# EXACT SOLUTIONS OF FLOW EQUATIONS OF AN INCOMPRESSIBLE FLUID OF VARIABLE VISCOSITY *VIA* ONE-PARAMETER GROUP

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## 1. INTRODUCTION

We will present some exact solutions of flow equations of an incompressible fluid of variable viscosity.

The basic non-dimensional equations of motion governing steady plane flow of an incompressible fluid of variable viscosity, in the absence of external forces and with no heat addition, are [1]:

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = -P_x + \frac{1}{\text{Re}}\{(2\mu u_x)_x + (\mu(u_y + v_x))_y\} \tag{2}$$

$$uv_x + vv_y = -P_y + \frac{1}{\text{Re}}\{(2\mu v_y)_y + (\mu(u_y + v_x))_x\} \tag{3}$$

$$uT_x + vT_y = \frac{1}{\text{Re Pr}}(T_{xx} + T_{yy}) + \frac{Ec}{\text{Re}}\{2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2\}. \tag{4}$$

The various symbols used here have their usual meanings.

Equation (1) implies the existence of a stream-function $\psi$ such that:

$$u = \psi_y, \quad v = -\psi_x. \tag{5}$$

The system of Equations (1)–(4), utilizing Equation (5), transforms to the following system of partial differential equations:

$$\psi_\xi(\psi_{\xi\xi} + \psi_{\eta\eta}) = 2P_\xi - \frac{1}{\text{Re}}(\mu(\psi_{\xi\xi} - \psi_{\eta\eta}))_\eta \tag{6}$$

$$\psi_\eta(\psi_{\xi\xi} + \psi_{\eta\eta}) = 2P_\eta - \frac{1}{\text{Re}}(\mu(\psi_{\xi\xi} - \psi_{\eta\eta}) - \frac{4}{\text{Re}}(\mu\psi_{\xi\eta})_\eta \tag{7}$$

$$\psi_\xi T_\eta - \psi_\eta T_\xi = \frac{1}{\text{Re Pr}}(T_{\xi\xi} + T_{\eta\eta}) + \frac{Ec}{2\,\text{Re}}\mu\{(\psi_{\xi\xi} - \psi_{\eta\eta})^2 + 4\psi_{\xi\eta}^2\}; \tag{8}$$

in the variables $\xi = x + y$ and $\eta = x - y$. In Equations (6)–(7), the function $P$ is given by

$$P = p + \tfrac{1}{4}(\psi_\xi^2 + \psi_\eta^2) + \frac{\mu}{\text{Re}}\psi_{\xi\eta}. \tag{9}$$

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Once a solution of the system of Equations (6) – (8) is determined, the pressure $p$ is obtained from Equation (9).

## 2. SOLUTIONS

In this section, we determine some exact solutions of the system of Equations (6) – (8) using one parameter group of transformations. We give here only the one parameter group $\Gamma_1$ and its invariants that are employed to obtain exact solutions of the flow Equations (6) – (8). For details of one parameter group theory the reader is referred to references [2 – 7].

If $\Gamma_1$ is a group consisting of a set of transformations defined by

$$\bar{\xi} = \alpha^n \xi, \quad \bar{\eta} = \alpha^m \eta, \quad \bar{\psi} = \alpha^r \psi,$$
$$\bar{P} = \alpha^s P, \quad \mu = \alpha^q \mu, \quad \bar{T} = \alpha^t T$$

with parameter $\alpha \neq 0$, then the invariants of $\Gamma_1$ for system of the Equations (6) – (8), are:

$$\theta = \frac{\xi}{\eta}, \quad \psi = F(\theta), \quad P = \frac{G(\theta)}{\eta^2},$$
$$\mu = H(\theta), \quad T = \frac{L(\theta)}{\eta^2}. \tag{10}$$

Let us now use these invariants to determine exact solutions of Equations (6)–(8). The system of Equations (6)–(8), using invariants (10) of $\Gamma_1$, is transformed to the following system of ordinary differential equations:

$$F'[(1+\theta^2)F'' + 2\theta F']$$
$$= 2G' + \frac{1}{\mathrm{Re}}\{(2H + \theta H' - 6\theta^2 H - \theta^3 H')F''$$
$$-(6\theta H + 2\theta^2 H')F' + \theta(1-\theta^2)HF'''\}, \tag{11a}$$

$$\theta F'[(1+\theta^2)F'' + 2\theta F']$$
$$= 2\theta G' + 4G + \frac{1}{\mathrm{Re}}\{(1+3\theta^2)HF'''$$
$$+(6H + 2\theta H')F'$$
$$+(H' + 3\theta^2 H' + 12\theta H)F''\}, \tag{11b}$$

$$(1+\theta^2)L'' + 6\theta L' + (6 + 2\,\mathrm{Re}\,\mathrm{Pr}\,F')L$$
$$= -\frac{Ec\,\mathrm{Pr}\,H}{2}\{[(1-\theta^2)F'' - 2\theta F']^2$$
$$+4(\theta F'' + F')^2\}, \tag{11c}$$

for the four functions $F, G, H, L$ of $\theta$.

The system (11) is underdetermined since it involves one more unknown function than the number of equations. The system can be made determinate by choosing more than twenty forms [8] of $F'(\theta)$. These forms of $F'(\theta)$ are chosen by employing the particular methods for determining the solutions of linear differential equations of second order. Let us now determine solutions of system (11) for some of these forms of $F'(\theta)$.

We see that Equations (11$c$) reduces to

$$L'' + \frac{6\theta L'}{1+\theta^2} = -\frac{Ec\,\mathrm{Pr}\,H}{2(1+\theta^2)}$$
$$\{[(1-\theta^2)F'' - 2\theta F']^2 + 4(\theta F'' + F')^2\}$$

a linear differential equation in $L'$, by taking

$$\mathrm{Re}\,\mathrm{Pr}\,F'(\theta) = -3. \tag{12}$$

The solution of system (11), utilizing (12), is:

$$F = -\frac{3}{\mathrm{Re}\,\mathrm{Pr}}\theta + A^*$$
$$G = \frac{9}{4\,\mathrm{Re}^2\,\mathrm{Pr}^2}(1+\theta^2) + \frac{A_1}{1+\theta^2}$$
$$H = \frac{1}{2\,\mathrm{Pr}} + \frac{A_1}{\theta^2(1+\theta^2)} - \frac{A_1}{\theta^3}\tan^{-1}\theta + \frac{A_2}{\theta^3}$$
$$L = -\frac{18Ec}{\mathrm{Re}^2\,\mathrm{Pr}}\left\{\left(\frac{3}{16}A_1 - \frac{1}{4\,\mathrm{Pr}}\right)\frac{1}{(1+\theta^2)^2}\right.$$
$$-\left(\frac{1}{4\,\mathrm{Pr}} + \frac{5A_1}{48}\right)\frac{1+2\theta^2}{(1+\theta^2)^2}$$
$$+\frac{3}{4\,\mathrm{Pr}}\left(-\frac{\theta^4}{4(1+\theta^2)^2} - \frac{\theta^2}{8(1+\theta^2)} + \frac{1}{8}\ln(1+\theta^2)\right)$$
$$+\frac{1}{7\,\mathrm{Pr}}\left(-\frac{\theta^6}{4(1+\theta^2)^2} - \frac{3}{4}\frac{\theta^4}{(1+\theta^2)}\right.$$
$$\left.+\frac{3\theta^2}{2} - \frac{3}{2}\ln(1+\theta^2)\right) + \frac{3}{2}A_2\left(-\frac{\theta}{4(1+\theta^2)^2}\right.$$
$$\left.+\frac{\theta}{8(1+\theta^2)} + \frac{1}{8}\tan^{-1}\theta\right) - \frac{1}{2}A_2\left(-\frac{1}{\theta}\right.$$
$$\left.-\frac{15}{8}\tan^{-1}\theta - \frac{9}{8}\frac{\theta}{(1+\theta^2)} + \frac{1}{4}\frac{\theta^3}{(1+\theta^2)^2}\right)$$
$$+\frac{A_2}{4}\left(-\frac{\theta^3}{4(1+\theta^2)^2}\right.$$
$$\left.-\frac{3\theta}{8(1+\theta^2)^2} + \frac{3}{8}\tan^{-1}\theta\right)$$

$$+3A_2\left(\frac{\theta^5 \ln\theta}{(1+\theta^2)^3} + \frac{10\theta^5+5}{4(1+\theta^2)^2}\right)$$
$$-\frac{A_1}{2}\left(\ln\left(\frac{\theta}{\sqrt{1+\theta^2}}\right) - \frac{\theta^2}{(1+\theta^2)} + \frac{1}{4}\frac{\theta^4}{(1+\theta^2)^2}\right)$$
$$+A_3\left(\frac{\theta(1-\theta^2)}{8(1+\theta^2)^2} + \frac{\theta}{2(1+\theta^2)} + \frac{3}{8}\tan^{-1}\theta\right)$$
$$+\frac{A_1}{4}\left[\left(\frac{18\theta+3\theta^3}{8(1+\theta^2)^2} - \frac{3\theta}{2(1+\theta^2)} - \frac{2}{\theta}\right)\tan^{-1}\theta\right.$$
$$+\frac{9}{16(1+\theta^2)^2} - \frac{3}{2(1+\theta^2)}$$
$$\left.+2\ln\left(\frac{\theta}{\sqrt{1+\theta^2}}\right)\right]\Bigg\} + A_4 .$$

where $A^*$, $A_1$, $A_2$, $A_3$, $A_4$, are arbitrary constants.

The equation ($11c$), on substituting $L = X(\theta)Y(\theta)$ and choosing $X$ such that $(1+\theta^2)x' + 3\theta x = 0$, reduces to the normal form

$$Y''(\theta) + \left[2\,\mathrm{Re}\,\mathrm{Pr}\,F'(\theta) + \frac{3}{1+\theta^2}\right]Y(0)$$
$$= (1+\theta^2)^{3/2} \qquad \text{[R.H.S. of (11c)]}$$

This equation can easily be integrated by taking

$$F' = -\tfrac{3}{2}\,\mathrm{Re}\,\mathrm{Pr}(1+\theta^2). \tag{13}$$

The solution of system (11), employing (13), is

$$F = \frac{3}{2\,\mathrm{Re}\,\mathrm{Pr}}\tan^{-1}\theta + A_1$$
$$G = \frac{3A_2}{2\,\mathrm{Re}\,\mathrm{Pr}}\frac{(\theta^2-1)}{(1+\theta^2)^2}$$
$$H = A_2$$
$$L = (1+\theta^2)^{3/2}\left[-\frac{9EcA_2}{2\,\mathrm{Re}^2\,\mathrm{Pr}}(\theta\ln(\sqrt{1+\theta^2}+\theta)\right.$$
$$\left.\frac{X_x}{X^x} + \sqrt{1+\theta^2}) + A_3\theta + A_4\right],$$

wherein $A_1$, $A_2$, $A_3$, $A_4$, are arbitrary constants.

We know from the theory of differential equations that the equation

$$a_1(\theta)E''(\theta) + a_2(\theta)E'(\theta)$$
$$+a_3(\theta)E(\theta) = \phi(\theta)$$

is an exact equation if

$$a_3(\theta) - a_2'(\theta) + a_1''(\theta) = 0 \tag{14}$$

where prime denotes differentiation with respect to $\theta$. Employing (14) in ($11c$), we find

$$F' = -\frac{1}{\mathrm{Re}\,\mathrm{Pr}}. \tag{15}$$

For this case, the solution of system (11), is

$$F = -\frac{\theta}{\mathrm{Re}\,\mathrm{Pr}} + A_1$$

$$G = \frac{(1+\theta^2)(\theta^6 + 6\,\mathrm{Pr}\,D_2)}{24\theta^3\,\mathrm{Re}^2\,\mathrm{Pr}^2}$$
$$+\frac{D_1}{16\,\mathrm{Re}^2\,\mathrm{Pr}}\{(1+\theta^2)\tan^{-1}\theta - \theta\}$$

$$H = \frac{\theta^3}{6\,\mathrm{Pr}} + \frac{D_1}{4}\tan^{-1}\theta - \frac{1}{4}\frac{\theta D_1}{(1+\theta^2)} + \frac{D_2}{\theta^3}$$

$$L = \frac{-2Ec\,\mathrm{Pr}}{(1+\theta^2)^2}\left[\left(C_1 - 4\frac{D_2}{3}\right)\theta\right.$$
$$+\left(\frac{C_1}{3} - \frac{D_2}{9} - \frac{3}{108}D_1\right)\theta^3$$
$$+\left(\frac{1}{120\,\mathrm{Pr}} - \frac{1}{36}D_1\right)\theta^5 + \frac{1}{6\,\mathrm{Pr}}\left(\frac{35}{588}\theta^7 + \frac{\theta^9}{54}\right)$$
$$+\left(D_2\ln\theta - \frac{D_1}{12}\ln(1+\theta^2)\right)\left(\theta + \frac{\theta^3}{3}\right) + \frac{D_2}{3\theta}$$
$$\left.+D_1\left(\frac{\theta}{8} + \frac{\theta^4}{12} + \frac{\theta^6}{72}\right)\tan^{-1}\theta\right] + D_3$$

wherein $A, C, D_1, D_2, D_3$, are arbitrary constants.

When $F' = -\frac{6}{\mathrm{Re}\,\mathrm{Pr}}\theta^2$, the system (11) has the solution

$$F = \frac{6}{\mathrm{Re}\,\mathrm{Pr}\,\theta} + A_1$$

$$G = \frac{9}{\mathrm{Re}^2\,\mathrm{Pr}^2\,\theta^4} + \frac{6}{\mathrm{Re}^2\,\mathrm{Pr}}\int\frac{(H-\theta H')}{\theta^3}d\theta + A_2$$

$$H = \left(A - \frac{3}{2\,\mathrm{Pr}}\right)\frac{\theta^4}{2(1+\theta^2)}$$
$$+\left(A - \frac{3}{2\,\mathrm{Pr}}\right)\frac{\theta^3}{2}\tan^{-1}\theta + B\theta^3$$
$$+\frac{12}{\mathrm{Pr}}\theta^3\left(\frac{1}{12\theta^3} - \frac{\theta}{8(1+\theta^2)^2}\right)$$

$$L = -\frac{18Ec}{\mathrm{Re}^2\,\mathrm{Pr}}\Bigg[\left(-\frac{81}{2100}A + \frac{209}{280\,\mathrm{Pr}}\right)\theta^2$$
$$-\frac{B}{6}\theta^3 - \frac{B}{12}\theta + \frac{925}{336\,\mathrm{Pr}}$$
$$+\left(\frac{5A}{56} + \frac{12083}{672\,\mathrm{Pr}}\right)\frac{1}{\theta^2}$$
$$-\left(\frac{18185}{448\,\mathrm{Pr}} + \frac{5A}{56}\right)\frac{\tan^{-1}\theta}{\theta^3}$$
$$+\frac{1}{7}\left(\frac{A}{12} - \frac{25}{8\,\mathrm{Pr}}\right)\theta^4\,\frac{969}{64\,\mathrm{Pr}\,\theta^2}\,\frac{1}{(1+\theta^2)}$$
$$+\frac{3}{4\,\mathrm{Pr}}\left(-\frac{\theta^6}{2(1+\theta^2)^2}\right.$$
$$\left.-\frac{9\theta^4}{4(1+\theta^2)^2} - \frac{63\theta^2}{8(1+\theta^2)^2}\right)$$
$$+\left(\frac{3}{4\,\mathrm{Pr}} - \frac{A}{2}\right)\left(\frac{\theta^3}{6} + \frac{\theta}{12}\right)\tan^{-1}\theta$$
$$+\left(\frac{A}{21} - \frac{5}{14\,\mathrm{Pr}}\right)\theta^4\ln\left(\frac{\theta}{\sqrt{1+\theta^2}}\right)\Bigg]$$
$$+C\theta^3 + \frac{D}{\theta^3}$$

where $A, A_1, A_2, B, C, D$ are arbitrary constants.

## 3. CONCLUSIONS

Using one parameter group $\Gamma_1$, some exact solutions of equations governing the motion of an incompressible fluid of variable viscosity are determined. Some more exact solutions can be obtained by taking other forms of $F'(\theta)$ and other invariants of group $\Gamma_1$.

## REFERENCES


[1] J. L. Bansal, *Viscous Fluid Dynamics.* New Delhi: Oxford & IBH Publishing Co., 1977.

[2] A. J. A. Morgan, "The Reduction of One of the Number of Independent Variables in Some System of Partial Differential Equations", *Quart. Journal of Math.*, **2** (1952), p. 250.

[3] W. F. Ames, *Nonlinear Partial Differential Equations in Engineering.* New York: Academic Press, 1965.

[4] R. K. Naeem and Nasiruddin Khan, "Solutions for Unsteady Transverse MHD Flows *via* One-Parameter Group", Karachi University Journal of Science, 1994, in press.

[5] N. H. Ibragimov, *Groups of Transformations in Mathematical Physics (English Edn.).* Dordrecht: Reidel, 1983.

[6] P. J. Olver, *Applications of Lie Groups to Differential Equations.* New York: Springer, 1986.

[7] L. V. Ovsiannikov, *Group Analysis of Differential Equations.* Boston: Academic Press, 1982.

[8] J. N. Sharma and R. K. Gupta, *Differential Equations.* Meerut-2 (U.P.), India: Krishna Parakashan Mandir, 1984.


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