POLYNOMIAL APPROACH TO H^∞ CONTROL PROBLEM WITH ADDITIONAL CONSTRAINTS

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الخلاصـة :

تـمّ إعادة صياغة مسألة $\mathbf{H}^\infty$ للتحكم الأمثل ذات المعطيات التامة في حالة قيود تكاملية الشكل. وبمساعدة طريقة S الوظيفية أمكن تحويل هذه المسألة الى مسألة قياسية وسطية غير شرطية في $\mathbf{H}^\infty$ – للتحكم الأمثل ذات دالة نوعية غير سالبة. وفي حالة القياس التام فإنه من الضروري فقط استخدام التحويل الطيفي إلى جداءات. إنّ المدخل كثير الحدودية المقترح يستند إلى المماثل كثير الحدودية لمعادلة (ريكاتي) ويسمح باجراء مختلف العمليات على المصفوفات كثيرات الحدوديـة الأصلية فقط أي الواردة في معادلات النموذج وبدون تحويلات مسبقة إلى فضاء الحالة، وكذلك بدون عمليات مختلفة في التحويل الطيفي إلى جداءات. كما توصف خوارزمية تكرارية لحساب المنظم الأمثل بدون استخدام أشعة الحالات، إنما تتضمن المصفوفات كثيرات الحدوديـة الـواردة في معادلات النموذج وحل معادلة (ريكاتي). وتمّ أيضاً حل المسألة في الحالة العامة وذلك عندما لايفترض عدم سلبية الصيغة التربيعية الداخله وفي دالة النوعية وتجدر الإشارة إلى أن النتائـج الحاصلة يمكن استخدامها في فرضية التحكم الذاتي.

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ABSTRACT

The $\mathbf{H}^\infty$ control problem for linear stationary systems was studied in detail in a great number of articles beginning from the work by G. Zames [1]. Nowadays different approaches to its solution were proposed. They include spectral methods [2], state-space solution [3], polynomial approaches [4, 5]. All these methods are based on the main procedure which solves one the following equivalent problems: Nehari problem in the spectral method, J-spectral factorization in polynomial approaches and Riccati equation for the state-space solution.

We show in this article that only standard spectral factorization is needed for to solve the standard problem of minimization of nonnegative cost function in the case of full information. The polynomial approach proposed below is based on a polynomial analogue of the Riccati equation. This approach allows one to fulfil all operations with initial polynomial matrices only, without their preliminary transformations into state-space form or different factorizations. The problem is solved also for the general case in which the quadratic form in the cost function is not assumed to be nonnegative.

In the first part of this paper the conditional problem of $\mathbf{H}^\infty$-minimization is considered. It is reduced to $\mathbf{H}^\infty$ control problems with parameters but without additional restrictions. The method of this reduction is based on so-called S-procedure which was previously used in the absolute stability theory [6].

Keywords: $\mathbf{H}^\infty$ control problem, polynomial approach, nonconvex optimization, Riccati equation.

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1 PLANT EQUATION AND COST FUNCTION

The plant dynamics is described by the equation

$$a(p)y(t) = b(p)u(t) + c(p)v(t),$$

where $p = d/dt$ is the differential operator, $t \in [0, \infty)$, the matrix polynomials $a(z)$, $b(z)$, $c(z)$ have degrees M, $M-1$, $M-1$ and dimensions $n \times n$, $n \times m$, $n \times k$ respectively. Initial values are assumed to be equal to zero. The cost function is defined as

$$\mathcal{F} = \int_0^\infty F(y,u)dt,$$

where F is a given quadratic form in y and u,

$$F(y,u) = y^*Qy + u^*Ru.$$

Without loss of generality matrix Q may be considered to be not degenerate. Indeed, in the opposite case one can exclude unnecessary components in the plant equation and the new problem will have a reduced dimension.

The matrix $a(z)$ is assumed to be $a(z) = z^M I + \cdots + a_M$ so that $a(z)^{-1}b(z) \to 0$ and $a(z)^{-1}c(z) \to 0$ as $z \to \infty$.

2 ADDITIONAL MEASUREMENTS

In applications the plant equation is usually the mathematical model only, so the function $v(t)$ may be considered as the model error. This function contains systems noises as well as additional dynamics of the plant not taken into account in the model. It is natural to suppose that the model error $v(t)$ has some additional properties and to use these properties in the compensator design algorithm. Below these properties are specified as an energy boundedness condition of signals $\xi_i(t)$, $1 \le i \le p$, connected with the function $v(t)$ by equations

$$d_i(p)\xi_i(t) = e_i(p)v(t), \qquad 1 \le i \le p,$$

where d_i and e_i are polynomial matrices of corresponding dimensions and all $\det(d_i(z))$ are Hurwitz polynomials.

The functions $\xi_i(t)$ are additional measurements. The upper bounds of their energies may be computed in applications during preliminary tests or on-line. In the last case this approach may be used for adaptive control design.

3 ADMISSIBLE COMPENSATORS AND PERFORMANCE INDEX

The class of all admissible compensators is described by linear equations:

$$u = W_y(p)y + W_v(p)v,$$

with some transfer functions W_y, W_v for which the closed loop system is stable.

Under the conditions $y(t) = 0$, $u(t) = 0$, $v(t) = 0$ for $t < 0$ it is required to find the minimum of the performance index $\mathcal{F}$ with additional constraints on the functions $v(t)$ and $\xi_i(t)$. The problem is stated as follows: to find the conditional minimum

$$J = \sup_{v \in L^2(0,\infty)} \{ \mathcal{F} \mid \|v\|^2 \le C_0, \ \|\xi_i\|^2 \le C_i, \ 1 \le i \le p \} \to \min,$$

with given constants $C_i > 0$, $0 \le i \le p$. Here $\| \cdot \|$ denotes the norm in the space $L^2(0,\infty)$. The norm in the space $\mathbf{H}^\infty$ will be denoted by

$$\|f(\cdot)\|_\infty = \sup\{\|f(z)\|, Re(z) > 0\}.$$

4 FREQUENCY CONDITION AND RESTRICTIONS ON R AND Q

It is well known that the minimum value of the performance index is equal to $-\infty$ if the following frequency condition does not hold:

$$F(a(i\omega)^{-1}b(i\omega)u, u) \ge 0$$

for any $\omega \in \mathbf{R}$, $u \in \mathbf{C}^m$. We assume that the strict inequality holds for the plant considered. In particular, this implies that the matrix R is positive definite.

The matrix Q is not assumed to be nonnegative definite. This condition makes the difference between the problem considered and the standard $\mathbf{H}^\infty$ control problem [3].

5 THEOREM ON S-PROCEDURE

Let some stabilizing compensator be fixed and γ be some positive number. Then according to the definition of the performance index J the condition $J < \gamma^2$ may be rewritten in the form

$$\forall v \in L^2(0,\infty) \ \text{ if } \ (\|v\|^2 \le C_0, \|\xi_i\|^2 \le C_i, \ 1 \le i \le p) \ \text{ then } \ \mathcal{F} < \gamma^2.$$

The following considerations are based on the so-called S-procedure which was used previously in the absolute stability theory [6]. This procedure allows to reduce a condition

$$\forall x \ \ \{(\mathcal{F}_i(x) \ge 0, \ 1 \le i \le p) \Longrightarrow \mathcal{F}_0(x) \ge 0\} \tag{S1}$$

where $\mathcal{F}_i$ are quadratic forms in x, $0 \le i \le p$, to the condition

$$\exists \tau_1, \ldots, \tau_p \ge 0 \ : \ \forall x \ \mathcal{F}_0(x) \ge \sum_{i=1}^{p} \tau_i \mathcal{F}_i(x). \tag{S2}$$

It is clear that (S2) implies (S1). But in some problems they appear to be equivalent. In such a problem several quadratic inequalities can be replaced by one quadratic inequality that gives a good approach to solve the problem.

Until 1990 the S-procedure was used for $p = 1$, because only in this case conditions (S1) and (S2) were proved to be equivalent. Then Megretsky and Treil [7] extended this result to the case $p > 1$ for some classes of functions which are used often in the control theory. The formulation of their result needs some definitions.

A subspace $L \subseteq \mathbf{L}^2(0,\infty)$ is called time-invariant if for any $x \in L$, $s > 0$ the function x^s, defined by $x^s(t) = 0$ for $t \le s, x^s(t) = x(t-s)$ for $t > s$, belongs to L. Similarly, a functional $\mathcal{F} : L \to \mathbf{R}$ is called time-invariant if $\mathcal{F}(x^s) = \mathcal{F}(x)$ for every $x \in L$, $s > 0$.

Theorem 1 *(on S-procedure, [7]). Let $L \subseteq \mathbf{L}^2(0,\infty)$ be time-invariant subspace, $\mathcal{F}_i : L \to \mathbf{R}\,(i = 0,1,\dots,p)$ be continuous time-invariant quadratic forms. Suppose that there exists $x_* \in L$ such that $\mathcal{F}_i(x_*) > 0$ for $1 \le i \le p$. Then conditions (S1) and (S2) are equivalent.*

This result was extended in [8] on quadratic functions $\mathcal{F}_i$ under the same conditions. It has been used for $\mathbf{H}^\infty$ control design for systems with conic nonlinearities and with separate restrictions on plant disturbance and measurument noise [9].

6 REDUCTION TO THE H^∞ CONTROL PROBLEM

Theorem 1 gives the following result for the problem stated in Section 3: the conditional inequality $J < \gamma^2$ is equivalent to the existence of such numbers $\tau_i \ge 0$, $0 \le i \le p$, that it holds

$$\gamma^2 - \mathcal{F} > \tau_0(C_0 - ||v||^2) + \sum_{i=1}^{p} \tau_i(C_i - ||\xi_i||^2)$$

or

$$\mathcal{F} - \tau_0||v||^2 - \sum_{i=1}^{p} \tau_i||\xi_i||^2 < \gamma^2 - \sum_{i=0}^{p} \tau_i C_i$$

for any function $v \in L^2(0,\infty)$. Since the LHS of the last inequality equals to zero for $v = 0$, we have

$$\gamma^2 > \sum_{i=0}^{p} \tau_i C_i.$$

This inequality determines upper bounds for the numbers τ_i, $0 \le i \le p$. The lower bounds can be defined from the condition

$$\mathcal{F} - \tau_0||v||^2 - \sum_{i=1}^{p} \tau_i||\xi_i||^2 \le 0.$$

Indeed, the LHS is proportional to $||v||^2$. Therefore it can have a finite upper bound only if it is not positive for any function v. This proves the following assertion.

Corollary 1 . *Let an admissible compensator be fixed. Then the performance index is equal to*

$$J = \inf_{\tau_i \ge 0} \left\{ \sum_{i=0}^{p} \tau_i C_i \;\middle|\; \forall v \in L^2(0,\infty)\ \mathcal{F} - \sum_{i=1}^{p} \tau_i||\xi_i||^2 \le \tau_0||v||^2 \right\}.$$

7 H^∞ CONTROL PROBLEM WITH PARAMETERS

Let $\tau_i \ge 0, 1 \le i \le p$, be fixed. The problem of minimization of τ_0 for which it holds

$$\mathcal{F} - \sum_{i=1}^{p} \tau_i||\xi_i||^2 \le \tau_0||v||^2$$

obviously coincides with the $\mathbf{H}^\infty$ control problem.

This auxiliary $\mathbf{H}^\infty$ control problem has a greater dimension compared with initial problem. The state vector in this problem becomes equal to

$$\tilde{y} = \text{column}(y, \xi_1, \ldots, \xi_p).$$

and the new cost function is

$$\tilde{\mathcal{F}} = \mathcal{F} - \sum_{i=1}^{p} \tau_i \|\xi_i\|^2.$$

This quadratic form always changes signs if $\tau_i \not\equiv 0$. The equation of the extended plant takes the form

$$\begin{pmatrix} a(p) & 0 \\ 0 & D(p) \end{pmatrix} \tilde{y}(t) = \begin{pmatrix} b(p) \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} c(p) \\ E(p) \end{pmatrix} v(t)$$

where $D = \text{diag}(d_1, \ldots, d_p)$, $E = \text{column}(e_1, \ldots, e_p)$.

Denote

$$T(\tau_1, \ldots, \tau_p) = \inf\{\, \tau_0 \mid \sup_{v \in BL^2(0,\infty)} \tilde{\mathcal{F}} < \tau_0 \,\},$$

where $BL^2(0, \infty)$ is the unit ball in the space $L^2(0, \infty)$.

Lemma 1 . *The minimum of the performance index J is equal to*

$$J_{\min} = \min_{\tau_i \geq 0} \left\{ C_0 T(\tau_1, \ldots, \tau_p) + \sum_{i=1}^{p} \tau_i C_i \right\}.$$

If this minimum is reached for $\tau_1^0, \ldots, \tau_p^0$ and if the compensator

$$u(t) = W_y(p) y(t) + \sum_{i=1}^{p} W_{\xi i}(p) \xi_i(t) + W_v(p) v(t)$$

is optimal for the auxiliary problem with $\tau_1^0, \ldots, \tau_p^0$ fixed, then this compensator is optimal for the initial control problem too.

This assertion immediately implies from the corollary 1 if the cost function is minimized first by τ_0 and then by $\tau_1, \ldots, \tau_p$.

Thus the conditional minimization problem is reduced to the standard $\mathbf{H}^\infty$ control problem with several parameters and then to the minimization by these parameters. The complexity of computations rises very quickly together with the dimension of a state vector. Therefore it is important to reduce this complexity.

There are several approaches to the numerical solution of $\mathbf{H}^\infty$ control problem stated in terms "input–output". Below a new approach is proposed which does not include J-spectral factorization or solution to the Nehari problem for the standard problem statement (the matrix Q is nonnegative). It contains a usual spectral factorization and linear equations only. If the matrix Q is not nonnegative then the J-spectral factorization procedure remains, but the dimension of a matrix in this operation is equal to the dimension of the initial output vector. In the following Section the standard state-space solution is produced. This solution was presented in

[3] for the problem with nonnegative matrix Q. But the S-procedure leads to the problem in which the matrix Q changes signs. Therefore the solution is generalized for the case of an arbitrary matrix Q.

8 SOLUTION TO THE STATE-SPACE H^∞ CONTROL PROBLEM

Consider the plant equation

$$\dot{x} = Ax + Bu + Cv, \qquad x(0) = 0,$$

where x is the state vector, u is the control, v is the disturbance. It is required to minimize the functional

$$J^0 = \sup_{v \in BL^2(0,\infty)} \int_0^\infty (x^* Qx + \|u\|^2)\, dt$$

in the class of linear compensators of the type

$$u(t) = W_x(p)x(t) + W_v(p)v(t)$$

with rational functions W_x, W_v.

The solution is based on properties of Riccati equations. Let $\gamma > 0$. Define the Hamiltonian matrix

$$H(\gamma) = \begin{pmatrix} A & \gamma^{-2}CC^* - BB^* \\ -Q & -A^* \end{pmatrix}$$

and the Riccati equation connected with $H(\gamma)$:

$$A^*X + XA + X(\gamma^{-2}CC^* - BB^*)X + Q = 0.$$

We need the solution $X = X(\gamma)$ with the additional condition: $A + (\gamma^{-2}CC^* - BB^*)X$ is a stable matrix, all its eigenvalues lie in the LHP. Such a solution is called stabilizing.

It is not difficult to prove that all eigenvalues of $H(\gamma)$ are symmetric with respect to the real axis and to the imaginary axis. If they do not lie on the imaginary axis then the half is situated in the RHP and the half – in the LHP. Denote by

$$X_{12} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

a basis of the invariant subspace corresponding to eigenvalues of $H(\gamma)$ with negative real parts. If the matrix X_1 is degenerate then the Riccati equation is proved to have no stabilizing solutions $X(\gamma)$. If $\det(X_1) \neq 0$ then the solution $X(\gamma)$ exists, is unique and is defined by

$$X(\gamma) = X_2 X_1^{-1}.$$

Indeed, it is easy to verify that

$$H(\gamma)X_{12} = X_{12}(A + (\gamma^{-2}CC^* - BB^*)X)$$

and hence the matrix $A + (\gamma^{-2}CC^* - BB^*)X$ is stable.

Below dom(Ric) denotes the set of all Hamiltonian matrices H for which $\det(i\omega I - H) \neq 0$ for all real ω and $\det(X_1) \neq 0$.

The following classical result is proved in [3] for the problem with the additional condition: $Q \geq 0$.

Theorem 2 *[3]. An admissible compensator such that $J^0 < \gamma^2$ exists iff the following conditions hold: 1) $H(\gamma) \in dom(Ric)$, 2) $X(\gamma) \geq 0$. Under these conditions any admissible compensator which provides $J^0 < \gamma^2$ is defined by the equation*

$$u(t) = -BB^*X(\gamma)x(t) + G(p)(v(t) - \gamma^{-2}CC^*X(\gamma)x(t))$$

where $G(\cdot)$ is an arbitrary rational function in the space $\mathbf{H}^\infty$ with the norm in $\mathbf{H}^\infty$ less than γ.

Consider the general case: the matrix Q may be not nonnegative definite. It can be shown that a similar result is true but the condition 2 must be replaced by the condition

$$X(\gamma) > X(\infty),$$

where $X(\infty)$ is the solution of the Riccati equation for the Hamiltonian matrix $H(\infty)$ in which the term $\gamma^{-2}CC^*$ is absent. The matrix $X(\infty)$ corresponds to the usual LQ optimal control problem. Moreover, if the solution $X(\gamma)$ exists but the condition $X(\gamma) \geq X(\infty)$ fails then the matrix of the closed loop system $A - BB^*X(\gamma)$ has eigenvalues in the RHP.

9 REDUCTION OF THE INPUT–OUTPUT EQUATION TO THE STATE-SPACE FORM

Let the polynomials of the initial plant equation be

$$a(z) = z^M I + \sum_{i=0}^{M-1} a_i z^i, \qquad b(z) = \sum_{i=0}^{M-1} b_i z^i, \qquad c(z) = \sum_{i=0}^{M-1} c_i z^i.$$

Then the plant can be described by the state-space equation

$$\dot{x} = Ax + Bu + Cv, \qquad y = e^* x,$$

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ I & 0 & \dots & 0 & -a_1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & I & -a_{M-1} \end{pmatrix}$$

$$e^* = (0, \dots, 0, I).$$

The dimension of the vector x is N=Mn. Define the extended vector $\hat{x}$ by adding an auxiliary component x_0. Also define the corresponding extended matrices with $(M+1)n$ rows:

$$\hat{x} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{M-1} \\ x_M \end{pmatrix}, \quad \hat{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \\ I \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{M-1} \\ 0 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{M-1} \\ 0 \end{pmatrix}.$$

$$G(z) = \begin{pmatrix} I & zI & \dots & z^M I \\ 0 & I & \dots & z^{M-1} I \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is easy to verify that the vector $\hat{x}$ satisfies the equation

$$\hat{x}(t) = G(p)\hat{a}y(t) - G(p)\hat{b}u(t) - G(p)\hat{c}v(t)$$

and the plant equation takes the form $x_0 = 0$.

For arbitrary quasi-polynomial $p(z) = p_{-k}z^{-k} + \dots + p_l z^l,\ k, l \geq 0$, define the operation $[\cdot]_+$ of taking the polynomial part:

$$[p(z)]_+ = p_0 + \dots + p_l z^l.$$

Using this notation it is easy to prove that for $0 \leq k \leq M$:

$$x_k(t) = [p^{-k}a(p)]_+ y(t) - [p^{-k}b(p)]_+ u(t) - [p^{-k}c(p)]_+ v(t).$$

10 POLYNOMIAL ANALOGUE OF THE RICCATI EQUATION

The solution to the $\mathbf{H}^\infty$ control problem stated in the previous section is based on the Riccati equation

$$A^*X + XA - \mathrm{XBB}^*X + \gamma^{-2}\mathrm{XCC}^*X + \mathrm{eQe}^* = 0. \tag{RX}$$

Suppose that the inverse matrix $Y = X^{-1}$ exists. Then it satisfies the Riccati equation

$$YA^* + AY - BB^* + \gamma^{-2}CC^* + \mathrm{YeQe}^*Y = 0. \tag{RY}$$

Divide the matrix Y into blocks, each of the dimension $N \times n$:

$$Y = (Y_0, \dots, Y_{M-1}).$$

In particular, Y_{M-1}=Ye. Denote this matrix by $\hat{V} = Y_{M-1}$. For any polynomial matrix $f(z)$ of the degree K_1 and of the dimension $K_2 \times K_3$ we shall denote by $\hat{f}$ the matrix of the dimension $(K_1K_2) \times K_3$ and with constant entries which is a block column of the sequential coefficients of $f(z)$ beginning from the constant term. Below the symbol * means transposing and changing the sign of an argument: $f(z)^* = f(-z)^T$.

Define the matrix

$$L(z) = (I, zI, \dots, z^{M-1}I)$$

of the dimension $M \times$N. It is easy to verify that

$$L(z)B = b(z), \qquad L(z)C = c(z), \qquad L(z)A = zL(z) - a(z)e^*.$$

Obviously the matrix polynomial $V(z) = L(z)\hat{V}$ of the dimension $n \times n$ has the degree less than or equal to $M - 1$. Let us multiply the equation (RY) from the left by $L(z)$ and from the right by $L(z)^*$:

$$\begin{aligned} LY(-zL^* - ea^*) + (zL - ae^*)YL^* - bb^* + \gamma^{-2}cc^* + V Q V^* &= \\ = -Va^* - aV^* + VQV^* - bb^* + \gamma^{-2}cc^* &= 0. \end{aligned}$$

Hence the function $V(z)$ satisfies the equation

$$Va^* + aV^* - \mathrm{VQV}^* + bb^* - \gamma^{-2}cc^* = 0, \tag{PRE}$$

which will be called PRE - the polynomial analogue of the Riccati equation.

We shall show in Section 12 that the equation (RY) has a solution iff the equation (PRE) has a solution. The form of the equation (PRE) is very similar to the form of (RY). This is the reason for its name. The dimension of the equation (PRE) is equal to the dimension of the output $y(t)$ and it is less than the dimension of (RY). Only initial polynomials of the plant equation are included in (PRE). Therefore (PRE) can be considered as an analogue to (RY) adapted to the input–output plant equation.

We shall show in Sections 11 and 13 that the conditions of Theorem 2 can be expressed in terms of PRE and its solution $V(z)$. These conditions are: frequency inequality, additional condition on X to be stabilizing, existence of X^{-1}, verification of the closed loop system stability. Using these results the general equation of suboptimal regulator is derived in Section 14 and an iterative procedure for the performance index minimization is described in Section 15.

11 STABILIZING SOLUTION TO PRE

The coefficients of the polynomial matrix $V(z)$ are block components of the matrix $Y = X^{-1}$. Now we give a criterion for the solution X to be stabilizing in terms of $V(z)$.

Lemma 2 . *The solution X is stabilizing iff the matrix*

$$S(z) = a(z) - V(z)Q$$

is antistable, i.e. all roots of $\det(S(z))$ *are situated in the RHP.*

Proof. It is easy to make sure that matrices $A+(\gamma^{-2}CC^* - BB^*)X$ and $-A - X^{-1}\mathrm{eQe}^*$ are similar. Therefore the solution X is stabilizing iff all roots of the polynomial $\det(zI - A\text{-YeQe}^*)$ have negative real parts. Taking into account the property

$$e^*(zI - A)^{-1}Y_{M-1} = a(z)^{-1}V(z)$$

we obtain

$$\det(zI - A - \mathrm{YeQe}^*) = \det(zI - A - Y_{M-1}Qe^*) = \det(a(z) - V(z)Q).$$

This proves the assertion of Lemma 2.

12 COMPUTING THE MATRIX X^{-1}

Assume that the polynomial matrix $V(z)$ has been defined from the PRE. Thus, the last matrix column Y_{M-1} of the matrix Y is defined. We shall show in this section that the rest of columns are linear combinations of the components of Y_{M-1}. Let

$$Y(z) = L(z)Y = (Y_0(z), \ldots, Y_{M-1}(z)).$$

Multiplying (RY) from the left by $L(z)$ gives

$$(Y_0(z),\dots,Y_{M-1}(z))A^* + (zL(z) - a(z)e^*)Y + V(z)Q\hat{V}^* = b(z)B^* - \gamma^{-2}c(z)C^*.$$

For every block column we have the equations:

$$\begin{aligned} -V(z)a_0^* + zY_0(z) - a(z)V_0^* + V(z)QV_0^* &= b(z)b_0^* + \gamma^{-2}c(z)c_0^*. \\ Y_{k-1}(z) - V(z)a_k^* + zY_k(z) - a(z)V_k^* + V(z)QV_k^* &= b(z)b_k^* + \gamma^{-2}c(z)c_k^*, \qquad 1 \le k \le M-1. \end{aligned}$$

The last equation gives a recurrent procedure for calculation of polynomials $Y_{k-1}(z)$ for $k = M-1, M-2, \dots, 1$. When all these polynomials are expressed in terms of $V(z)$ the first equation turns into the PRE. The solution can be written using the operation $[\cdot]_+$:

$$Y_{k-1}(z) = V(z)[z^{-k}a(z)]_+^* + (a(z) - V(z)Q)[z^{-k}V(z)]_+^* + b(z)[z^{-k}b(z)]_+^* - \gamma^{-2}c(z)[z^{-k}c(z)]_+^*,$$

$$1 \le k \le M-1.$$

Substituting formally $k = M$ in this equation we get the identity: $Y_{M-1}(z) = V(z)$ but for $k = 0$ and with additional notation $Y_{-1} = 0$ we get the PRE for $V(z)$.

Thus, the last equation with $0 \le k \le M$ contains PRE for $V(z)$, notation $Y_{M-1}(z) = V(z)$ and the expression for all $Y_k(z), 0 \le k \le M-2$.

Define the operation $\circ$ for arbitrary polynomial matrices $f(z)$ and $g(z)$ of appropriate dimensions:

$$(f \circ g)(z) = [f^T(z^{-1})g(z)]_+ = \hat{f}^T G(z)\hat{g},$$

where $\hat{f}$ and $\hat{g}$ are block columns of coefficients of $f(z)$ and $g(z)$ and the matrix $G(z)$ is defined in Section 9.

Define the polynomial expression of the matrix Y:

$$Y(z) = (Y_0(z),\dots,Y_{M-1}(z)) = L(z)\mathrm{Y}.$$

Then the following assertion is proved.

Theorem 3 . *Let the solution $V(z)$ of the PRE exist. Then the solution Y of (RY) exists and it is defined by the equation*

$$Y(z) = V(z)(zL \circ a)(z)^* + (a(z) - V(z)Q)(zL \circ V)(z)^* + b(z)(zL \circ b)(z)^* - \gamma^{-2}c(z)(zL \circ c)(z)^*.$$

This is the expression of all entries of the matrix Y in terms of the initial matrices $a(z), b(z), c(z), Q$ and the solution $V(z)$ of the PRE.

Corollary 2 . *Consider the extended matrices $\tilde{L}(z) = (I, zI, \dots, z^M I)$ and $\tilde{Y}(z) = (0, Y_0(z), \dots, Y_{M-1}(z))$. Then the definition of $V(z)$, the PRE and the expression for $Y(z)$ can be written as one equation*

$$\tilde{Y}(z) = V(z)(\tilde{L} \circ a)(z)^* + (a(z) - V(z)Q)(\tilde{L} \circ V)(z)^* + b(z)(\tilde{L} \circ b)(z)^* - \gamma^{-2}c(z)(\tilde{L} \circ c)(z)^*.$$

13 FREQUENCY CRITERION FOR EXISTENCE OF SOLUTION TO PRE

Lemma 3 . *Let $\gamma > 0$ and $\det(Q) \neq 0$. Then the following conditions are equivalent:*

1. *The Hamiltonian matrix $H(\gamma)$ has no eigenvalues on the imaginary axis.*
2. *There exists such a solution $V(z)$ to the PRE that the matrix polynomial $a(z) - V(z)Q$ is antistable.*
3. *It holds*

$$\det(aQ^{-1}a^* + bb^* - \gamma^{-2}cc^*) \neq 0 \tag{FC}$$

on the imaginary axis.

Proof. $3 \Rightarrow 2$. We can rewrite the PRE in the form

$$(V - aQ^{-1})Q(V - aQ^{-1})^* = aQ^{-1}a^* + bb^* - \gamma^{-2}cc^*.$$

If $z \to \infty$ then the signature of the RHS of this equation coincides with the signature of Q. Since the matrix in the RHS is not degenerate its signature is constant on the imaginary axis. For the standard problem with positive definite matrix Q it is sufficient to fulfil usual spectral factorization:

$$\Pi^* Q \Pi = aQ^{-1}a^* + bb^* - \gamma^{-2}cc^*$$

with the additional condition: the matrix polynomial $\Pi(z)$ is stable. After that define

$$V(z) = (a(z) - \Pi(z)^*)Q^{-1}.$$

If the matrix Q changes signs then such a factorization exists too [3, 5]. It is the J-spectral factorization of the matrix polynomial and its dimension coincides with the dimension of $y(t)$.

$2 \Rightarrow 1$. It follows from Theorem 3 and Lemma 2.

$1 \Rightarrow 3$. Straightforward calculations give for $|z| = 1$:

$$\begin{aligned}
\det(zI - H(\gamma)) &= \det(zI - A)\det(zI + A^* + eQe^*(zI - A)^{-1}(\gamma^{-2}CC^* - BB^*)) \\
&= \det(Q) \mid \det(a(z)) \mid^2 \det(Q^{-1} + a(z)^{-1}(b(z)b(z)^* - \gamma^{-2}c(z)c(z)^*)(a(z)^{-1})^*) \\
&= \det(a(z)Q^{-1}a(z)^* + b(z)b(z)^* - \gamma^{-2}c(z)c(z)^*).
\end{aligned}$$

This completes the proof of Lemma 3.

It is not difficult to obtain frequency conditions for the general case of an arbitrary matrix Q. If Q is degenerate then special linear transformations can be used to reduce the problem to one considered above.

14 GENERAL FORM OF SUBOPTIMAL COMPENSATOR

According to Theorem 2 the "central" compensator which provides $J < \gamma^2$ has the form

$$u(t) = K^T x(t), \qquad K = -XB,$$

where x is the state vector defined in Section 9.

Define the polynomial matrix $K(z) = zL(z)K$. Using the operation $\circ$ defined in Section 12, one can rewrite the compensator equation as

$$u(t) = (K \circ a)(p)y(t) - (K \circ b)(p)u(t) - (K \circ c)(p)v(t).$$

The matrix K satisfies the equation

$$YK = -B.$$

Multiplying it by $L(z)$ from the left, we have

$$-b(z) = Y(z)K = V(z)(K \circ a)(z)^* + (a(z) - V(z)Q)(K \circ V)(z)^* + b(z)(K \circ b)(z)^* - \gamma^{-2}c(z)(K \circ c)(z)^*.$$

Introduce the notation:

$$\phi_y(z) = (K \circ a)(z), \qquad \phi_u(z) = (K \circ b)(z) + I, \qquad \phi_v(z) = (K \circ c)(z).$$

Then the equation which determines the gain coefficients K and the compensator equation can be written with the same notation:

$$\begin{aligned} V\phi_y^* + b\phi_u^* - \gamma^{-2}c\phi_v^* &= (VQ - a)(K \circ V)^*, \\ \phi_u(p)u(t) &= \phi_y(p)y(t) - \phi_v(p)v(t). \end{aligned}$$

General equation of the suboptimal compensator presented in Theorem 2 contains the term $C^*Xx(t)$. It looks very similar to the term $B^*Xx(t)$ and indeed it can be transformed in the same way as above. This leads to the following assertion.

Theorem 4 . *Let $\gamma > 0$ and (FC) holds. Then any suboptimal admissible compensator with $J < \gamma^2$ if it exists is defined by the equation*

$$\phi_u(p)u(t) = \phi_y(p)y(t) - \phi_v(p)v(t) + G(\Psi(y, u, v)), \tag{RI}$$

where G is an arbitrary rational matrix function such that $\|G\|_\infty < \gamma$,

$$\Psi(y, u, v) = \psi_y(p)y(t) - \psi_u(p)u(t) - \psi_v(p)v(t).$$

The polynomial matrices are defined as

$$\begin{aligned} \phi_y(z) &= (K \circ a)(z), & \phi_u(z) &= (K \circ b)(z) + I, & \phi_v(z) &= (K \circ c)(z), \\ \psi_y(z) &= (\kappa \circ a)(z), & \psi_u(z) &= (\kappa \circ b)(z), & \psi_v(z) &= (\kappa \circ c)(z) + I, \end{aligned}$$

the polynomial matrices $K = K(z)$ and $\kappa = \kappa(z)$ of the degree M are defined by the equations $K(0) = 0, \kappa(0) = 0$ and

$$\begin{aligned} &V\phi_y^* + b\phi_u^* - \gamma^{-2}c\phi_v^* = (VQ - a)(K \circ V)^*, \qquad \text{(K)} \\ &V\psi_y^* + b\psi_u^* - \gamma^{-2}c\psi_v^* = (VQ - a)(\kappa \circ V)^*, \end{aligned}$$

where $V = V(z)$ is the solution of PRE such that the polynomial $\det(a(z) - V(z)Q)$ is antistable.

15 RECURSIVE ALGORITHM FOR SEARCH THE OPTIMAL VALUE OF γ

According to Theorem 2 the optimal value $\gamma_{\min}$ is the infinum of positive numbers γ for which the Hamiltonian matrix $H(\gamma)$ has no eigenvalues on the imaginary axis and there exists a solution $X(\gamma) \geq 0$ to the Riccati equation (RX). If Q is not nonnegative then the condition $X(\gamma) \geq 0$ must be replaced by $X(\gamma) \geq X(\infty)$. Moreover, if the condition $X(\gamma) \geq X(\infty)$ fails then the "central" regulator $u = Kx$ defined in Theorem 2 does not stabilize the system. Combining this property with the conclusion of Theorem 4 we obtain the following assertion.

Lemma 4 . *Let* $\det(Q) \neq 0$. *Then*

1. The optimal value $\gamma^2_{\min}$ *of the performance index* J *is equal to the maximal number* γ^2 *for which either the polynomial*

$$\det(aQ^{-1}a^* + bb^* - \gamma^{-2}cc^*)$$

has roots on the imaginary axis or the equation (K) has no solution.

2. If $\gamma > \gamma_{\min}$ *then the equation (R) give all stabilizing compensators with* $J < \gamma^2$. *If* $\gamma < \gamma_{\min}$ *and* $V(z)$ *exists then the compensator (R) does not stabilize the closed loop system even for* $G = 0$.

Stability condition for $G = 0$ together with the inequality (FC) indicates that $\gamma > \gamma_{\min}$. If one of these conditions fails then $\gamma \leq \gamma_{\min}$. The stability condition can be verified using various criteria, for example, by Hurwitz theorem. The alternative way is to compute the matrices $Y(\gamma)$ and $Y(\infty)$ using Theorem 3 and to verify the condition $Y(\gamma)^{-1} \geq Y(\infty)^{-1}$. The iterative procedure for estimation of $\gamma_{\min}$ can be easily designed on the basis of these properties.

CONCLUSION

The $\mathbf{H}^\infty$ control problem with full information was stated with additional restrictions of integral type on the model error process. Using of the S-procedure allowed to reduce it to the parametric unconditional problems of $\mathbf{H}^\infty$-optimization. In these problems the cost function appears to be not nonnegative and the dimension of the state vector increases together with the numerical complexity.

A new polynomial approach was proposed to solve these problems. It is based on the polynomial analogue of the Riccati equation (PRE). An iterative algorithm is described to find the optimal compensator. This algorithm does not use state vector, it contains initial polynomial matrices and the solution of the PRE. This result may be used in an adaptive control design.

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