisfying:

(*i*) $\int_\Omega r(\mathbf{x})\,|f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} < \infty$;

(*ii*) $\int_\Omega |f(\mathbf{x})|^2 \mathrm{d}s < \infty$;

where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\, \mathrm{d}x_2 ... \mathrm{d}x_n$ is the volume element corresponding to Ω, while $\mathrm{d}s$ is the surface element corresponding to $\partial\Omega$.

Definition 2.1. We define a Hilbert space H of two component vectors by

$$H = L^2(\Omega; r) \oplus L^2(\partial\Omega); \tag{2.1}$$

with inner product

$$[f, g]_H = \langle f_1, g_1 \rangle_\Omega + \langle f_2, g_2 \rangle_{\partial\Omega}$$

$$= \int_\Omega r(\mathbf{x}) f_1(\mathbf{x}) \overline{g_1(\mathbf{x})}\; \mathrm{d}x + \int_{\partial\Omega} f_2(\mathbf{x}) \overline{g_2(\mathbf{x})}\; \mathrm{d}s; \tag{2.2}$$

and norm

$$\|f\|_H^2 = \int_\Omega r(\mathbf{x}) |f_1(\mathbf{x})|^2\; \mathrm{d}\mathbf{x} + \int_{\partial\Omega} |f_2(\mathbf{x})|^2\; \mathrm{d}s; \tag{2.3}$$

for each $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in H.

Definition 2.2. Let H^* be a set of all those elements $f(\mathbf{x})$ satisfying

(*i*) $f \in C^1(\overline{\Omega}) \cap C^2(\Omega)$;

(*ii*) $\frac{1}{r}(-E_n + q) f \in L^2(\Omega; r)$.

We define a linear operator A: $D(A) \to H$ by

$$Af = \left(\frac{1}{r}(-E_n + q) f_1, f_{1\nu} + \sigma(\mathbf{x}) f_1\right); \tag{2.4}$$

for each $f = (f_1, f_2)$ in $D(A)$, in which the domain $D(A)$ of A is defined as follows:

$$D(A) = \left\{\left(f|_\Omega, f|_{\partial\Omega}\right) \in H : f \in H^*\right\}, \tag{2.5}$$

where $f|_\Omega$ (or $f|_{\partial\Omega}$) is a restriction of f on Ω (or on $\partial\Omega$) respectively.

Remark 2.1. The parameter λ is an eigenvalue and f_1 is a corresponding eigenfunction of problem (1.1) – (1.2) if and only if

$f = (f_1, f_2) \in D(A)$ and $Af = \lambda f$.

Therefore, the eigenvalues and the eigenfunctions of problem (1.1) – (1.2) are equivalent to the eigenfunctions of A in H.

Lemma 2.1. $D(A)$ is a dense subset of H with respect to the inner product (2.2).

Proof. Suppose that $D(A)$ is not a dense subset of H with respect to the inner product (2.2), then there exists a non-zero element $\mathbf{0} \neq f = (f_1, f_2) \in H$ such that

$$[f, g]_H = 0 \text{ for all } g = (g_1, g_2) \in D(A).$$

In particular

$$\langle f_1, g_1 \rangle_\Omega = 0 \text{ for all } g_1 \in C^1(\overline{\Omega}) \cap C^2(\Omega),$$

which means that $f_1 \equiv 0$. Hence, using (2.5) we obtain

$$0 = \langle f_2, g_2 \rangle_{\partial\Omega} = \langle f_2, g_1|_{\partial\Omega} \rangle_{\partial\Omega} \text{ for all } g_1 \in C^1(\overline{\Omega}) \cap C^2(\Omega),$$

which means that $f_2 \equiv 0$. Thus $f \equiv \mathbf{0}$. This is a contradiction. Hence $D(A)$ is dense in H.

Lemma 2.2. The linear operator A in H is symmetric.

Proof. Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be any two elements in $D(A)$, then

$$[Af, g]_H = \int_\Omega \left\{ -\sum_{i,j=1}^{n} \left(p_{ij} f_{1x_j} \right)_{x_j} + q f_1 \right\} \overline{g}_1 \, d\mathbf{x} + \int_{\partial\Omega} \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \nu_i + \sigma f_1 \right\} \overline{g}_2 \, ds. \tag{2.6}$$

Making use of the formula (4) of Section 4.3 in [13], the above formula (2.6) becomes

$$[Af, g]_H = \int_\Omega \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \overline{g}_{1x_i} + q f_1 \overline{g}_1 \right\} d\mathbf{x} + \int_{\partial\Omega} \sigma f_1 \overline{g}_2 \, ds + \int_{\partial\Omega} \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \nu_i \right\} \left(\overline{g}_2 - \overline{g}_1 \right) ds. \tag{2.7}$$

Because of (2.5) the last integral in (2.7) vanishes, and consequently

$$[Af, g]_H = \int_\Omega \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \overline{g}_{1x_i} + q f_1 \overline{g}_1 \right\} d\mathbf{x} + \int_{\partial\Omega} \sigma f_1 \overline{g}_1 \, ds. \tag{2.8}$$

Applying a similar argument, it follows that

$$[f, Ag]_H = \int_\Omega \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \overline{g}_{1x_i} + q f_1 \overline{g}_1 \right\} d\mathbf{x} + \int_{\partial\Omega} \sigma f_1 \overline{g}_1 \, ds. \tag{2.9}$$

From (2.8) and (2.9) we find that

$$[Af, g]_H = [f, Ag]_H. \tag{2.10}$$

Therefore A is a symmetric linear operator in H.

Remark 2.2.

(*i*) Since A in H is symmetric, then it has only real eigenvalues

(*ii*) Since A in H is symmetric and its domain; $D(A)$, is densely defined in H, then A is self-adjoint in H.

(*iii*) The density of the domain $D(A)$ in H gives us the completeness of the orthonormal system of eigenfunctions of A.

Lemma 2.3. Let $f(\mathbf{x})$ be a complex-valued function such that $|f(\mathbf{x})| \in C^1(\overline{\Omega})$, then

$$\int_\Omega |f_1(\mathbf{x})|^2 \, d\mathbf{x} \leq 4\mu^2 \int_\Omega |\text{grad } f_1(\mathbf{x})|^2 \, d\mathbf{x} + 2\mu \int_{\partial\Omega} |f_1(\mathbf{x})|^2 \, ds, \tag{2.11}$$

where

$$\mu = \sup\{|x_1| : \mathbf{x} = (x_1, \dots, x_n) \in \Omega\}.$$

Proof. Since $|f_1(\mathbf{x})|$ is a real-valued function and $|f_1(\mathbf{x})| \in C^1(\overline{\Omega})$, then by using Theorem 2 in [13, p.67], we have

$$\int_{\Omega} |f_1(\mathbf{x})|^2 \, d\mathbf{x} \le 4\mu^2 \int_{\Omega} \sum_{i=1}^{n} \left\{ |f_1(\mathbf{x})|_{x_i} \right\}^2 d\mathbf{x} + 2\mu \int_{\partial\Omega} |f_1(\mathbf{x})|^2 \, ds. \tag{2.12}$$

Substituting the well known inequality:

$$\left\{ |f_1(\mathbf{x})|_{x_i} \right\}^2 \le \left| f_{1x_i}(\mathbf{x}) \right|^2, \ \mathbf{x} \in \Omega, \tag{2.13}$$

into (2.12) we arrive at (2.11).

Lemma 2.4. The linear operator A in H is bounded from below.

Proof. Let $f = (f_1, f_2)$ be any element in $D(A)$. We have

$$[Af, f]_H = \int_{\Omega} \left\{ \sum_{i,j=1}^{n} p_{ij} f_{1x_j} \bar{f}_{1x_i} + q|f_1|^2 \right\} d\mathbf{x} + \int_{\partial\Omega} \sigma |f_1|^2 \, ds. \tag{2.14}$$

Making use of the ellipticity condition (1.3), the above formula (2.14) reduces to

$$[Af, f]_H \ge c_o \int_{\Omega} |\text{grad } f_1|^2 \, d\mathbf{x} + \int_{\Omega} q|f_1|^2 \, d\mathbf{x} + \int_{\partial\Omega} \sigma |f_1|^2 \, ds. \tag{2.15}$$

With $\beta = \max(4\mu^2, 2\mu)$, Lemma 2.3. gives the inequality

$$\frac{1}{\beta} \int_{\Omega} |f_1|^2 \, d\mathbf{x} - \int_{\partial\Omega} |f_1|^2 \, ds \le \int_{\Omega} |\text{grad } f_1|^2 d\mathbf{x}. \tag{2.16}$$

Substituting (2.16) into (2.15) we have

$$[Af, f]_H \ge \int_{\Omega} \left\{ \frac{c_o + \beta q(\mathbf{x})}{\beta r(\mathbf{x})} \right\} r(\mathbf{x}) |f_1(\mathbf{x})|^2 \, d\mathbf{x} + \int_{\partial\Omega} \{\sigma(\mathbf{x}) - c_o\} |f_1(\mathbf{x})|^2 \, ds. \tag{2.17}$$

Define a real number γ as follows:

$$\gamma = \min \left\{ \inf_{\mathbf{x}\in\Omega} \left[\frac{c_o + \beta q(\mathbf{x})}{\beta r(\mathbf{x})} \right], \ \inf_{\mathbf{x}\in\partial\Omega} [\sigma(\mathbf{x}) - c_o] \right\}. \tag{2.18}$$

Then (2.17) can be written in the form

$$[Af, f]_H \ge \gamma \|f\|_H^2. \tag{2.19}$$

This proves that the linear operator A in H is bounded from below.

Remark 2.3.

(*i*) If $q(\mathbf{x}) \ge 0 \ \forall \mathbf{x} \in \overline{\Omega}$ and if $\sigma(\mathbf{x}) > c_o \forall \mathbf{x} \in \partial\Omega$, then $\gamma > 0$. Under these assumptions we deduce that A in H is strictly positive and consequently $\lambda = 0$ is not an eigenvalue of A in H.

(*ii*) Since A in H is symmetric and bounded from below, then for every eigenvalue λ of A in H, $\lambda \ge \gamma$.

Lemma 2.5. The linear operator A in H is unbounded from above.

Proof. Let $\Phi(\mathbf{x})$ be a test function with compact support on $\overline{\Omega}$. We define a sequence this test function by

$$\Phi_m(\mathbf{x}) = \Phi(m\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}, \quad m = 1,2,\ldots. \tag{2.20}$$

On using the same argument of Lemma 2.4, we find that

$$[A\Phi_m,\Phi_m]_H \geq \int_\Omega \left\{\frac{m^2}{\beta} + q(\mathbf{x})\right\}|\Phi_m(\mathbf{x})|^2\,d\mathbf{x}, \tag{2.21}$$

Taking the limit as $m \to \infty$ in (2.21), we obtain

$$\lim_{m\to\infty} [A\Phi_m,\Phi_m]_H = \infty. \tag{2.22}$$

This proves that A in H is unbounded from above.

3. THE RESOLVENT OPERATOR AND THE EXPANSION THEOREM

Under the assumptions that $q(\mathbf{x}) \geq 0$ $\forall\mathbf{x}\in\overline{\Omega}$ and $\sigma(\mathbf{x}) > c_o\forall\mathbf{x}\in\partial\Omega$, we have shown that $\lambda = 0$ is not an eigenvalue of A in H. Then the inverse operator A^{-1} of A exists in H. To study the operator A^{-1} it is convenient to give an explicit formula for it in terms of the Green's function for problem (1.1) – (1.2) with $q(\mathbf{x}) \geq 0$.

Here it is difficult to characterize $D(A^{-1}) = R(A)$, the range of A exactly. In any case, it is not true that

$$D(A^{-1}) = \left\{\left(f|_\Omega, f|_{\partial\Omega}\right) \in H: f\in C^\circ(\overline{\Omega})\right\};$$

because for such an f we cannot in general find $u = (u_1,u_2)\in D(A)$ with $Au = f$. Hence, with reference to [13, Section 4.4] if $G_o(\mathbf{x}, \mathbf{y})$ denotes the Green's function for problem (1.1) – (1.2) with $q(\mathbf{x}) = 0$ and if $G(\mathbf{x}, \mathbf{y})$ denotes the Green's function for the same problem with $q(\mathbf{x}) > 0$, then we have define A^{-1} as follows:

$$D(A^{-1}) = \left\{\left(f|_\Omega, f|_{\partial\Omega}\right) \in H: f\in C^1(\overline{\Omega})\cap C^2(\Omega)\right\}; \tag{3.1}$$

$$A^{-1}: D(A^{-1}) \to H;$$

$$A^{-1} f = \left(\int_\Omega G(\mathbf{x},\mathbf{y})\ f_1(\mathbf{y})\ r(\mathbf{y})\ d\mathbf{y},\ \int_{\partial\Omega} G_o(\mathbf{x},\mathbf{y})\ f_2(\mathbf{y})\ ds\right); \tag{3.2}$$

for each $f = (f_1, f_2) \in D(A^{-1})$; where

$$G(\mathbf{x},\mathbf{y}) = G_o(\mathbf{x},\mathbf{y}) - \int_\Omega G_o(\mathbf{x},\mathbf{z})\ G(\mathbf{z},\mathbf{y})\ q(\mathbf{z})\ d\mathbf{z}. \tag{3.3}$$

Remark 3.1.

(i) Applying a similar argument of Lemma 2.1., we can show that $D(A^{-1})$ is dense in H.

(ii) A^{-1} is a linear operator in H.

Remark 3.2. The Green's function $G_0(\mathbf{x}, \mathbf{y})$ for fixed $\mathbf{x} \in \overline{\Omega}$ is a fundamental solution of $\mathbf{y}$ with respect to Ω (see [13, Section 4.3]):

$$G_o(\mathbf{x},\mathbf{y}) = S(\mathbf{x},\mathbf{y}) + K(\mathbf{x},\mathbf{y}). \tag{3.4}$$

where $S(\mathbf{x}, \mathbf{y})$ is a singularity function defined as follows:

$$S(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{1}{(n-2)\omega_n}|\mathbf{x}-\mathbf{y}|^{2-n} & \text{for } n > 2, \\ -\dfrac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| & \text{for } n = 2. \end{cases} \tag{3.5}$$

which is the solution of the equation $\Delta_n u = 0$ for $\mathbf{x} \neq \mathbf{y}$, where ω_n denotes the surface of the unit ball in R^n, while $K(\mathbf{x}, \mathbf{y})$ is a regular function satisfying the following:

$$K(\mathbf{x},\mathbf{y}) \in C^1(\overline{\Omega}) \cap C^2(\Omega);$$

$$\Delta_n K(\mathbf{x},\mathbf{y}) = 0 \text{ in } \Omega;$$

$$K_\nu(\mathbf{x},\mathbf{y}) + \sigma(\mathbf{y})K(\mathbf{x},\mathbf{y}) = -\{S_\nu(\mathbf{x},\mathbf{y}) + \sigma(\mathbf{y})S(\mathbf{x},\mathbf{y})\} \quad \text{on} \quad \partial\Omega.$$

Note that $G_0(\mathbf{x}, \mathbf{y})$ and also $G(\mathbf{x}, \mathbf{y})$ will, by Section 4.4. in [13], satisfy estimates similar to (3.5); that is,

$$|G(\mathbf{x},\mathbf{y})| \leq \begin{cases} c_1|\mathbf{x}-\mathbf{y}|^{2-n} + c_2 & \text{for } n > 2, \\ c_1|\log|\mathbf{x}-\mathbf{y}|| + c_2 & \text{for } n = 2, \end{cases} \tag{3.6}$$

where c_1 and c_2 can be determined. We can get (3.6) by using the maximum principle. See, for example, G. Hellwig [14].

Definition 3.1. We define linear operators B_1 and B_2 as follows:

(*i*) $D(B_1) = \{u \in L^2(\Omega; r) : u \in C^1(\overline{\Omega}) \cap C^2(\Omega)\};$

$$B_1 u = \int_\Omega G(\mathbf{x},\mathbf{y}) \; u(\mathbf{y}) \; r(\mathbf{y}) \; d\mathbf{y};$$

for each $u \in D(B_1)$.

(*ii*) $D(B_2) = \{u \in L^2(\partial\Omega) : u \in C^1(\overline{\Omega})\};$

$$B_2 u = \int_{\partial\Omega} G_o(\mathbf{x},\mathbf{y}) \; u(\mathbf{y}) \; ds;$$

for each $u \in D(B_2)$.

Remark 3.3.

(*i*) With reference to [13, Section 7.4], we conclude that the linear operators B_1 and B_2 are compact in $L^2(\Omega; r)$ and $L^2(\partial\Omega)$ respectively. Consequently, formula (3.2) shows that A^{-1} is a compact linear operator in H.

(*ii*) Since A in H is symmetric, then A^{-1} in H is also symmetric.

(*iii*) Since $D(A^{-1}) \neq H$, then A^{-1} is an essentially self-adjoint operator.

The results of our investigations are summarized in the following expansion theorem:

Theorem 3.1. The spectrum of A consists of an unbounded sequence of real eigenvalues of finite multiplicity without accumulation point in $(-\infty,\infty)$. Denoting them by

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \text{ with } \lim_{k\to\infty} \lambda_k = \infty,$$

and the corresponding eigenfunctions by $\Phi_1, \Phi_2, \Phi_3, \ldots$, we have $\{\Phi_k\}_{k=1}^{\infty}$ forms a complete fundamental system in H and for every $f \in H$ we have the expansion formula

$$f = \sum_{k=1}^{\infty} [f, \Phi_k]_H \Phi_k; \tag{3.7}$$

in the sense of strong convergence in H.

The above theorem has some corollaries for particular choices of the function $f \in H$.

Corollary 3.1. If $f = (f_1, 0) \in H, f_1 \in L^2(\Omega; r)$ then

$$f_1 = \sum_{k=1}^{\infty} \left(\int_{\Omega} r(\mathbf{x}) f_1(\mathbf{x}) \Phi_{k1}(\mathbf{x}) \ \mathrm{d}\mathbf{x} \right) \Phi_{k1}(\mathbf{x}),$$

and

$$0 = \sum_{k=1}^{\infty} \left(\int_{\Omega} r(\mathbf{x}) f_1(\mathbf{x}) \Phi_{k1}(\mathbf{x}) \ \mathrm{d}\mathbf{x} \right) \Phi_{k2}(\mathbf{x}),$$

where $\Phi_k = (\Phi_{k1}, \Phi_{k2})$ are eigenfunctions of Theorem 3.1.

Corollary 3.2. If $f = (0, f_2) \in H, f_2 \in L^2(\partial\Omega)$, then

$$0 = \sum_{k=1}^{\infty} \left(\int_{\partial\Omega} f_2(\mathbf{x}) \Phi_{k2}(\mathbf{x}) \ \mathrm{d}s \right) \Phi_{k1}(\mathbf{x}),$$

and

$$f_2 = \sum_{k=1}^{\infty} \left(\int_{\partial\Omega} f_2(\mathbf{x}) \Phi_{k2}(\mathbf{x}) \ \mathrm{d}s \right) \Phi_{k2}(\mathbf{x}).$$

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