THE BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL – DIFFERENCE EQUATION OF THE SECOND ORDER

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الخلاصـــة :

يهتم هذا البحث بمسائل القيم الحديِّـة (م ق ح) التفاضلية بالنسبة لأحد متغيراتها والفرقية بالنسبة للمتغير الثاني . لقد أثبتت النظرية قابلية الحل لهذه ألـ (م ق ح) ؛ إذا كان المجال الذي يبحث فيه الحل مستطيلاً فإنَّ ألـ (م ق ح) قابلة للحل اعتيادياً ، أما إذا كان المجال غير مستطيل فإن المسألة غير قابلة للحل اعتيادياً . والمؤثر المناظر لهذه ألـ (م ق ح) ليس من نوع مؤثرات (فريدهولم) ويمكن أن يكون له نواة لانهائية .

ABSTRACT

The paper is concerned with the boundary value problem (BVP) for equations, which are differential with respect to one variable and difference with respect to the other variable.

The solvability theorem of this BVP is proved. If the domain in which the investigated solutions occur is rectangular, the BVP is normally solvable, if the domain is nonrectangular, the BVP is not normally solvable. The operator corresponding to this BVP is not a Fredholm type operator, and can have an infinite kernel.

THE BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL – DIFFERENCE EQUATION OF THE SECOND ORDER

INTRODUCTION

The paper is concerned with the boundary value problem (BVP) for equations, which are differential with respect to one variable (t) and difference with respect to the other variable (s). For example

$$x_{tt}(t,s)+a_0x(t,s)+a_1(x(t,s+1)+x(t,s-1))$$
$$=f(t,s),\ (t,s)\in Q,\ x(t,s)=0,\ (t,s)\in\mathbb{R}^2\backslash Q. \quad (1)$$

Here

$$(t,s)\in\mathbb{R}^2,\quad f(t,s)\in L_2(Q),\quad Q=(0,T)\times(0,a).$$

Investigations of different equations of this type and their applications in earlier papers (see [1]) deal with discrete variation of $s(x(t,s)=x_s(t),\ s\in\mathbb{Z})$. Problems where s is a continuous variable have appeared more recently in several biological and ecological models [2]. The theory of the investigated BVP is connected with the theory of the BVP for strongly elliptic differential – difference equations, which are difference and differential with respect to the same variables [3]. In contrast to the theory of this BVP (and discrete BVP), the operator, that corresponding to the BVP (1) is not of Fredholm type, and can have an infinite dimensional kernel (see Example 1). The initial value problem for differential – difference equations of this type was studied in references [1] and [4].

In this paper we prove a theorem of solvability of the BVP for difference – differential equations.

NOTATIONS AND RESULTS

Consider the equation:

$$-(R_0x_t(t,s))_t+R_1x_t(t,s)+R_2x(t,s)=f(t,s),$$
$$(t,s)\in Q \quad (2)$$

with boundary conditions

$$x(t,s)=0,\qquad (t,s)\in\mathbb{R}^2\backslash Q. \quad (3)$$

Here $(t,s)\in\mathbb{R}^2$, $f(t,s)\in L_2(Q)$, $Q=(0,T)\times(0,a)$, ($a=N+\theta$, N-integer, $0<\theta\le 1$); R_k: $L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)$ are difference operators

$$(R_kx)(t,s)=\sum_{i=-N}^{N}a_k^i(t)\times(t,s+i), \quad (4)$$

$k=0,1,2$, $a_0^i\in C^2(0,a)$, $a_1^i\in C^1(0,a)$, $a_2^i\in C(0,a)$, R_0-one-to-one operator, *i.e.* $R_0x\ne 0$ for $\forall\, x\in L_2(\mathbb{R}^2)$, $x\ne 0$.

Consider the operators I_Q, P_Q, R_Q^i, $i=1,2$,

$$I_Q\colon\ L_2(Q)\to L_2(\mathbb{R}^2),\ (I_Qx)(t,s)=x(t,s),$$
$$(t,s)\in Q,\ (I_Qx)(t,s)=0,\ (t,s)\in\mathbb{R}^2\backslash Q;$$

$$P_Q\colon\ L_2(\mathbb{R}^2)\to L_2(Q),\ (P_Qx)(t,s)=x(t,s),$$
$$(t,s)\in Q;$$

$$R_Q^i\colon L_2(Q)\to L_2(Q),\ R_Q^i=P_QR_iI_Q,\ i=1,2.$$

Lemma 1. Operators R_Q^i: $L_2(Q)\to L_2(Q)$ are bounded.

The proof is obvious.

Denote $L_2\left(\bigcup_\iota Q_{r\iota}\right)=\left\{x(t,s)\in L_2(Q)\,|\,x(t,s)=0\right.$ for $\left.(t,s)\in Q\backslash\bigcup_\iota Q_{r\iota}\right\}$; here, if $\theta<1$, then $r=1,2$,

$$Q_{1\iota}=\ (0,T)\times(\iota-1,\iota-1+\theta),\ (\iota=1,\dots,N+1),$$
$$Q_{2\iota}=\ (0,T)\times(\iota-1+\theta,\iota),\ (\iota=1,\dots,N);$$

if $\theta=1$, then $r=1$,

$$Q_{1\iota}=\ (0,T)\times(\iota-1,\iota),\ (\iota=1,\dots,N+1).$$

We introduce the isomorphism of Hilbert spaces:

U_r: $L_2\left(\bigcup_{\iota=1}^m Q_{r\iota}\right)\ \to\ L_2^m(Q_{r1})$ by the formula $(U_rx)_\iota(t,s)=x(t,s+\iota-1),(t,s)\in Q_{r1}$, $\iota=1,\dots,m$, where $L_2^m=\prod_{r=1}^m L_2(Q_{r1})$, $m=N+1$, if $r=1$; $m=N$, if $r=2$.

Obviously, the operator $R_{Qr}^k=U_rR_Q^kU_r^{-1}$: $L_2^m(Q_{r1})\to L_2^m(Q_{r1})$, is the operator of multiplication by an $m\times m$ dimensional matrix R_{Qr}^k with the elements $b_{1p}(t)$ according to the formula:

$$b_{1p}(t)=a_{1-p}(t). \quad (5)$$

Assertion 1. Let R be a one-to-one operator. Then Equation (2) can be reduced to the form (in the old notation):

$$-x_{tt}(t,s)+R_1x_t(t,s)+R_2x(t,s)=f(t,s),$$
$$(t,s)\in Q. \tag{6}$$

Proof is in the supplement.

Remark 1. If Q is a nonrectangular domain Equation (2) cannot be simplified to the form (6).

Let $H^{p,0}(Q)$ be a Sobolev space $H^{p,0}(Q)=\left\{x\in L_2(Q) \mid \frac{\partial^k x}{\partial t^k}\in L_2(Q)\ k=1,\dots,p\right\}$ (see Section 7, III of reference [5]) with the inner product:

$$(x,y)_p=\sum_{i=0}^{p}\int_Q x_t^{(i)}y_t^{(i)}\,\mathrm{d}t\,\mathrm{d}s.$$

Let H be a closure of a set $C_0^\infty(Q)$ in $H^{1,0}(Q)$.

Definition 1. We say that the function $x\in H$ is a solution of the boundary value problem (6), (3), if for $\forall\ v\in H$

$$(x_t,v_t)+(R_Q^1x_t,v)+(R_Q^2x,v)=(f,v), \tag{7}$$

where $(.,.)$ is the inner product in the space $L_2(Q)$.

Consider the operator $A:H\to L_2(Q)$, $Ax=R_Q^1x_t+R_Q^2x$. Then Equation (7) takes the form

$$(x_t,v_t)+(Ax,v)=(f,v). \tag{8}$$

Consider also homogeneous equation

$$(x_t,v_t)+(Ax,v)=0, \tag{9}$$

and its adjoint equation

$$(x_t,v_t)+(A^+x,\ v)=0, \tag{10}$$

where A^+ is formally adjoint to operator A

$$(A^+x)(t,s)=-(R_Q^1)^*x_t+[(R_Q^2)^*-\frac{\partial}{\partial t}(R_Q^1)^*]x;$$

$$(R_Q^k)^*=P_Q(R_k^*)I_Q;$$

$$(R_k^*x)(t,s)=\sum_{i=-N}^{N}a_k^i(t)x(t,s-i),\ k=1,2.$$

From Lemma 1.10, Section 1.8, 1 of reference [6] there follows:

$$\|f\|_{L_2(Q)}<k_Q\|f_t\|_{L_2(Q)},\qquad k_Q=\text{const}<0.$$

Lemma 2. For any function $f\in H$

$$\|f\|_{L_2(Q)}<k_Q\|f_t\|_{L_2(Q)},\qquad k_Q=\text{const}>0.$$

It follows from Lemma 2 that in H we can introduce the equivalent inner product by the formula:

$$(x,y)_1=\int_Q x_tv_t\,\mathrm{d}s\,\mathrm{d}t.$$

Lemma 3. Equations (8) − (10) correspond to the operator equations in H with the norm $\|.\|_1$:

$$x+Gx=F, \tag{11}$$

$$x+Gx=0, \tag{12}$$

$$x+G^*x=0, \tag{13}$$

where $G\colon H\to H$ is a linear bounded operator.

Proof. For any fixed $x\in H$ the linear functional $\phi_x(v)=(Ax,v)$ is bounded in H (see Lemma 1). According to the Riesz theorem on the general form of the functional in Hilbert space, there exists a unique solution $w=w(x)\in H$, such that $\|w\|_1=\|\phi_x\|\le c\|x\|_1$, $(c=\text{const}>0)$, and $\phi_x(v)=(w,v)_1$. Hence, $w=w(x)$ defines the linear bounded operator $G\colon H\to H$, such that $(Ax,v)=(Gx,v)_1$ for every $v,x\in H$. And for every $v,x\in H$ $(A^+x,\ v)=(x,\ Av)=(x,Gv)_1=(G^*x,v)_1$, G^* is the adjoint to G operator. In the same way we define F from the formula

$$(f,v)=(F,v)_1,\quad \forall\ v\in H \tag{14}$$

and $\|F\|_1<c_2\|f\|$, $c_2=\text{const}>0$. □

Remark 2. In contrast to the theory of elliptic differential − difference equations which are difference and differential in respect to the same variables [3], operator G is not compact and the theory of Fredholm type operator does not apply to Equation (11).

Consider $H^p(0,T)$, $p\ge0$ — the Sobolev space:

$$H^p(0,T)=\left\{x\in L_2(0,T) \mid \frac{\partial^k x}{\partial t^k}\in L_2(0,T)\quad k=1,\dots,p\right\},$$

$$\mathring{H}^1(0,T)=\{x\in H^1(0,T) \mid x(0)=x(T)=0\}.$$

Denote W operator $W\colon H\to H$ by the formula $Wx=x+Gx$.

Lemma 4. Operator W is normally solvable, *i.e.* $\overline{Im(W)} = Im(W)$ [7].

Proof.

(1). We prove that some auxiliary equation is normally solvable. Using operators U_r, R^k_{Qr}, we write Equation (8) in the form:

$$((U_r x)_t, (U_r v)_t)_{N_r} + (R^1_{Qr}(U_r x)_t, (U_r v))_{N_r} + (R^2_{Qr}(U_r x), (U_r v))_N = ((U_r f), (U_r v))_{N_r}, \quad r = 1,2, \tag{15}$$

where $(.,.)_{N_r}$ is the inner product in $L_2^{N_r}(Q_{r1})$.

Suppose that $\theta = 1$, $a = N+1$ (in case $\theta < 1$, the proof is analogous). Let $s_0 \in (0,1)$ be a fixed number. Denote $U = U_1$,

$$R^k_Q = R^k_{Q1},\ \tilde{x}(t) = (Ux)(t, s_0) \in \mathring{H} = \prod_{\iota=1}^{N+1} \mathring{H}^1(0,T),$$
$$\tilde{v}(t) = (Uv)(t,s_0) \in \mathring{H},$$
$$\tilde{f}(t) = (Uf)(t,s_0) \in L_2^{N+1}(0,T). \tag{16}$$

If $x(t,s)$ is a solution of (15), then for almost all $s_0 \in (0,1)$ $\tilde{x}(t)$ is a solution of the system of the ordinary differential equations in $(0,T)$:

$$(\tilde{x}_t, \tilde{v}_t)^{N+1} + (R^1_Q \tilde{x}_t, \tilde{v})^{N+1} + (R^2_Q \tilde{x}, \tilde{v})^{N+1} = (\tilde{f}, \tilde{v})^{N+1}, \quad \forall\, \tilde{v} \in \mathring{H}, \tag{17}$$

where $(.,.)^{N+1}$ is the inner product in $L_2^{N+1}(0,T)$.

In the same way as for the proof of Lemma 1, Section 1, IV in reference [5], one can prove that Equation (17) is equivalent to the equation:

$$Cy \triangleq By + y = \Psi \tag{18}$$

in the Hilbert space $\mathring{H}$ with the inner product $(y,v)_\circ \triangleq (y_t, v_t)^{N+1}$, where B: $\mathring{H} \to \mathring{H}$ is a compact operator, and Ψ is defined by the formula

$$(\Psi, \tilde{v})_\circ = (\tilde{f}, \tilde{v})^{N+1} \qquad (\forall\ v \in \mathring{H}). \tag{19}$$

Since B is a compact operator, C is normally solvable (see Theorem 4.23 of reference [8]).

(2). We now show if $\Psi = (UF)(t,s_0) \in Im(C)$ for almost all $s_0 \in (0,1)$ then $F \in Im(W)$. We use Theorem 2.3 of reference [7]:

Result. The closed operator A $(y = Ax)$ is normally solvable if and only if $\forall y \in Im(A)$ $\exists x \in D(A)$: $y = Ax$, $\|x\| \le k\|y\|$, $k = \text{const} > 0$.

Suppose that $\Psi \in Im(C)$. Since C is normally solvable $\exists \tilde{x} \in \mathring{H}$: $\Psi = C\tilde{x}$, $\|\tilde{x}\|_\circ \le k\|\Psi\|_\circ$, $k = \text{const} > 0$, k does not depend on Ψ and s_0. Then for x and F such that $(Ux)(t,s_0) = \tilde{x}(t)$, $(UF)(t,s_0) = \Psi(t)$ for almost all $s_0 \in (0,1)$, $\|x\|_1 \le k\|F\|_1$. Therefore, using (14), (16), (19), we get $Wx = F$ *i.e.* $F \in Im(W)$.

It follows from the above that for any $F \in Im(W)$ there exists $x \in H$ such that $Wx = F$ and $\|x\|_1 \le k\|F\|_1$; therefore, using the Theorem 2.3 of reference [7] we obtain $\overline{Im\ W} = Im\ W$.

Theorem 1. If Equation (9) has only the zero solution, then for any $f \in L_2(Q)$ Equation (8) has a unique solution $x \in H$ and $\|x\|_1 \le c\|f\|$, $c = \text{const} > 0$.

If Equation (9) has nonzero solutions then Equation (8) has a solution if and only if

$$(f, \hat{x}) = 0 \tag{20}$$

for all solutions of Equation (10) in $\hat{x}$. In the case the solution spaces of (9), (10) have the infinite dimensions.

Proof. Since the operator W in accordance with Lemma 5 is bounded and defined on all space H, so W is closed. At the same time, from lemma 6 it follows that $\overline{Im\ W} = Im\ W$. For closed operators normal solvability is equivalent to correct solvability ($\|x\|_1 \le c\|F\|_1$) (see Theorem 2.1, of reference [7]). From this and the formulae (16), (19) the first part of the theorem follows.

For a normally solvable operator, $Im(W)$ is an orthogonal complement of $Ker(W^*)$ (see Theorem 3.2 of reference [7]). Therefore, using (16) and (19) we get (20).

Suppose that (9) has nonzero solutions. We show that the solution spaces of Equations (9) and (10) have infinite dimensions. In this case Equation (19) for $\Psi = 0$ also has nonzero solutions and since B is a compact operator the general solution of equation $\tilde{x} + B\tilde{x} = 0$ has the form $\tilde{x}(t) = \sum_{i=1}^{m} c_i y_i(t)$, where c_i are arbitrary constants. Then the common solution of Equation (9) has the form $x(t,s) = \sum_{i=1}^{m} U^{-1}(c_i(s) y_i(t))$, where $c_i(s) \in L_2(0,1)$ are arbitrary functions. We can represent $c_i(s)$ in the form $c_i(s)=\sum_{j=1}^{\infty} c_{ij}v_j(s)$, $(i=1,\ldots,m)$, where $\{v_j(s)\}_{j=1}^{\infty}$ is the basis set in $L_2(0,1)$ and $\sum_{j=1}^{\infty}|c_{ij}|^2<\infty$. Therefore the general solution of Equation (9) $x(t,s)=\sum_{i=1}^{m}\sum_{j=1}^{\infty} c_{ij}\; U^{-1}(v_j(s)y_i(t))$, *i.e.* the solution space of Equation (9) has an infinite dimension.

Since the solution spaces dimensions of Equation (9) and (10) are equal, the last assertion of the theorem is proved.

Example 1. Let Equation (6) have the form:

$$x_{tt}(t,s)+Rx(t,s)=f(t,s),\quad (t,s)\in Q,$$

where

$$(Rx)(t,s)=2/3\,x(t,s)+1/3[x(t,s-1)+x(t,s+1)],$$

$$Q=(0,\pi)\times(0,2).$$

Then Equation (8) takes the form:

$$-\int_Q x_t v_t \mathrm{d}t\mathrm{d}s+\int_Q P_Q[2/3(I_Qx)(t,s)+1/3(I_Qx)(t,s+1)+1/3(I_Qx)(t,s-1)]v(t,s)\mathrm{d}t\mathrm{d}s=\int_Q f(t,s)v(t,s)\mathrm{d}t\mathrm{d}s. \tag{21}$$

Obviously, $\theta=1$, $N=1$, $Q_1=(0,\pi)\times(0,1)$, $Q_2=(0,\pi)\times(1,2)$. Denote $x_i=(Ux)_i$, $f_i=(Uf)_i$, $i=1,2$. Equation (21) takes the form

$$\begin{cases}(x_1)_{tt}+2/3\,x_1+1/3\,x_2=f_1,\\ (x_2)_{tt}+1/3\,x_1+2/3\,x_2=f_2,\qquad (t,s)\in Q_1,\end{cases} \tag{22}$$

and Equations (9) and (10) take the form:

$$\begin{cases}(x_1)_{tt}+2/3\,x_1+1/3\,x_2=0,\\ (x_2)_{tt}+1/3\,x_1+2/3\,x_2=0,\qquad (t,s)\in Q_1.\end{cases} \tag{23}$$

It is easy to check that (23) has only these nonzero solutions $\hat{x}_1^i=\hat{x}_2^i=c_i\sin t\; v_i(s)\in H(Q_1)$, $i=1,2,\ldots$, where $\{v_j(s)\}_{j=1}^{\infty}$ is the basis set in $L_2(0,1)$, c_i — arbitrary constants. Hence Equation (21) has a solution for $f\in L_2(Q)$ if and only if $\int_{Q_1} v_i(s)\;\sin t(f_1(t,s)+f_2(t,s))\mathrm{d}t\mathrm{d}s=0$, $i=1,2,\ldots$, $((f_1,f_2)=Uf)$ or for almost all $s_0\in(0,1)$ $\int_0^{\pi}\sin t(f_1(t,s_0)+f_2(t,s_0))\mathrm{d}t=0$.

Remark 3. If in Equation (6), the difference operator coefficients depend also on s: $a_k^i=a_k^i(t,s)$, then:

(*a*) operator W is not normally solvable and Theorem 1 can be false even if $a_k^i=c+\varepsilon\mu(s)$, where $c=\text{const}$, $\varepsilon=$ a samll parameter (see Example 2 below);

(*b*) the condition $\Psi=(UF)(t,s_0)\in Im(C)$ for almost all $s_0\in(0,1)$ is necessary for $F\in Im(W)$, but generally speaking it is not sufficient.

The same facts apply when a_k^i cannot depend on s, but the domain Q is nonrectangular.

Example 2. Consider the boundary value problem

$$x_{tt}(t,s)+(\pi(1-\varepsilon)+\varepsilon_s)^2x(t,s)=t(\pi(1-\varepsilon)+\varepsilon_s)^3,\qquad (t,s)\in Q, \tag{24}$$

$$x(t,s)=0,\qquad (t,s)\in\mathbb{R}^2\backslash Q, \tag{25}$$

where $\varepsilon>0$ is a small parameter, $Q=(0,1)\times(0,2\pi)$.

The common solution of the homogeneous equation:

$$x_0(t,s)=c_1(s)\sin(\pi(1-\varepsilon)+\varepsilon_s)t+c_2(s)\cos(\pi(1-\varepsilon)+\varepsilon_s)t,\; c_1(s),\; c_2(s)\in L_2(0,2\pi)-\text{arbitrary functions.}$$

Using boundary conditions, we get: $x_0(t,s)=0$ for $s\neq\pi$ or $x_0(t,s)=0$ for almost all $s\in(0,2\pi)$. If there exists a general solution of (24), then from (*b*) of Remark 3 it has the form:

$$x(t,s)=c_1(s)\sin(\pi(1-\varepsilon)+\varepsilon_s)t+c_2(s)\cos(\pi(1-\varepsilon)+\varepsilon_s)t+(\pi(1-\varepsilon)+\varepsilon_s)t.$$

Using the boundary conditions we get

$$x(t,s)=\frac{-(\pi(1-\varepsilon)+\varepsilon_s)}{\sin(\pi(1-\varepsilon)+\varepsilon_s)}\sin(\pi(1-\varepsilon)+\varepsilon_s)t+(\pi(1-\varepsilon)+\varepsilon_s)t.$$

We show that $x\notin L_2(Q)$. Denote $\alpha=\varepsilon(s-\pi)$, $v=(\pi+\alpha)t$, $u=\dfrac{(\pi+\alpha)}{\sin(\pi+\alpha)}\sin(\pi+\alpha)t$. Then $x=-u+v$. Obviously $v\in L_2(Q)$. Therefore $x\in L_2(Q)$ if and only if $u\in L_2(Q)$. Let us prove that $u\notin L_2(Q)$. Indeed,

$$\|u\|^2 = \int_0^{2\pi}\int_0^1 u^2 dt ds = \frac{1}{\varepsilon}\int_{\pi\varepsilon}^{\pi\varepsilon}\int_0^1 u^2 dt d\alpha$$
$$= \frac{1}{\varepsilon}\int_{\pi\varepsilon}^{\pi\varepsilon}\frac{(\pi+\alpha)^2}{\sin^2(\pi+\alpha)}d\alpha\int_0^1 \sin^2(\pi+\alpha)t dt$$
$$= \frac{1}{4\varepsilon}\int_{\pi\varepsilon}^{\pi\varepsilon}\frac{(\pi+\alpha)(2(\pi+\alpha)-\sin2(\pi+\alpha))}{\sin^2(\pi+\alpha)}d\alpha$$
$$= \frac{1}{4\varepsilon}\int_{\pi\varepsilon}^{\pi\varepsilon}\frac{(\pi+\alpha)(2(\pi+\alpha)-\sin2\alpha)}{\sin^2\alpha}d\alpha = \infty,$$

since $\frac{(\pi+\alpha)(2(\pi+\alpha)-\sin2\alpha)}{\sin^2\alpha} = O(\alpha^{-2})$ for $\alpha\to 0$, *i.e.* $u \notin L_2(Q)$.

Therefore $x \notin L_2(Q)$ and BVP (24), (25) have no solutions, though the homogeneous BVP has only the zero solution.

SUPPLEMENT

(Proof of Assertion 1). □

Denote $L_Q = \{x \in L_2(\mathbb{R}^2) \mid x(t,s) = 0, (t,s) \notin Q\}$.

Lemma 5. Let R_1, R_2 be the operators in the form (4).

Then the operator R_3: $L_Q \to L_2(\mathbb{R}^2)$, $R_3 x = R_2(R_1 x)$ also is the operator in the form (4).

Proof. Let the operators R_1, R_2 be defined by the formula (4). Consider the operator $(R_3x)(t,s) = (R_2R_1x)(t,s)$. Denote $y = R_1x$, and, using (3), we obtain

$$(R_3x)(t,s) = \sum_{i=-N}^{N} a_2^i(t)y(t,s+i)$$
$$= \sum_{i=-N}^{N}\sum_{j=-N}^{N} a_2^i(t)a_1^j(t)x(t,s+i+j)$$
$$= \sum_{\iota=-2N}^{2N} a_3^\iota(t)x(t,s+1) = \sum_{i=-N}^{N} a_3^\iota(t)x(t,s+1),$$

where

$$a_3^\iota(t) = \sum_{\substack{i,j\\ i+j=\iota}} a_2^i(t)a_1^j(t). \quad \square \qquad (26)$$

Lemma 6. Let R be one-to-one operator in the form (4).

Then there exists a unique operator R^{-1}: $L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ in the form (4) such that $\forall\, x \in L_Q$ $(R^{-1}Rx)(t,s) = x(t,s)$, $(t,s) \in Q$.

Proof. Let R_1 be the operator, defined by (4). We construct the operator R_2 such that $\forall\, x \in L_Q$ $(R_2R_1x)(t,s) = R_3x(t,s) = x(t,s)$, $(t,s) \in Q$, *i.e.* $a_3^\iota = 0$, $|\iota| \le N$, $\iota \ne 0$, $a_3^0 = 1$. Using (26) we get the system of equations in a_2^i:

$$Aa_2 = a_3,$$

where $a_2 = (a_2^{-N},\dots, a_2^N)$, $a_3 = (a_3^{-N},\dots, a_3^N)$, $A = \|a_{ij}\|_{i,j=1}^{2N+1}$, $a_{ij} = a_1^{i-j}$, $a_1^i = 0$, $i > N$, $i < -N$. From (5), the action of the operator R_1 in L_{2Q}, where $2Q = (0,T)\times(0,2a)$, is equivalent to multiplication by matrix A^*. Since R_1 is one-to-one operator so det $A \ne 0$ and the system has a unique solution. □

Lemma 7. Let R be one-to-one operator in the form (4) and $(Rx) \in H^{k,0}(Q)$, $k > 0$.

Then $x \in H^{k,0}(Q)$ and $(Rx)_t = R_t x + Rx_t$.

The proof is evident.

Using Lemmas 5–7 and multiply Equation (2) by R_0^{-1}, we write (2) in the form:

$$-x_{tt}(t,s) + \tilde{R}_1 x_t(t,s) + \tilde{R}_2 x(t,s) = \tilde{f}(t,s),\ (t,s) \in Q,$$
$$\tilde{R}_1 = R_0^{-1}(R_2 - (R_0)_t),\ \tilde{R}_2 = R_0^{-1}R_2,\ \tilde{f} = R_0^{-1}f. \quad \square$$

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