A REGIONALLY IMPLICIT METHOD FOR SOLVING LARGE SETS OF SIMULTANEOUS RESERVOIR EQUATIONS

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الخلاصة

تم طرح طريقة قوية وجديدة لحل المجموعات الكبيرة لمعادلات المكمن من نوع فوريير على هيئة فوارق محدودة ، وتعتمد الطريقة نوعاً ما على اختيار عشوائي لمنطقة في حيز ذو بعدين حيث يتم استنتاج حل المتغير المجهول ضمنياً بناء على عدة شروط تفرض على حدود المنطقة ويستعمل أسلوب مكرر لازاحة المنطقة على الحيز الثنائي الابعاد حتى يتم اشباع المقاييس الخاصة بنقطة الالتقاء ، ومع أنه يصعب الوصول الى اثبات دقيق جداً في نقطة الالتقاء الا أنه تم عرض الحالة في بعض النماذج في المعادلات ، ويؤكد استعمال الكمبيوتر بأن الطريقة المذكورة أعلاه تتطلب حيز نخزين صغير نسبياً كما تتضمن امكانية معالجة سلسلة طويلة من المعاملات وآلافاً من المعادلات مع السرعة العالية .

ABSTRACT

A novel and powerful method or solving large sets of reservoir equations of the Fourier type in finite difference form is proposed. The technique is based upon a somewhat arbitrary selection of a region in two-dimensional space in which the solution of the unknown variable is obtained implicity subject to certain conditions imposed upon the boundary of the region. An iterative technique is used to move the region over the entire two-dimensional space until convergence criteria are satisfied. Although a rigorous proof of convergence is difficult, a demonstration of convergence is presented for a sample problem. Computer usage confirms that the method requires relatively little storage space and has the capability to handle a wide range of coefficients and thousands of equations with high speed.

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INTRODUCTION

The numerical simulation of reservoir equations of the Fourier type in two-dimensions frequently requires the solution of a large number of equations. This can require considerable time and expense even on our very large, third generation, computers; and occasionally the problem is so large or expensive that even these computers are deemed indequate. For this reason considerable work has been underway to develop new or improved techniques of solving the large sets of equations involved. Desirable solution methods should be stable so that the user can be assured of convergence even for great difference in adjacent coefficients. The machine solution time should be small and core storage space should be small. The solution technique should be simple, easily programmed and readily checked out.

In 1955 a very useful implicit technique for solving large sets of equations was proposed [1]. This methods, known as the alternating direction implicit method, has been of great utility and is still being used. In 1968 Stone presented a strongly implicit procedure as a solution technique [2]. The Stone method is now referred to as SIP. The latter method is rated superior to ADIP, yet SIP requires considerable knowledge and experience in its application. Both methods require iteration parameters and both methods may fail to converge for some problems.

In the paper by Stone it was pointed out that for large sets of equations iterative methods are normally considered superior to direct solution methods. The latter require more storage and more arithmetic operations and can become quite involved in programming. By contrast, Gauss-Seidel and successive over-relaxation iterative methods require little core storage and are easily programmed. They are normally used with single iterative parameters. Frankel [3] and Young [4] reported on convergence rates of the early iterative methods.

In this paper we provide the foundation to support a new method of solution. The new method presented here, when coupled with an accelerator, runs faster than SIP and requires less storage. The method is easily understood, readily programmed and has never yet failed to converge even for greatly different coefficients.

MATHEMATICAL DEVELOPMENT

The general fluid flow equation is of the same form as dimensional heat flow and is given by

$$\vec{\nabla}\left[K(x,y)\vec{\nabla}T(x,y,t)\right]+Q(x,y)=\rho(x,y)c(x,y)\frac{\partial T(x,y,t)}{\partial t} \tag{1}$$

where K is the thermal conductivity of the material, ρ is its density, c is the specific heat capacity, Q is the source or sink term, and T is the temperature. This heat nomenclature is readily understood and is used in this paper.

Using central differences in space and backward differences in time we can write Equation (1) as

$$\frac{K_{i,j+1/2}}{\Delta x^2}\left(T^{n+1}_{i,j+1}-T^{n+1}_{i,j}\right)-\frac{K_{i,j-1/2}}{\Delta x^2}\left(T^{n+1}_{i,j}-T^{n+1}_{i,j-1}\right)$$
$$+\frac{K_{i+1/2,j}}{\Delta y^2}\left(T^{n+1}_{i+1,j}-T^{n+1}_{i,j}\right)-\frac{K_{i-1/2,j}}{\Delta y^2}\left(T^{n+1}_{i,j}-T^{n+1}_{i-1,j}\right)$$
$$+Q_{i,j}=\rho_{i,j}c_{i,j}\frac{\left(T^{n+1}_{i,j}-T^{n}_{i,j}\right)}{\left(t_{n+1}-t_n\right)} \tag{2}$$

where we have used a rectangular grid defined by

$$x_j=j\Delta x \qquad j=0,1,2,\ldots,(J-1)$$

$$y_i=i\Delta y \qquad i=0,1,2,\ldots,(I-1)$$

and $T^{n+1}_{i,j}=T(x_j,y_i,t_{n+1})$.

Compacting our notation we write (2) as

$$B_{i,j}T^{n+1}_{i,j-1}+D_{i,j}T^{n+1}_{i-1,j}+E'_{i,j}T^{n+1}_{i,j}+F_{i,j}T^{n+1}_{i+1,j}+H_{i,j}T^{n+1}_{i,j+1}$$
$$+Q'_{i,j}=G_{i,j}\frac{\left(T^{n+1}_{i,j}-T^{n}_{i,j}\right)}{\left(t_{n+1}-t_n\right)} \tag{3}$$

where we have used

$$B_{i,j}=\frac{K_{i,j-1/2}}{\Delta x^2} \quad \text{et cetera.}$$

Incorporating the right hand side of Equation (3) into the left hand side we define

$$E_{i,j}=E'_{i,j}-\frac{G_{i,j}}{(t_{n+1}-t_n)}$$

$$Q_{i,j}=Q'_{i,j}+\frac{G_{i,j}T^n_{i,j}}{(t_{n+1}-t_n)}$$

to get

$$B_{i,j}T^{n+1}_{i,j-1}+D_{i,j}T^{n+1}_{i-1,j}+E_{i,j}T^{n+1}_{i,j}+F_{i,j}T^{n+1}_{i+1,j}+H_{i,j}T^{n+1}_{i,j+1}+Q_{i,j}=0 \tag{4}$$

The superscript (n+1) is at this point superfluous, and we discard it with the note that Equation (4) is fully implicit in the unknown T.

Two of the currently popular methods of solving finite difference equations similar to Equation 4 are ADIP [1] and SIP [2]. SIP has been shown to be superior to ADIP but requires the boundary condition $\frac{\partial T}{\partial x}=\frac{\partial T}{\partial y}=0$ in order to generate its algorithm. Both ADIP and SIP are dependent upon iteration parameters to increase the rate of convergence, and the choice of iteration parameters is not always completely reliable, i.e., divergence may sometimes result. Our method requires neither a special choice of boundary conditions nor iteration parameters.

Furthermore, we note that ADIP uses equations which are alternately implicit in the x and y directions. Although this formulation has certain mathematical advantages, it seems intutively more advantageous to construct equations which are regionally implicit.

Consider the thirteen block grid shown in Figure 1. If we write an equation similar to Equation (4) for each of the five unshaded blocks we obtain five equations and thirteen unknowns. We can, however, easily reduce the number of unknowns to five by defining an iteration scheme such that the temperatures in the shaded blocks are known either from initial values or the previous iteration. In block (i,j−1) we write

$$B_{i,j-1}T^k_{i,j-2}+D_{i,j-1}T^k_{i-1,j-1}+E_{i,j-1}T^{k+1}_{i,j-1}+F_{i,j-1}T^k_{i+1,j-1}+H_{i,j-1}T^{k+1}_{i,j}+Q_{i,j-1}=0 \tag{5}$$

where the superscript k denotes the iteration number. In block (i−1,j)

$$B_{i-1,j}T^k_{i-1,j-1}+D_{i-1,j}T^k_{i-2,j}+E_{i-1,j}T^{k+1}_{i-1,j}+F_{i-1,j}T^{k+1}_{i,j}+H_{i-1,j}T^k_{i-1,j+1}+Q_{i-1,j}=0. \tag{6}$$

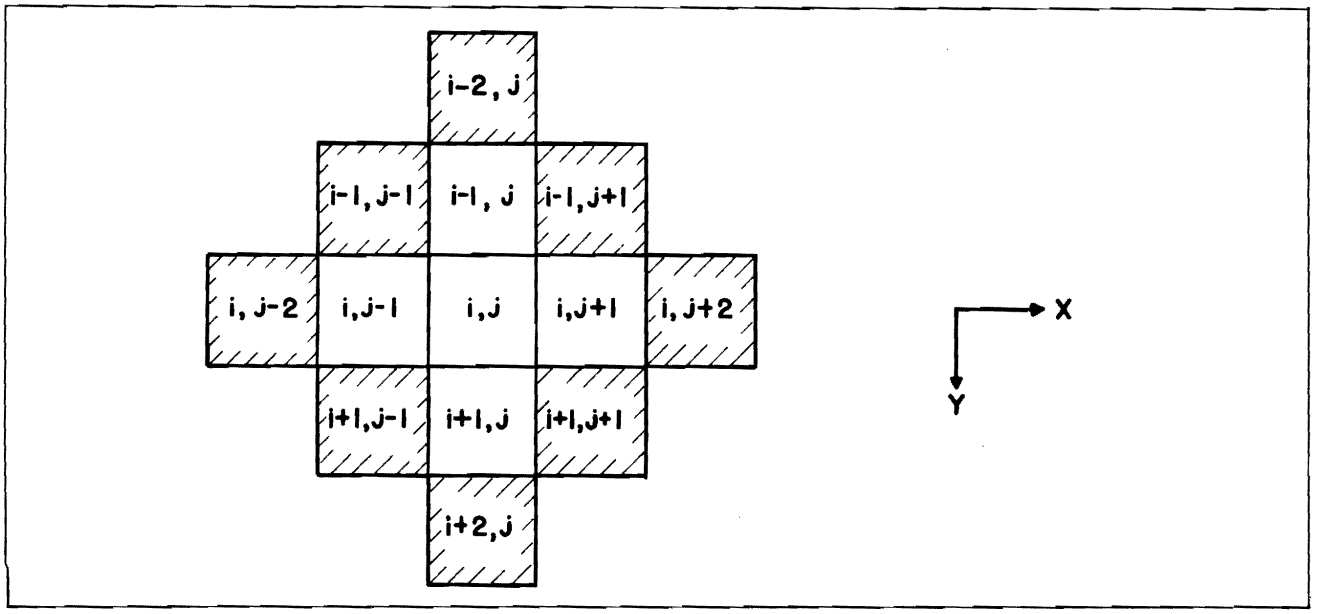

Figure 1. Basic Grid Used in Developing New Method. Hachured Blocks are Those Having Constant Potential During One Computation. Interior Blocks are Solved Implicity for the New Potentials.

In block (i,j)

$$B_{i,j}T^{k+1}_{i,j-1}+D_{i,j}T^{k+1}_{i-1,j}+E_{i,j}T^{k+1}_{i,j}+F_{i,j}T^{k+1}_{i+1,j}+H_{i,j}T^{k+1}_{i,j+1}+Q_{i,j}=0 \tag{7}$$

In block (i+1,j)

$$B_{i+1,j}T^{k}_{i+1,j-1}+D_{i+1,j}T^{k+1}_{i,j}+E_{i+1,j}T^{k+1}_{i+1,j}+F_{i+1,j}T^{k}_{i+2,j}H_{i+1,j}T^{k}_{i+1,j+1}+Q_{i+1,j}=0. \tag{8}$$

In block (i,j+1)

$$B_{i,j+1}T^{k+1}_{i,j}D_{i,j+1}T^{k}_{i-1,j+1}+E_{i,j+1}T^{k+1}_{i,j+1}+F_{i,j+1}T^{k}_{i+1,j+1}+H_{i,j+1}T^{k}_{i,j+2}+Q_{i,j+1}=0. \tag{9}$$

All of the Q terms are known, as are the values of T at the k^{th} iteration. So we define

$$C_{1,1}=-\frac{1}{E_{i,j-1}}\left[Q_{i,j-1}+B_{i,j-1}T^{k}_{i,j-2}+D_{i,j-1}T^{k}_{i-1,j-1}+F_{i,j-1}T^{k}_{i+1,j-1}\right] \tag{10}$$

$$C_{2,1}=\frac{-1}{E_{i-1,j}}\left[Q_{i-1,j}+B_{i-1,j}T^{k}_{i-1,j-1}+D_{i-1,j}T^{k}_{i-2,j}+H_{i-1,j}T^{k}_{i-1,j+1}\right] \tag{11}$$

$$C_{3,1}=-Q_{i,j} \tag{12}$$

$$C_{4,1}=-\frac{1}{E_{i+1,j}}\left[Q_{i+1,j}+B_{i+1,j}T^{k}_{i+1,j-1}+F_{i+1,j}T^{k}_{i+2,j}+H_{i+1,j}T^{k}_{i+1,j+1}\right] \tag{13}$$

$$C_{5,1}=\frac{-1}{E_{i,j+1}}\left[Q_{i,j+1}+D_{i,j+1}T^{k}_{i-1,j+1}+F_{i,j+1}T^{k}_{i+1,j+1}H_{i,j+1}T^{k}_{i,j+2}\right] \tag{14}$$

Then Equations (5) - (9) can be written in matrix notation as in (15).

$$\begin{bmatrix} 1 & 0 & \frac{H_{i,j-1}}{E_{i,j-1}} & 0 & 0 \\ 0 & 1 & \frac{F_{i-1,j}}{E_{i-1,j}} & 0 & 0 \\ B_{i,j} & D_{i,j} & E_{i,j} & F_{i,j} & H_{i,j} \\ 0 & 0 & \frac{D_{i+1,j}}{E_{i+1,j}} & 1 & 0 \\ 0 & 0 & \frac{B_{i,j+1}}{E_{i,j+1}} & 0 & 1 \end{bmatrix} \begin{bmatrix} T^{k+1}_{i,j-1} \\ T^{k+1}_{i-1,j} \\ T^{k+1}_{i,j} \\ T^{k+1}_{i+1,j} \\ T^{k+1}_{i,j+1} \end{bmatrix} = \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ C_{3,1} \\ C_{4,1} \\ C_{5,1} \end{bmatrix} \tag{15}$$

The solution to Equation (15) can be easily obtained by making use of the symmetrical properties of the coefficient matrix on the left hand side of the equation.

Define

$$U=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2=\begin{bmatrix} \frac{H_{i,j-1}}{E_{i,j-1}} \\ \\ \frac{F_{i-1,j}}{E_{i-1,j}} \end{bmatrix} \qquad V_8=\begin{bmatrix} \frac{D_{i+1,j}}{E_{i+1,j}} \\ \\ \frac{B_{i,j+1}}{E_{i,j+1}} \end{bmatrix}$$

$$V_4=\begin{bmatrix} B_{i,j} & D_{i,j} \end{bmatrix}$$

$$V_5=\begin{bmatrix} E_{i,j} \end{bmatrix}$$

$$V_6=\begin{bmatrix} F_{i,j} & H_{i,j} \end{bmatrix}$$

$$c_1=\begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix} \quad c_2=\begin{bmatrix} C_{3,1} \end{bmatrix} \quad c_3=\begin{bmatrix} C_{4,1} \\ C_{5,1} \end{bmatrix}$$

$$p_1=\begin{bmatrix} T^{k+1}_{i,j-1} \\ T^{k+1}_{i-1,j} \end{bmatrix} \quad p_2=\begin{bmatrix} T^{k+1}_{i,j} \end{bmatrix} \quad p_3=\begin{bmatrix} T^{k+1}_{i+1,j} \\ T^{k+1}_{i,j+1} \end{bmatrix}$$

Then Equation (15) become:

$$Up_1+V_2p_2=c_1 \tag{16a}$$

$$V_4p_1+V_5p_2+V_6p_3=c_2 \tag{16b}$$

$$V_8 p_2 + U p_3 = c_3 \quad (16c)$$

Solving Equations (16) for $p_2 = T_{i,j}^{k+1}$ we obtain

$$T_{i,j}^{k+1} = N/D, \quad (17)$$

where

$$N = -(Q_{i,j} + B_{i,j}C_{1,1} + D_{i,j}C_{2,1} + F_{i,j}C_{4,1} + H_{i,j}C_{5,1}) \quad (18)$$

$$D = E_{i,j} - \frac{B_{i,j}H_{i,j-1}}{E_{i,j-1}} - \frac{D_{i,j}F_{i-1,j}}{E_{i-1,j}} - \frac{F_{i,j}D_{i+1,j}}{E_{i+1,j}} - \frac{H_{i,j}B_{i,j+1}}{E_{i,j+1}} \quad (19)$$

and in addition we obtain

$$T_{i,j-1}^{k+1} = C_{1,1} - \frac{H_{i,j-1}T_{i,j}^{k+1}}{E_{i,j-1}} \quad (20)$$

$$T_{i-1,j}^{k+1} = C_{2,1} - \frac{F_{i-1,j}T_{i,j}^{k+1}}{E_{i-1,j}} \quad (21)$$

$$T_{i+1,j}^{k+1} = C_{4,1} - \frac{D_{i+1,j}T_{i,j}^{k+1}}{E_{i+1,j}} \quad (22)$$

$$T_{i,j+1}^{k+1} = C_{5,1} - \frac{B_{i,j+1}T_{i,j}^{k+1}}{E_{i,j+1}} \quad (23)$$

Equations (10) - (14) and Equations (17) - (23) constitute the solution to the temperatures in the regionally implicit grid. The easiest way to program these equations for computer solution is simply to allow two extra blocks on each side of the rectrangular grid so that there results $(I+4)\times(J+4)$ blocks instead of IJ blocks. The equations are then nested inside DO loops over the i and j indices such that the (i,j) block in Figure 1 passes over each block in the problem once per iteration.

If we use only Equation (17) then the temperature in each block is updated once per iteration. However, the inclusion of Equations (20) - (23) means that for each iteration the temperature in corner blocks is updated three times, that in the boundary blocks four times, and that in the interior blocks five times.

In general it is definitely more advantageous to include Equations (20) - (23) than to use Equations (17) - (19) alone. For example, in a 7×9 homogeneous, steady state test case it was found that using only Equations (17) - (19) required 23 iterations for convergence, while the inclusion of Equations (20) - (23) reduced the number of iterations to 14.

DEMONSTRATION OF CONVERGENCE

A rigorous proof of the convergence of the new technique is particularly difficult because of the repeated updating of the temperature in each block per iteration. We can however, demonstrate convergence if we solve only for the temperature in the center block (i,j) of our thirteen block grid. We further assume a homogeneous, steady-state problem in which case $B_{i,j} = D_{i,j} = F_{i,j} = H_{i,j} = 1$ and $E_{i,j} = -4$. Substituting these values into Equations (10) - (14) and then using Equations (17) - (19) we obtain

$$T_{i,j}^{k+1} = \frac{Q_{i,j}}{3} + \frac{1}{12}\Big[Q_{i-1,j} + Q_{i,j-1} + Q_{i,j+1} + T_{i,j-2}^{k+1} + 2T_{i-1,j-1}^{k+1} + 2T_{i+1,j-1}^{k+1} + T_{i-2,j}^{k+1} + T_{i+2,j}^{k} + 2T_{i-1,j+1}^{k} + 2T_{i+1,j+1}^{k} + T_{i,j+2}^{k}\Big] \quad (24)$$

where we move the thirteen block grid such that it passes down the first column of blocks, then down the second, third, etc.

Now suppose we apply Equation (24) to a 5×5 problem with the blocks labeled as in Figure 2. Using matrix notation we can write Equation (24) as

$$\vec{T}^{k+1} = \vec{A}_1\vec{T}^{k+1} + \vec{A}_2\vec{T}^{k} + \vec{A}_3\vec{Q} \quad (25)$$

where the terms $\vec{A}_1\vec{T}^{k+1}$ and $\vec{A}_2\vec{T}^{k}$ are given in Figures 3 and 4, respectively. The term $\vec{A}_3\vec{Q}$ includes the source terms of Equation (24) but its exact form does not concern us because we shall subsequently show that this term cancels out.

If the iterative scheme defined by Equation (25) does indeed converge to the true solution $\vec{T}$, then at convergence we have

$$\vec{T} = \vec{A}_1\vec{T} + \vec{A}_2\vec{T} + \vec{A}_3\vec{Q} \quad (26)$$

Subtracting Equation (25) from Equation (26) we get

$$\vec{T} - \vec{T}^{k+1} = \vec{A}_1(\vec{T} - \vec{T}^{k+1}) + \vec{A}_2(\vec{T} - \vec{T}^{k}). \quad (27)$$

Define an error vector

$$\vec{e}^{k} = \vec{T} - \vec{T}^{k}, \quad (28)$$

then Equation (27) becomes

$$\vec{e}^{k+1} = \vec{A}_1\vec{e}^{k+1} + \vec{A}_2\vec{e}^{k}. \quad (29)$$

or

$$(\vec{U}-\vec{A}_1)\vec{e}^{k+1}=\vec{A}_2\vec{e}^{k} \qquad (30)$$

where U is the unit matrix. Hence

$$\begin{aligned}
\vec{e}^{1} &= (\vec{U}-\vec{A}_1)^{-1}\vec{A}_2\vec{e}^{o} \\
\vec{e}^{2} &= [(\vec{U}-\vec{A}_1)^{-1}\vec{A}_2]^{2}\vec{e}^{o} \\
&\cdot \quad\quad \cdot \\
&\cdot \quad\quad \cdot \\
&\cdot \quad\quad \cdot \\
\vec{e}^{k} &= [(\vec{U}-\vec{A}_1)^{-1}\vec{A}_2]^{k}\vec{e}^{o}.
\end{aligned} \qquad (31)$$

We need only to establish that the spectral radius ρ of $(\vec{U}-\vec{A}_1)^{-1}\vec{A}_2$ is less than unity to demonstrate convergence [5]. This we do by use of a computer and find that $\rho=0.839$. We expect however, that had we been able to include Equations (20)-(23) in this analysis the spectral radius would have been smaller, indicating a more rapid rate of convergence.

1.1	1.2	1.3	1.4	1.5
2.1	2.2	2.3	2.4	2.5
3.1	3.2	3.3	3.4	3.5
4.1	4.2	4.3	4.4	4.5
5.1	5.2	5.3	5.4	5.5

Figure 2. Block Notation for Sample Problem Demonstrating Convergence.

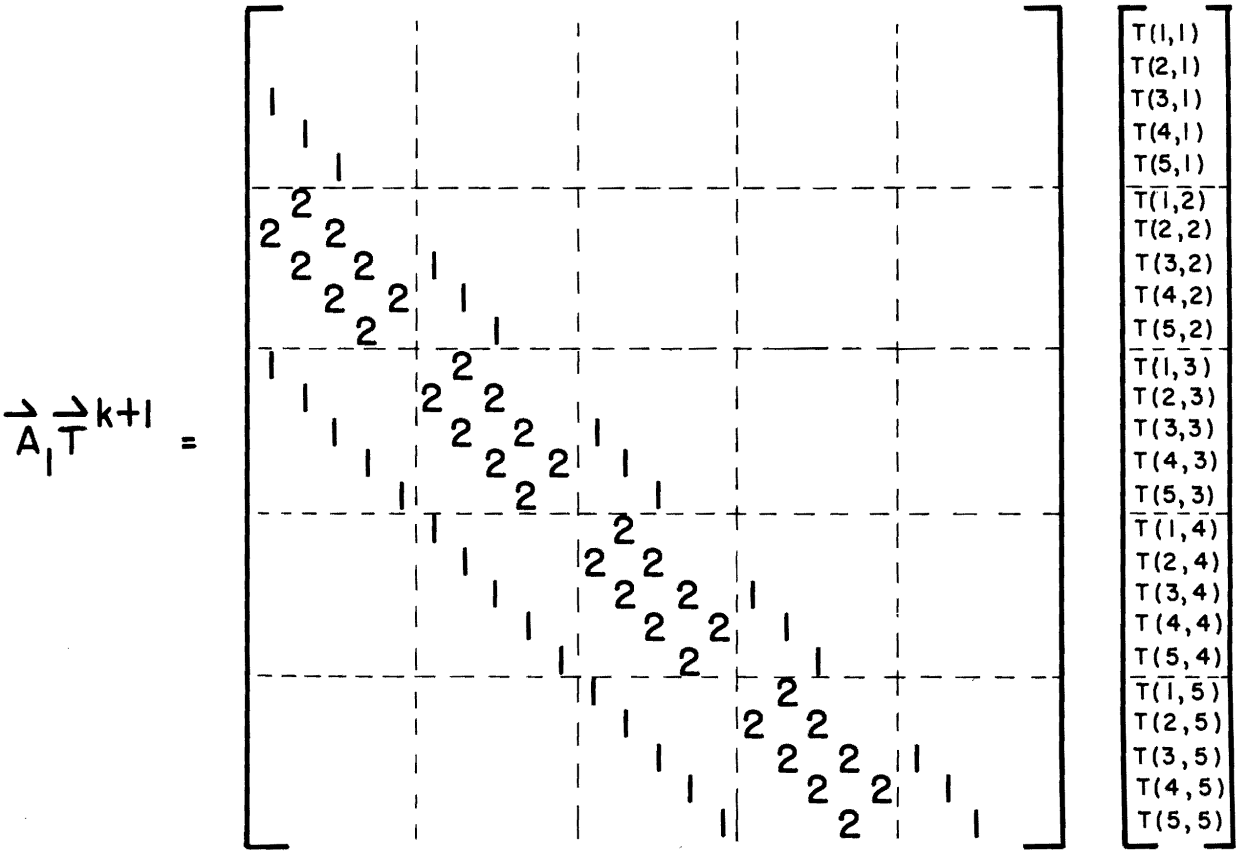

Figure 3. Graphical Display of Matrix Product A_1T Appearing in Equation (25) Showing the lower Triangular Form.

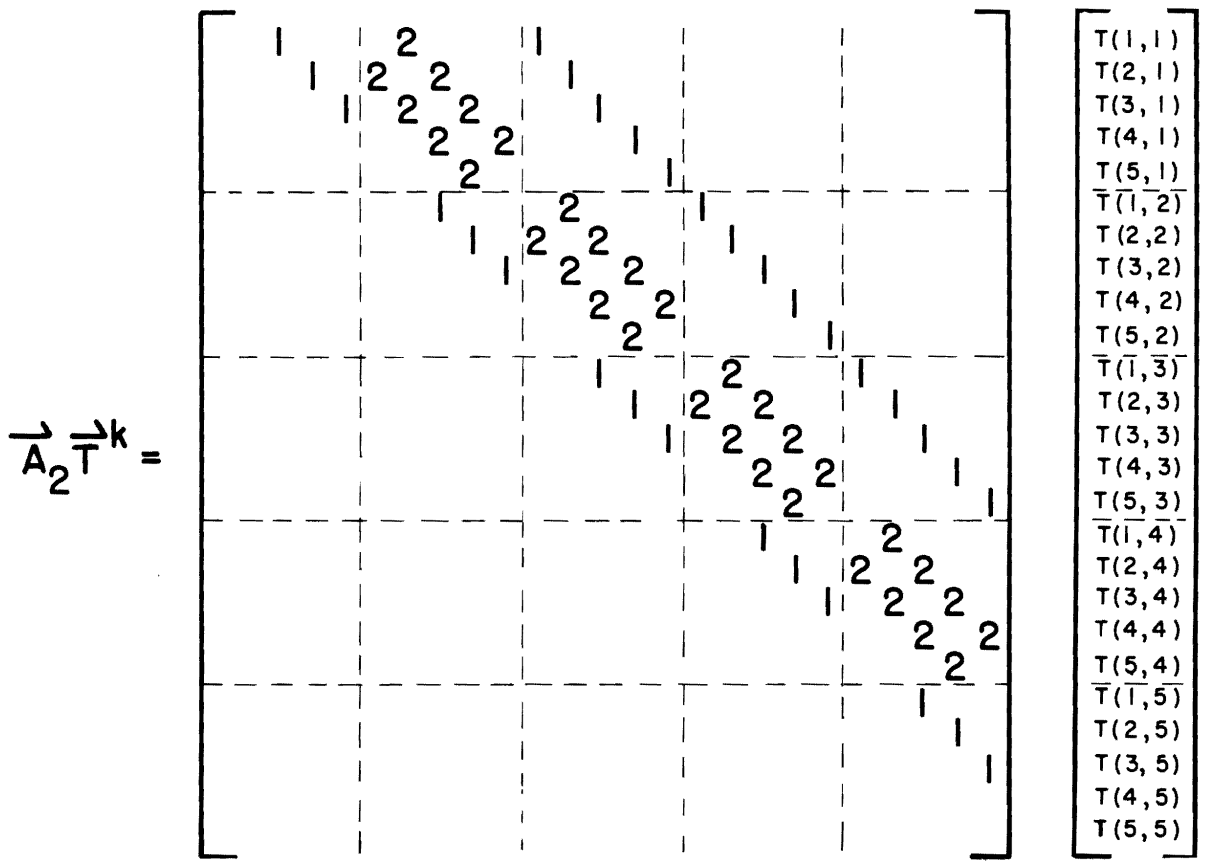

Figure 4. Graphical Display of Matrix Product A_2T Appearing in Equation (25) Showing the Upper Triangular Form.

CONCLUSIONS

The purpose of this paper was to present a new concept and develop the mathematical foundation of a regionally implicit method for solving the general Fourier equation in finite difference form. It becomes immediately apparent that the thirteen block grid that we have described is by no means the only useful grid. Indeed, from limited experience with other grids of varying geometries and numbers of blocks it is probably safe to say that a number of other useful grids can be chosen in addition to that described.

With regard to computer usage the new method compiles faster, requires less storage space, and is more reliable than what is considered to be the best of other available methods.

The new method converges for adjacent coefficients varying in excess of a million to one, a case in which many solution methods fail. If additional speed is desired for large problems this new method may be accelerated mathematically.

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