AN ERROR ANALYSIS FOR THE SECANT METHOD UNDER GENERALIZED ZABREJKO–NGUEN-TYPE ASSUMPTIONS

Ioannis K. Argyros

Cameron University
Department of Mathematics
Lawton, OK 73505, U.S.A.

الخلاصة :

استعملت في هذا البحث طريقة « زابرجكو - نوين » المُعَمَّمة للحصول على تحليل الخطأ بطريقة القاطع في فضاء بناخ .

ABSTRACT

An error analysis for the secant method in Banach spaces is provided under generalized Zabrejko–Nguen Assumptions.

Keywords and Phrases: Secant method, Banach space.

AMS (MOS) Subject Classifications: 47H17, 65H10, 65J15.

AN ERROR ANALYSIS FOR THE SECANT METHOD UNDER GENERALIZED ZABREJKO–NGUEN-TYPE ASSUMPTIONS

INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) + G(x) = 0, \tag{1}$$

where F, G are nonlinear operators defined on some convex subset D of a Banach space E_1 with values in a Banach space E_2.

Sufficient conditions for the convergence of the secant method

$$x_{n+1} = x_n - \delta F(x_{n-1}, x_n)^{-1}(F(x_n) + G(x_n)), x_{-1},\ \ x_0 \in D,\ \ n \geq 0 \tag{2}$$

have been given by many authors under various assumptions (see, *e.g.* [1–14] and the references there). Here the divided differences $\delta F(x_{n-1}, x_n) \in L(E_1, E_2)$ for all $n \geq 0$.

We assume that $\delta F(x_{-1}, x_0)^{-1}$ exists for $x_{-1} \neq x_0$, and

$$\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x + h_1, y + h_2) - \delta F(x, y))\| \leq D_1(t_1 + \|h_1\|, t_1) + D_2(t_2 + \|h_2\|, t_2), \tag{3}$$

and

$$\|\delta F(x_{-1}, x_0)^{-1}(G(z + h_3) - G(x))\| \leq c(t_3 + \|h_3\|) - c(t_3) \tag{4}$$

for all $x \in U(x_0, t_1) = \{x \in E_1 \mid \|x - x_0\| \leq t_1\}$, $y \in U(x_0, t_2)$, $z \in U(x_0, t_3)$, $\|h_1\| \leq R - t_1$,

$\|h_2\| \leq R - t_2$, and $\|h_3\| \leq R - t_3$, for some fixed $R > 0$.

D_1, D_2 are nonnegative and continuous functions of two variables such that if one of the variables is fixed, then D_1, D_2 are nondecreasing functions of the other on the interval $[0, R]$, with $D_1(0,0) = D_2(0,0) = 0$. The function c is nonnegative and nondecreasing on $[0, R]$, with $c(0) = 0$.

Using the majorant method and the above conditions we will provide an error analysis for the secant method. Our estimates on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$, generalize earlier ones [2–14], when $G = 0$ on D (or not). We also show how to choose the functions D_1, D_2, and c.

CONVERGENCE ANALYSIS

We will need to introduce the constants

$$r_{-1} = 0,\ \ r_0 = \|x_{-1} - x_0\| > 0,\ \ r_1 = r_0 + \|x_1 - x_0\| > 0, \tag{5}$$

$$a = 1 - [D_1(R,0) + D_2(R,0) + D_1(r_0, 0)], \tag{6}$$

the sequences for all $n \geq 0$

$$r_{n+2} = r_{n+1} + \frac{1}{a_{n+1}}\left\{\int_{r_n}^{r_{n+1}} [D_1(t, r_{n-1}) + D_2(t, r_n)]\mathrm{d}t + c(r_{n+1}) - c(r_n)\right\}, \tag{7}$$

$$a_{n+1} = 1 - [D_1(r_n, 0) + D_2(r_{n+1}, 0) + D_1(r_0, 0)], \tag{8}$$

and the function T on $[r_0, R]$ by

$$T(r) = r_1 + \frac{1}{b(r)}\left\{\int_{r_0}^{r}[D_1(t,r) + D_2(t,r)]\mathrm{d}t + c(r) - c(r_0)\right\}, \tag{9}$$

where

$$b(r) = 1 - [D_1(r, 0) + D_2(r, 0) + D_1(r_0, 0)]. \tag{10}$$

We will now state and prove the main result:

Theorem. Let $F, G\colon D \subseteq E_1 \to E_2$ be nonlinear operators satisfying conditions (3) and (4). Assume:

(*i*) the inverse of the linear operator $\delta F(x_{-1}, x_0)$ exists for $x_{-1}, x_0 \in D$, with $x_{-1} \neq x_0$;

(*ii*) there exists a minimum positive number R_1 such that

$$T(R_1) \leq R_1. \tag{11}$$

(*iii*) there exist R with $R_1 \leq R$ such that the constant a, given by (6) is positive;

Then

(*a*) the scalar sequence $\{r_n\}$ $n \geq -1$ generated by (7) is monotonically increasing and bounded above by its limit, which is number R_1.

(*b*) the sequence $\{x_n\}$ $n \geq -1$ generated by the secant method (2) is well defined, remains in $U(x_0, R_1)$ for all $n \geq -1$, and converges to a solution x^* of equation $F(x) + G(x) = 0$, which is unique in $U(x_0, R)$ (if $G = 0$ on D).

Moreover, the following estimates are true for all $n \geq 0$:

$$\|x_n - x_{n-1}\| \leq r_n - r_{n-1}, \tag{12}$$

$$\|x_n - x^*\| \leq R_1 - r_n, \tag{13}$$

$$\left\|\delta F(x_{-1}, x_0)^{-1}(F(x_{n+1}) + G(x_{n+1}))\right\| \leq v_{n+1} = \int_{r_n}^{r_{n+1}}[D_1(t, r_{n-1}) + D_2(t, r_n)]\mathrm{d}t + c(r_{n+1}) - c(r_n), \tag{14}$$

$$\left\|x_{n+1} - x^*\right\| \leq \frac{\bar{v}_{n+1}}{I_{n+1}}, \text{ (if } G = 0) \tag{15}$$

$$I_{n+1} = 1 - \int_0^1\left[D_1\left((1-t)\left\|x_0 - x^*\right\| + t\left\|x_{n+1} - x_0\right\|, 0\right) + D_2\left((1-t)\left\|x_0 - x^*\right\| + t\left\|x_{n+1} - x_0\right\|, 0\right)\right]\mathrm{d}t - D_1(r_0, 0)$$

$$\left\|x_{n+1} - x_n\right\| \leq \left\|x_n - x^*\right\| + \frac{p_n}{s_n}, \tag{16}$$

where

$$p_n = \int_0^1\left[D_1\left(\left\|x_n - x_{n-1}\right\| + t\left\|x_n - x^*\right\|, \left\|x_{n-1} - x_0\right\|\right) - D_2\left(\left\|x_n - x_0\right\| + t\left\|x_n - x^*\right\| \left\|x_n - x_0\right\|\right)\right]\mathrm{d}t$$

$$+ c\left(\left\|x_n - x_0\right\| + \left\|x_n - x^*\right\|\right) - c\left(\left\|x_n - x_0\right\|\right), \tag{17}$$

$$s_n = 1 - [D_1(\|x_{n-1} - x_0\|, 0) + D_2(\|x^* - x_0\|, 0) + D_1(r_0, 0)]. \tag{18}$$

and

$$\bar{v}_{n+1} = \int_{r_n}^{r_{n+1}} \left[D_1(t, r_{n-1}) + D_2(t, r_n)\right] dt.$$

Proof.

(*a*) By (5), (7), (8) and the monotonicity of the function D_1, D_2, and c, we deduce that the sequence $\{r_n\}$ $n \geq -1$ is monotonically increasing and nonnegative. Using (5), (7), (8), we obtain r_{-1}, r_0, $r_1 \leq R_1$. Let us assume that $r_{k+1} \leq R_1$ for $k = -1, 0, 1, 2, \ldots, n$. Then by (7) and the induction hypothesis

$$r_{k+2} \leq r_{k+1} + \frac{1}{b(R_1)}\left\{\int_{r_k}^{r_{k+1}} \left[D_1(t, r_{k-1}) + D_2(t, r_k) dt + c(r_{n+1}) - c(r_k)\right]\right\}$$

$$\leq r_k + \frac{1}{b(R_1)}\left\{\int_{r_{k-1}}^{r_{k+1}} \left[D_1(t, R_1) + D_2(t, R_1)\right] dt + c(r_{k+1}) - c(r_{k-1})\right\}$$

$$\leq \ldots \leq r_1 + \frac{1}{b(R_1)}\left\{\int_{r_0}^{r_{k+1}} \left[D_1(t, R_1) + D_2(t, R_1)\right] dt + c(R_1) - c(r_0)\right\}$$

$$= T(R_1) \leq R_1 \text{ by } (11).$$

That is the scalar sequence $\{r_n\}$ $n \geq -1$ is bounded above by R_1. By (*ii*) and (*iii*) R_1 is the minimum zero of equation $T(r) - r = 0$ in $(0, R_1]$, and from the above $R_1 = \lim_{n \to \infty} r_n$.

(*b*) By (5) and (11) it follows that $x_{-1}, x_1 \in U(x_0, R_1)$, and (12) is true for $n = 0, 1$. Let us assume that $x_{k+1} \in U(x_0, R_1)$ and (12) is true for $k = -1, 0, 1, \ldots, n$. We first show that $\delta F(x_k, x_{k+1})$ is invertible. In fact, by the induction hypothesis, and (12)

$$\left\|x_{k+1} - x_0\right\| \leq \sum_{j=1}^{k+1} \left\|x_j - x_{j-1}\right\| \leq \sum_{j=1}^{k+1} (r_j - r_{j-1}) = r_{k+1} - r_0 \leq R_1\,, \tag{19}$$

and hence, by (3) and (4)

$$\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_k, x_{k+1}) - \delta F(x_{-1}, x_0))\| \leq \|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_k, x_{k+1}) - \delta F(x_0, x_0))\|$$

$$+ \|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_0, x_0) - \delta F(x_{-1}, x_0))\|$$

$$\leq \|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_0, x_0) - \delta F(x_0 + (x_k - x_0), x_0 + (x_{k+1} - x_0)))\|$$

$$+ \|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_0 + (x_{-1} - x_0), x_0 + (x_0 - x_0)) - \delta F(x_0, x_0))\|$$

$$\leq D_1(\|x_k - x_0\|, 0) + D_2(\|x_{k+1} - x_0\|, 0) + D_1(r_0, 0) + D_2(0, 0)$$

$$\leq D_1(r_k, 0) + D_2(r_{k+1}, 0) + D_1(r_0, 0)$$

$$\leq D_1(R_1, 0) + D_2(R_1, 0) + D_1(r_0, 0) < 1, \tag{20}$$

by the choice of $a > 0$ and hypothesis (*iii*). It now follows from the Banach lemma on invertible operators that

$$\left\|\delta F(x_k, x_{k+1})^{-1} \delta F(x_{-1}, x_0)\right\| \leq \frac{1}{a_{k+1}} \leq \frac{1}{b(R_1)} \leq \frac{1}{a}\,, \tag{21}$$

where a, a_{k+1} are given by (6) and (8) respectively.

Using the estimates

$$\|h_1\| = \|x_k + t(x_{k+1} - x_k) - x_{k-1}\| \le \|x_k - x_{k-1}\| + t\|x_{k+1} - x_k\|, \tag{22}$$

$$\|h_2\| = \|x_k + t(x_{k+1} - x_k) - x_k\| \le t\|x_{k+1} - x_k\|, \tag{23}$$

$$\|h_3\| = \|x_k - x_{k+1}\|, \tag{24}$$

relations (2), (3), (4), (21), (22), (23), and (24) we obtain in turn for all $k \ge 0$

$$\|x_{k+2} - x_{k+1}\| \le \|\delta F(x_k, x_{k+1})^{-1}\delta F(x_{-1}, x_0)\|$$

$$\cdot \|\delta F(x_{-1}, x_0)^{-1} [(F(x_{k+1}) - F(x_k) - \delta F(x_{k-1}, x_k))(x_{k+1} - x_k) + G(x_{k+1}) - G(x_k)]\|$$

$$\le \frac{1}{a_{k+1}}\left[\int_0^1 \left\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x_k + t(x_{k+1} - x_k), x_k + t(x_{k+1} - x_k)) - \delta F(x_{k-1}, x_k))(x_{k+1} - x_k)\mathrm{d}t + \left\|\delta F(x_{-1}, x_0)^{-1}(G(x_{k+1}) - G(x_k))\right\|\right]$$

$$\le \frac{1}{a_{k+1}}\left\{\int_0^1 [D_1(r_k - r_0 + t(r_{k+1} - r_k), r_{k-1})] + D_2(r_k - r_0 + t(r_{k+1} - r_k), r_k - r_0)](r_{k+1} - r_k)\mathrm{d}t + D_3(r_{k+1} - r_0, r_k - r_0)\right\} \tag{25}$$

$$\le \frac{1}{a_{k+1}}\left\{\int_{r_k}^{r_{k+1}} [D_1(t, r_{k-1}) + D_2(t, r_k)]\mathrm{d}t + D_3(r_{k-1}, r_k)\right\}$$

$$\le \frac{1}{a_{k+1}}\left\{\int_{r_k}^{r_{k+1}} [D_1(t, r_{k-1}) + D_2(t, r_k)]\mathrm{d}t + c(r_{k-1}) - c(r_k)\right\}$$

$$= r_{k+2} - r_{k+1}, \tag{26}$$

which shows (12) for all $n \ge 0$, where we used $\delta F(x,x) = F'(x)$ for all $x \in U(x,R)$.

It now follows from (a), (19) and (26) that the secant iteration $\{x_n\}$, $n \ge -1$ is Cauchy, well defined and remains in $U(x_0, R_1)$ for all $n \ge -1$. Hence, it converges to some x^* in such a way that (13) is satisfied. For $n = 0$, (13) gives $x^* \in U(x_0, R_1)$. By taking the limit as $n \to \infty$ in (2) we obtain $F(x^*) + G(x^*) = 0$, which shows that x^* is a solution of Equation (1). To show uniqueness, we assume that there exists another solution y^* of Equation (1) in $U(x_0, R)$.

Then, using (26) for $x_k = x_{k+1} = y^* + t(x^* - y^*)$, we obtain

$$\| \delta F(x_{-1}, x_0)^{-1}\left[\left(\int_0^1 [F'(y^* + t(x^* - y^*)) - \delta F(x_0, x_0)]\mathrm{d}t + (\delta F(x_0, x_0) - \delta F(x_{-1}, x_0)\right)\right]$$

$$\le \int_0^1 \left[D_1((1-t)R + tR_1, 0) + D_2((1-t)R + tR_1, 0)\right]\mathrm{d}t + D_1(r_0, 0) + D_2(0,0) < 1 \tag{27}$$

by the choice of a and hypothesis (ii), where we also used the estimates

$$\|x_0 - y^* - t(x^* - y^*)\| = \|(1-t)(x_0 - y^*) + t(x_0 - x^*)\| \le (1-t)R + tR_1.$$

It now follows from (27) that the linear operator $\int_0^1 F'(y^* + t(x^* - y^*))\mathrm{d}t$ is invertible. By using the approximation (if $G = 0$)

$$F(x^*) - F(y^*) = \int_0^1 F'(y^* + t(x^* - y^*))(x^* - y^*)\mathrm{d}t$$

we obtain $x^* = y^*$, which shows that x^* is the unique solution of Equation (1) in $U(x_0, R)$.

Using the approximation

$$x_{n+1} - x_n = x^* - x_n + \left(\delta F(x_{n-1}, x_n)^{-1}\delta F(x_{-1}, x_0)\right)$$
$$\cdot[\delta F(x_{-1}, x_0)^{-1}((F(x^*) - F(x_n) - \delta F(x_{n-1}, x_n)(x^* - x_n))) + (G(x^*) - G(x_n))],$$

estimates (3), (4), and the triangle inequality, as before, we get

$$\|x_{n+1} - x_n\| \le \|x_n - x^*\| + \frac{p_n}{s_n},$$

which shows (16) for all $n \ge 0$.

Moreover, from the estimate

$$\int_0^1 \|\delta F(x_{-1}, x_0)^{-1}((F'(x^* + t(x_{n+1} - x^*)) - \delta F(x_0, x_0) + (\delta F(x_0, x_0) - \delta F(x_1, x_0)))\|\,\mathrm{d}t$$
$$\le \int_0^1 \left[D_1\left((1-t)\|x_0 - x^*\| + t\|x_{n+1} - x_0\|, 0\right) + D_2\left((1-t)\|x_0 - x^*\| + t\|x_{n+1} - x_0\|, 0\right)\right]\mathrm{d}t + D_1(r_0, 0)$$
$$\le \int_0^1 \left[D_1((1-t)R_1 + tR_1, 0) + D_2((1-t)R_1 + tR_1, 0)\right]\mathrm{d}t + D_1(r_0, 0) \tag{28}$$

since $a > 0$.

It now follows from (28) that the linear operators $\int_0^1 F'(x^* + t(x_{n+1} - x^*))\mathrm{d}t$ is invertible, and

$$\left\|\left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))\mathrm{d}t\right]^{-1}\delta F(x_{-1}, x_0)\right\| \le \frac{1}{I_{n+1}} \le \frac{1}{a}, \tag{29}$$

Furthermore, using the approximation (if $G = 0$)

$$F(x_{n+1}) - F(x^*) = \left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))\mathrm{d}t\right](x_{n+1} - x^*),$$

estimates (21) and (29), we obtain

$$\|x_{n+1} - x^*\| \le \left\|\left[\int_0^1 F'(x^* + t(x_{n+1} - x^*))\mathrm{d}t\right]^{-1}\delta F(x_{-1}, x_0)\right\|$$
$$\cdot\|\delta F(x_{-1}, x_0)^{-1}F(x_{n+1})\| \le \frac{v_{n+1}}{I_{n+1}} \le \frac{v_{n+1}}{a}$$

where v_{n+1} is given by (14) for all $n \geq 0$.

That completes the proof of the theorem.

Remarks

(a) Let us assume that the following stronger conditions are satisfied instead of (3) and (4)

$$\|(\delta F x_{-1}, x_0)^{-1}(\delta F(x,y) - \delta F(z,z))\| \leq q_1(r)\|x-z\| + q_2(r)\|y-z\| \tag{30}$$

and

$$\|\delta F(x_{-1}, x_0)^{-1}(G(x)-G(y))\| \leq q_3(r)\|x-y\| \tag{31}$$

for all $x, y, z \in U(x_0, r) \subseteq U(x_0, R) \subseteq D$. The functions q_1, q_2, and q_3 are nondecreasing on the interval $[0, R]$.

Then we can show

$$\|\delta F(x_{-1}, x_0)^{-1}(\delta F(x+h_1, y+h_2) - \delta F(x,y))\|$$
$$\leq (w_1(t_1 + \|h_1\|) - w_1(t_1)) + (w_2(t_2 + \|h_2\|) - w_2(t_2)) \tag{32}$$

and

$$\|\delta F(x_{-1}, x_0)^{-1}(G(z+h_3) - G(z))\| \leq w_3(t_3 + \|h_3\|) \tag{33}$$

for all $x \in U(x_0, t_1)$, $y \in U(x_0, t_2)$, $z \in U(x_0, t_3)$,

$\|h_1\| \leq R - t_1$, $\|h_2\| \leq R - t_2$, and $\|h_3\| \leq R - t_3$,

with

$$w_1(r) = \int_0^r q_1(t)\mathrm{d}t, \quad w_2(r) = \int_0^r q_2(t)\mathrm{d}t, \quad \text{and} \quad w_3(r) = \int_0^r q_3(t)\mathrm{d}t. \tag{34}$$

Proof. We will only show (32), since (33) can then easily follow. Set

$g = \delta F(x_{-1}, x_0)^{-1}\delta F$, let $x \in U(x_0, t_1)$, $y \in U(x_0, t_2)$, $m \in N$, then from (30) we obtain

$$\|g(x+h_1, y+h_2) - g(x,y)\| \leq \sum_{j=1}^{m} \|g(x + m^{-1}jh_1, y + m^{-1}jh_2) - g(x + m^{-1}(j-1)h_1, y + m^{-1}(j-1)h_2)\|$$
$$\leq \sum_{j=1}^{m} q_1(t_1 + m^{-1}j\|h_1\|)m^{-1}\|h_1\| + \sum_{j=1}^{m} q_2(t_2 + m^{-1}j\|h_2\|)m^{-1}\|h_2\|$$
$$\leq \int_{t_1}^{t_1+\|h_1\|} q_1(t)\mathrm{d}t + \int_{t_2}^{t_2+\|h_2\|} q_2(t)\mathrm{d}t \quad \text{as} \quad m \to \infty,$$

by the monotonicity of q_1, q_2, and the definition of the Riemann integral. That completes the proof for (32) and (33).

Several authors have studied the convergence of the secant method using conditions (30) and (31) for $q_1(r) = k_1$, $q_2(r) = k_2$, and $q_3(r) = k_3$ on $[0, R]$ $G = 0$, (or not) for some positive constants k_1 and k_2 (see, *e.g.* [3], [5–7]). If we now choose

$$D_1\left(t+\|h_1\|,t_1\right)=\int_{t_1}^{t_1+\|h_1\|}q_1(t)\mathrm{d}t,$$

$$D_2\left(t+\|h_2\|,t_2\right)=\int_{t_2}^{t_2+\|h_2\|}q_2(t)\mathrm{d}t,$$

and

$$C\left(t_3+\|h_3\|\right)-C(t_3)=\int_{t_3}^{t_3+\|h_3\|}q_3(t)\mathrm{d}t,$$

then conditions (3) and (4) will be satisfied.

Moreover,

$$D_1(t_1+\|h_1\|,\,t_1)\le k_1\|x-z\|,$$

$$D_2(t_2+\|h_2\|,\,t_2)\le k_2\|y-z\|,$$

$$C(t_3+\|h_3\|)-C(t_3)\le k_3\|x-y\|,$$

which suggest that our estimates on the distances $\|x_{n+1}-x_n\|$ and $\|x_n-x^*\|$ will be smaller than the corresponding ones in [2–14], ((For $G=0$, or not) and the references there).

(*b*) Furthermore, if we choose D_1, D_2 and D_3 as in the remark, then

$$D_1\left(t+\|h_1\|,t_1\right)\le\int_{t_1}^{t_1+\|h_1\|}q_1(t)\mathrm{d}t,$$

$$D_2\left(t+\|h_2\|,t_2\right)\le\int_{t_2}^{t_2+\|h_2\|}q_2(t)\mathrm{d}t,$$

and

$$D_3\left(t+\|h_3\|,t_3\right)\le\int_{t_3}^{t_3+\|h_3\|}q_3(t)\mathrm{d}t,$$

then our estimates on the distances $\|x_{n+1}-x_n\|$ and $\|x_n-x^*\|$ will be smaller than the corresponding ones in [1–14], ((for $G=0$, or not), and the references there).

(*c*) Estimates (15) and (16) can sometimes be solved explicitly for $\|x_{n+1}-x^*\|$ and $\|x_n-x^*\|$ respectively, when for example conditions (30) and (31) are true instead of (3) and (4) for $q_1(r)=k_1$, $q_2(r)=k_2$, and $q_3(r)=k_3$ on $[0,R]$. Estimate (15) will then provide an upper bound on $\|x_{n+1}-x^*\|$, whereas (16) will provide a lower bound on the estimate $\|x_n-x^*\|$ for all $n\ge 0$.

(*d*) Finally, note that by (19) and (25), it can easily be seen that a stronger result can immediately follow if by making the appropriate changes the estimate $\|x_k-x_0\|\le r_k-r_0$ is used instead of $\|x_k-x_0\|\le r_k$ for all $k\ge 0$ in the proof of the theorem.

(*e*) The uniqueness of the solution x^* of Equation (1) in $U(x_0,R)$ was established only when $G=0$ on D. We assume that $G\ne 0$ on D, and define the iterations

$$y_{n+1}=y_n-\delta F(x_{-1},x_0)^{-1}(F(y_n)+G(y_n)),\ \text{for any}\ y_0\in U(x_0,R_1)\ n\ge 0$$
$$z_{n+1}=z_n-\delta F(x_{-1},x_0)^{-1}(F(z_n)+G(z_n)),\ z_0=x_0,\ \ z_{-1}=x_{-1}\ \ n\ge 0$$

$$s_{n+1} = s_n + \int_{s_{n-1}}^{s_n} \left[D_1(t,s_0) + D_2(t,0)\right]dt + c(s_n) - c(s_{n-1}) \quad n \geq 1$$

$$s_{-1} = 0, s_0 = \|y_1 - y_0\|, s_1 = s_0 + \|y_1 - y_0\|$$

$$t_{n+1} = t_n + \int_{t_{n-1}}^{t_n} \left[D_1(t,t_0) + D_2(t,0)\right]dt + c(t_n) - c(t_{n-1}) \quad n \geq 0$$

$$t_{-1} = R, \quad s_0 \leq t_0 < R$$

$$\delta_n = t_n - s_n + \int_{t_{n-1}}^{t_n} \left[D_1(t,t_0) + D_2(t,0)\right]dt + c(s_{n-1}) - c(t_{n-1})$$

$$-\int_{s_{n-1}}^{s_0} \left[D_1(t,s_0) + D_2(t,0)\right]dt - \int_{t_n}^{s_0} \left[D_1(t,s_0) + D_2(t,0)\right]dt \quad n \geq 0$$

and the function

$$T_1(r) = s_1 + \int_{r_0}^{r} \left[D_1(t,s_0) + D_2(t,0)\right]dt + c(r) - c(r_0).$$

Moreover, we assume that in addition to the hypotheses of the above theorem, there exists a minimum positive number R_1^* with $R_1^* \leq R$ such that

$$T_1(R_1^*) \leq R_1^*,$$

and

$$\delta n \geq 0 \qquad n \geq 0.$$

Then as in the theorem above, we can show:

(*i*) the sequence $\{s_n\}$ $n \geq -1$ is monotonically increasing, whereas the sequence $\{t_n\}$ $n \geq -1$ is monotonically decreasing and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = R_1^* \leq R_1 \quad \text{and} \quad T_1(R_1) \leq R_1.$$

(*ii*) the sequence $\{z_n\}$ $n \geq -1$ is well defined, remains in $U(x_0, R_1^*)$ for all $n \geq 0$, and converges to a solution z^* of Equation (1), which is unique in $U(x_0, R)$, with $z^* = x^*$.

Moreover, the following estimates are true:

$$\|z_n - z_{n-1}\| \leq s_n - s_{n-1} \qquad n \geq 0$$

$$\|z_n - x^*\| \leq R_1^* - s_n \qquad n \geq 0$$

and

$$\|z_n - y_n\| \leq t_n - s_n \qquad n \geq 0.$$

The condition on the sequence $\{\delta_n\}$ can be dropped if we define the sequences

$$\bar{s}_{n+1} = \int_0^{\bar{s}_n} \left[D_1(t,\bar{s}_0) + D_2(t,0)\right]\mathrm{d}t + c(\bar{s}_n) + \bar{s}_1, \bar{s}_0 = 0 \quad n \geq 0$$

$$\bar{t}_{n+1} = \int_0^{\bar{t}_n} \left[D_1(t,\bar{t}_0) + D_2(t,0)\right]\mathrm{d}t + c(\bar{t}_n) + \bar{s}_1, \bar{t}_0 = R \quad n \geq 0$$

instead of the sequences $\{s_n\}$ and $\{t_n\}$ respectively. The conclusions (*i*) and (*ii*) will then also hold for the new sequences $\{\bar{s}_n\}$ and $\{\bar{t}_n\}$.

Moreover, the following estimates are true:

$$s_n - s_{n-1} \leq \bar{s}_n - \bar{s}_{n-1}$$

and

$t_n - s_n \leq \bar{t}_n - \bar{s}_n$ for all $n \geq 0$.

REFERENCES

[1] J. Appel, E. Pascale, and P. P. Zabrejko, "On the Application of the Newton–Kantorovich Method to Nonlinear Equations of Uryson Type", *Numer. Funct. Anal. and Optimiz.*, **12**(**3 and 4**) (1991), pp. 271–283.

[2] I. K. Argyros, "On Newton's Method and Nondiscrete Mathematical Induction", *Bull. Austral. Math. Soc.*, **38** (1988), pp. 131–140.

[3] I. K. Argyros, "The Secant Method and Fixed Points of Nonlinear Operators", *Monatshefte für Math.*, **106** (1988), pp. 85–94.

[4] I. K. Argyros and F. Szidarovszky, *The Theory and Applications of Iteration Methods*. Boca Raton, Florida, U.S.A.: CRC Press, Inc., 1993.

[5] M. Balazs and G. Goldner, "On the Method of the Cord and on a Modification of it for the Solution of Nonlinear Operator Equations", *Stud. Cerc. Mat.*, **20** (1968), pp. 981–990.

[6] F. A. Potra, "Sharp Error Bounds for a Class of Newton–Like Methods", *Libertas Mathematica*, **5** (1985), pp. 72–84.

[7] F. A. Potra and V. Ptak, "Sharp Error Bounds for Newton's Process", *Numer. Math.*, **34** (1980), pp. 63–72.

[8] F. A. Potra and V. Ptak, Nondiscrete Induction and Iterative Processes. Boston: Pitman, 1984.

[9] J. W. Schmidt, "Regula-Falsi Verfahren mit Konsistenter Steigung und Majoranten Printzipp", *Periodica Mathematica Hungarica*, **5**(**3**) (1974), pp. 187–193.

[10] T. Yamamoto, "A Method for Finding Sharp Error Bounds for Newton's Method under the Kantorovich Assumptions", *Numer. Math.*, **49** (1986), pp. 203–220.

[11] T. Yamamoto, "Error Bounds for Newton-Like Methods under Kantorovich-Type Assumptions", *Japan. J. Appl. Math.*, **3** (1986), pp. 295–313.

[12] T. Yamamoto and X. Chen, "Ball-Convergence Theorems and Error Estimates for Certain Iterative Methods for Nonlinear Equations", *Japan. J. Appl. Math.*, **7** (1990), pp. 131–143.

[13] P. P. Zabrejko and D. F. Nguen, "The Majorant Method in the Theory of Newton– Kantorovich Approximations and the Ptak Error Estimates", *Numer. Funct. Anal. and Optimiz.*, **9** (1987), pp. 671–684.

[14] A. I. Zincenko, "Some Approximate Methods of Solving Equations with Nondifferentiable Operators (Ukrainian)", *Dopovidi Akad. Nauk. Ukrain, RSR*, (1963), pp. 156–161.

Paper Received 29 September 1993; Revised 8 May 1994; Accepted 3 July 1994.