θ-CONNECTEDNESS AND δ-CONNECTEDNESS IN BITOPOLOGICAL SPACES

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الخلاصـــة :

في هذا البحث تَـمَّ تقديم مفهومي الترابط من النوعين (θ) و (δ) في الفراغات ثنائية التوبولوجي . كما تَـمَّـتْ دراسة بعض خواصهما والعلاقة بينهما وبين الترابط الثنائي . أخيراً تَـمَّـتْ دراسة سلوك هذين النوعين من الترابط تحت تاثير بعض الدوال في الفراغات (*ij* – المنتظمة) ، و (*ij* – شبه المنتظمة) و (*ij* – المنتظمة تقريباً) .

ABSTRACT

In this paper θ-connectedness and δ-connectedness have been introduced in bitopological spaces by utilizing the ij-θ-closure and ij-δ-closure operators.

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θ-CONNECTEDNESS AND δ-CONNECTEDNESS IN BITOPOLOGICAL SPACES

1. INTRODUCTION

The notion of bitopological spaces was introduced by Kelly [1] in 1963. In 1967 Pervin [2] presented the notion of pairwise connectedness for bitopological spaces. In 1987, Banerjee [3] introduced the notion of ij-θ-closure and ij-δ-closure operators in bitopological spaces.

In this paper we present and investigate the notion of a θ-connected (resp. δ-connected) subset relative to a bitopological space (X, T_1, T_2) by utilizing the ij-θ-closure (resp. ij-δ-closure) operators. Then we investigate the relationship between pairwise connected subsets, θ-connected and δ-connected subsets. Moreover, we discuss the behavior of θ-connectedness and δ-connectedness under functions between bitopological spaces. Finally, we compare all these forms of connectedness and investigate their properties in ij-almost-regular, ij-semi-regular, and ij-regular spaces.

Throughout the paper, by a space (X, T_1, T_2) (or for short X), we mean a bitopological space (X, T_1, T_2). The interior and the closure of a subset A of X with respect to T_i will be denoted by i-$int\, A$ and i-$cl\, A$, respectively. Also, $i, j = 1, 2$ and $i \neq j$.

A space (X, T_1, T_2) is called *pairwise connected* if X cannot be expressed as the union of two nonempty subsets A and B such that $(A \cap i\text{-}cl\, B) \cup (B \cap j\text{-}cl\, A) = \varnothing$. A subset K of (X, T_1, T_2) is pairwise connected if the bitopological space (K, T_{1k}, T_{2k}) is pairwise connected, where T_{ik} is the relative topology on K induced by T_i [2]. A point $x \in X$ is called an ij-θ-closure (resp. ij-δ-closure) point of a subset A of (X, T_1, T_2) [3] if $A \cap j\text{-}cl\, U \neq \varnothing$ (resp. $A \cap i\text{-}int(j\text{-}cl\, U) \neq \varnothing$) for any i-open neighborhood U of x. The set of all ij-θ-closure (resp. ij-δ-closure) points of A is called the ij-θ-closure (resp. ij-δ-closure) of A and is denoted by $ij\text{-}cl_\theta A$ (resp. $ij\text{-}cl_\delta A$). If $ij\text{-}cl_\theta A = A$ (resp. $ij\text{-}cl_\delta A = A$), then A is called ij-θ-closed (resp. ij-δ-closed). If the subset A is ij-θ-closed and ji-θ-closed (resp. ij-δ-closed and ji-δ-closed) then A is called pairwise θ-closed (resp. pairwise δ-closed). The complement of an ij-θ-closed set (resp. ij-δ-closed set) is called ij-θ-open (resp. ij-δ-open). A subset A of a space (X, T_1, T_2) is called ij-regular open (ij-$r.o$) [4] if $A = i\text{-}int(j\text{-}cl\, A)$ and it is ij-regular closed (ij-$r.c$) if $A = i\text{-}cl(j\text{-}int\, A)$. A is called pairwise $r.o$ (resp. pairwise $r.c$) if it is both ij-$r.o$ and ji-$r.o$ (resp. ij-$r.c$ and ji-$r.c$). Clearly the complement of an ij-$r.o$ set is ij-$r.c$.

A space (X, T_1, T_2) is said to be ij-regular [1] if and only if for each i-open set V of X and each $x \in V$ there exists an i-open set U of X such that $x \in U \subset j\text{-}cl\, U \subset V$. A space (X, T_1, T_2) is called ij-almost-regular [4] if and only if for each $x \in X$ and each i-open set V containing x, there exists an i-open set U such that $x \in U \subset j\text{-}cl\, U \subset i\text{-}int(j\text{-}cl\, V)$. A space (X, T_1, T_2) is said to be ij-semi-regular [5] if for each $x \in X$ and each i-open set V containing x, there exists an i-open set U such that $x \in U \subset i\text{-}int(j\text{-}cl\, U) \in V$.

A function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise continuous* [2] if and only if the inverse image of each i-open set in Y is an i-open set in X.

2. SETS θ-CONNECTED RELATIVE TO BITOPOLOGICAL SPACES

Definition 2.1. A pair (A, B) of nonempty subsets of (X, T_1, T_2) is said to be θ-separation (denoted by $[A|B]_\theta$) if $(A \cap ij\text{-}cl_\theta B) \cup (B \cap ji\text{-}cl_\theta A) = \varnothing$.

Definition 2.2. A subset S of a space (X, T_1, T_2) is said to be θ-connected relative to X if it cannot be expressed as the union of two nonempty subsets A and B which form a θ-separation relative to X. The space (X, T_1, T_2) is called θ-connected if it admits no θ-separation relative to X itself.

Lemma 2.3. The collection $T_{i\theta}$ of all ij-θ-open sets in (X, T_1, T_2) is a topology on X.

Proof.

(*i*) X and $\varnothing$ are ij-θ-open sets.

(*ii*) Suppose that A and B are ij-θ-open sets in (X, T_1, T_2). Let $x \in A \cap B$, then $x \notin X - (A \cap B) = A^c \cup B^c$ (where $A^c = X - A$ and $B^c = X - B$). Now A^c and B^c are ij-θ-closed sets. Thus there exist i-open sets U and V of X such that $x \in U$, $j\text{-}cl\, U \cap A^c = \varnothing$ and $x \in V$, $j\text{-}cl\, V \cap B^c = \varnothing$. Therefore $x \in U \cap V$, $j\text{-}cl(U \cap V) \subset j\text{-}cl\, U \cap j\text{-}cl\, V$. Since $j\text{-}cl\, U \subset A$ and $j\text{-}cl\, V \subset B$, we have $j\text{-}cl\, U \cap j\text{-}cl\, V \subset A \cap B$. Thus $x \in ij\text{-}int_\theta(A \cap B)$ and so $A \cap B$ is an ij-θ-open set.

(*iii*) Let $\{A_k\}$, $k \in I$, be any family of ij-θ-open subsets of X and $x \in \bigcup_k A_k$. Then $x \in A_k$ for some k. Since A_k is an ij-θ-open set, there exists an i-open set U such that $x \in j\text{-}cl\, U \subset A_k$ and so

$x \in$ j-cl $U \subset \bigcup_k A_k$. Hence $x \in ij\text{-}int_\theta(\bigcup_k A_k)$ and thus $\bigcup_k A_k$ is an ij-θ-open set. Therefore the collection of all ij-θ-open sets in (X, T_1, T_2) is a topology on X.

Theorem 2.4. A space (X, T_1, T_2) is θ-connected if and only if the space $(X, T_{1\theta}, T_{2\theta})$ is pairwise connected.

Lemma 2.5. Let A and B be subsets of a space (X, T_1, T_2). If $[A|B]_\theta$, $\varnothing \neq A_o \subset A$ and $\varnothing \neq B_o \subset B$, then $[A_o|B_o]_\theta$.

Proof. Since $[A|B]_\theta$ is θ-separation, we have $[A \cap ij\text{-}cl_\theta B] \cup [B \cap ji\text{-}cl_\theta A] = \varnothing$. Now $ij\text{-}cl_\theta B_o \subset ij\text{-}cl_\theta B$ and $ji\text{-}cl_\theta A_o \subset ji\text{-}cl_\theta A$. Thus, $[A_o|B_o]_\theta$ is θ-separation.

Theorem 2.6. For the space (X, T_1, T_2) and a subset S of X the following are equivalent:

(*a*) S is θ-connected relative to X.

(*b*) For each two points x, y of S, there exists a subset D of S such that $x, y \in D$ and D is θ-connected relative to X.

(*c*) For any θ-separation $[A|B]_\theta$ relative to X such that $S \subset A \cup B$, either $S \subset A$ or $S \subset B$.

Proof.

$(a) \to (b)$. It is obvious.

$(b) \to (c)$. Let $[A|B]_\theta$ be a θ-separation relative to X and $S \subset A \cup B$. Suppose that $a \in A \cap S$ and $b \in B \cap S$. There exists a set D θ-connected relative to X and containing a, b. Let $A_o = A \cap D$ and $B_o = B \cap D$ then A_o and B_o are disjoint and $A_o \cup B_o = D$. By Lemma 2.5, we have $[A_o|B_o]_\theta$. This is a contradiction.

$(c) \to (a)$. It is straightforward.

Theorem 2.7. Let (X, T_1, T_2) be a space and S a subset θ-connected relative to X. If $S \subset Z \subset ij\text{-}cl_\theta S \cap ji\text{-}cl_\theta S$, then Z is θ-connected relative to X.

Proof. Suppose that Z is not θ-connected relative to X. There exists a $[A|B]_\theta$ such that $Z = A \cup B$. Since S is θ-connected relative to X and $S \subset Z$, by Theorem 2.6, we have $S \subset A$ or $S \subset B$. If $S \subset A$, then we obtain $B = B \cap Z \subset B \cap ji\text{-}cl_\theta S \subset B \cap ji\text{-}cl_\theta A = \varnothing$. This is a contradiction. The case $S \subset B$ is proved in the same way.

Theorem 2.8. Let (X, T_1, T_2) be a space and S a pairwise θ-closed subset of X. If $[A|B]_\theta$ is a θ-separation relative to X and $S = A \cup B$, then A is ji-θ-closed and B is ij-θ-closed.

Proof. Suppose that A is not ji-θ-closed. Let $y \in ji\text{-}cl_\theta A - A$. Since S is ji-θ-closed and $S = A \cup B$, then $y \in B \cap ji\text{-}cl_\theta A$. This is a contradiction. Thus A must be ji-θ-closed. To prove that B is ij-θ-closed, just use a similar argument.

Corollary 2.9. Let (X, T_1, T_2) be a space. If $[A|B]_\theta$ is a θ-separation of X itself, then A is ji-θ-closed and B is ij-θ-closed.

Lemma 2.10. Let B be a subset of space (X, T_1, T_2). If B is j-open, then $i\text{-}cl\, B = ij\text{-}cl_\theta B$.

Proof. Let $x \in i\text{-}cl\, B$, therefore for every i-open neighborhood U of x, $B \cap U \neq \varnothing$ and so $B \cap j\text{-}cl\, U \neq \varnothing$, this means $x \in ij\text{-}cl_\theta B$ thus $i\text{-}cl\, B \subset ij\text{-}cl_\theta B$.

Now, let $x \notin i\text{-}cl\, B$, there exists an i-open set U such that $U \cap B = \varnothing$. Since B is j-open we get $j\text{-}cl\, U \cap B = \varnothing$. Thus $x \notin ij\text{-}cl_\theta B$, and so $ij\text{-}cl_\theta B \subset i\text{-}cl\, B$.

Theorem 2.11. For a space (X, T_1, T_2) the following properties hold:

(*a*) Pairwise-connected subsets are θ-connected subsets relative to X.

(*b*) A pair (A, B) of nonempty disjoint subsets of X such that A, $B \in T_1 \cap T_2$ results θ-separation relative to X.

Proof.

(*a*) Let $[A|B]_\theta$ be a θ-separation relative to X and suppose that $S = A \cup B$. Since $i\text{-}cl\, Z \subset ij\text{-}cl_\theta Z$ for any subset Z of X, we have $A \cap T_{is} - cl\, B = (A \cap i\text{-}cl\, B) \cap S \subset A \cap ij\text{-}cl_\theta B = \varnothing$ (where T_{is} is the relative toplogy on S induced by T_i and $T_{is}\text{-}cl\, A$ is the closure of A with respect to T_{is}). Similarly, we have $B \cap T_{js} - cl\, A = \varnothing$. This shows that the space (S, T_{1s}, T_{2s}) is not pairwise-connected.

(*b*) Since $A \cap B = \varnothing$ and $A \in T_i$ for $i = 1, 2$, we have $A \cap i\text{-}cl\, B = \varnothing$. Since $B \in T_1 \cap T_2$ we have $i\text{-}cl\, B = ij\text{-}cl_\theta B$ and hence $A \cap ij\text{-}cl_\theta B = \varnothing$. Similarly, we have $B \cap ji\text{-}cl_\theta A = \varnothing$ and so $[A|B]_\theta$.

Theorem 2.12. Let (X, T_1, T_2) be a space and $\{F_m | m \in I\}$ be a family of sets θ-connected relative to X. If the pair (F_m, F_n) is not θ-separation relative to X for any $m, n \in I$, then $\cup\{F_m | m \in I\}$ is θ-connected relative to X.

Proof. Suppose that there exists a $[A|B]_\theta$ such that $\cup\{F_m|m \in I\} = A \cup B$. Since for each $m \in I$, F_m is θ-connected relative to X and $F_m \subset A \cup B$, by Theorem 2.6 $F_m \subset A$ or $F_m \subset B$. Now put $I_r = \{m \in I|F_m \subset A\}$ and $I_s = \{m \in I|F_m \subset B\}$. Then $I_r \neq \emptyset$, $I_s \neq \emptyset$ and $I_r \cup I_s = I$. Let $m_a \in I_r$ and $m_b \in I_s$, then $F_{m_a} \subset A$ and $F_{m_b} \subset B$. By Lemma 2.5, we obtain $[F_{m_a}|F_{m_b}]_\theta$. This is a contradiction.

Definition 2.13. [3] A function $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij-δ-continuous* (*ij-δ-c*) if and only if the inverse image of each *ij*-δ-open subset V of Y is an *ij*-δ-open subset of X. Equivalently, for every $x \in X$ and for every *i*-open neighborhood V of $f(x)$, there exists an *i*-open neighborhood U of x such that $f(i\text{-}int(j\text{-}cl\ U)) \subset i\text{-}int(j\text{-}cl\ V)$.

Definition 2.14. [6] A function $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij-strongly-θ-continuous* (*ij-s.θ.c*) if and only if the inverse image of each *i*-open subset V of Y is *ij*-θ-open subset of X. Equivalently, for every $x \in X$ and for *i*-open neighborhood V of $f(x)$, there exists an *i*-open neighborhood U of x such that $f(j\text{-}cl\ U) \subset V$.

Lemma 2.15. A function $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-s.θ.c* if and only if $f:(X, T_{1\theta}, T_{2\theta}) \to (Y, \sigma_1, \sigma_2)$ is pairwise continuous.

Theorem 2.16. If $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-s.θ.c* and K is θ-connected relative to X, then $f(K)$ is a pairwise-connected subset of Y.

Proof. Follows from Theorem 2.4 and Lemma 2.15.

Corollary 2.17. If $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is an *ij-s.θ.c* surjection and (X, T_1, T_2) is θ-connected, then (Y, σ_1, σ_2) is pairwise-connected.

The following theorem offers a classifying space for θ-connected bitopological spaces.

Theorem 2.18. Let $Y = \{0,1\}$ and $\sigma_1 = \{Y, \emptyset, \{0\}\}$, $\sigma_2 = \{Y, \emptyset, \{1\}\}$. The space (X, T_1, T_2) is θ-connected if and only if every *ij-s.θ.c* function $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is constant.

Proof.

Necessity: If $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is an *ij-s.θ.c* surjection, then $A = f^{-1}(0)$ and $B = f^{-1}(1)$ form a θ-separation $[A|B]_\theta$ of the whole space X, by Lemma 2.10. Therefore (X, T_1, T_2) is not θ-connected.

Sufficiency: Suppose that (X, T_1, T_2) is not θ-connected. There exists a θ-separation $[A|B]_\theta$ such that $X = A \cup B$. Define a function $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ as follows: $f(A) = 0$ and $f(B) = 1$. Since $A \cap ij\text{-}cl_\theta B = \emptyset$ we have $ij\text{-}cl_\theta(f^{-1}(1)) = ij\text{-}cl_\theta B = (A \cup B) \cap ij\text{-}cl_\theta B = B = f^{-1}(1)$. Therefore, $f^{-1}(1)$ is 12-θ-closed where $\{1\}$ is 1-closed. Similarly, $f^{-1}(0)$ is 21-θ-closed where $\{0\}$ is 2-closed. This shows that $f:(X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-s.θ.c*. However, f is not constant.

3. SETS δ-CONNECTED RELATIVE TO BITOPOLOGICAL SPACES

Several proofs of the theorems stated in this section parallel to those of Section 2 and therefore are omitted.

Definition 3.1. A pair (A, B) of nonempty subsets of a space (X, T_1, T_2) is said to be a *δ-separation* relative to X (denoted by $[A|B]_\delta$) if $[A \cap ij\text{-}cl_\delta B] \cup [B \cap ji\text{-}cl_\delta A] = \emptyset$.

Definition 3.2. A subset S of a space (X, T_1, T_2) is said to be *δ-connected* relative to X if it cannot be expressed as the union of two nonempty subsets A and B which form a δ-separation relative to X. A space (X, T_1, T_2) is said to be δ-connected if it admits no δ-separation relative to X.

Lemma 3.3. The collection $T_{i\delta}$ of all *ij*-δ-open sets in (X, T_1, T_2) is a topology on X.

Proof.

(*i*) X and $\emptyset$ are *ij*-δ-open sets.

(*ii*) Suppose that A and B are *ij*-δ-open sets in (X, T_1, T_2). Let $x \in A \cap B$, then $x \notin A^c \cup B^c$. Now A^c and B^c are *ij*-δ-closed sets. Thus there exist *i*-open sets U and V such that $x \in U$, $i\text{-}int(j\text{-}cl\ U) \cap A^c = \emptyset$ and $x \in V$, $i\text{-}int(j\text{-}cl\ V) \cap B^c = \emptyset$. Therefore $i\text{-}int(j\text{-}cl\ U) \subset A$ and $i\text{-}int(j\text{-}cl\ V) \subset B$, thus $i\text{-}int(j\text{-}cl\ U \cap j\text{-}cl\ V) \subset A \cap B$. Since $x \in U \cap V$, where $U \cap V$ is an *i*-open set, $x \in ij\text{-}int_\delta(A \cap B)$ and therefore $A \cap B$ is an *ij*-open set.

(*iii*) Let $\{A_k\}$, $k \in I$, be any family of *ij*-δ-open subsets of X and $x \in \bigcup_k A_k$. Then $x \in A_k$ for some k. Since A_k is an *ij*-δ-open set, there exists an *i*-open set U such that $x \in i\text{-}int(j\text{-}cl\ U) \subset A_k$ and so $x \in i\text{-}int(j\text{-}cl\ U) \in \bigcup_k A_k$. Hence $x \in ij\text{-}int_\delta(\bigcup_k A_k)$, thus $\bigcup_k A_k$ is an *ij*-δ-open set. Therefore the collection of all *ij*-δ-open sets in (X, T_1, T_2) is a topology on X.

Theorem 3.4. A space (X, T_1, T_2) is δ-connected if and only if the space $(X, T_{1\delta}, T_{2\delta})$ is pairwise connected.

Lemma 3.5. Let A and B be subsets of a space (X, T_1, T_2). If $[A|B]_\delta$, $\varnothing \neq A_0 \subset A$ and $\varnothing \neq B_0 \subset B$, then $[A_0|B_0]_\delta$.

Theorem 3.6. The following are equivalent for a space (X, T_1, T_2) and a subset S of X.

(*a*) S is δ-conntected relative to X.

(*b*) For each two points x, y of S, there exists a subset D of S such that $x, y \in D$ and D is δ-connected relative to X.

(*c*) For any δ-separation $[A|B]_\delta$ relative to X such that $S \subset A \cup B$, either $S \subset A$ or $S \subset B$.

Theorem 3.7. Let (X, T_1, T_2) be a space and S a subset δ-connected relative to X. If $S \subset Z \subset ij\text{-}cl_\delta S \cap ji\text{-}cl_\delta S$, then Z is δ-connected relative to X.

Theorem 3.8. Let (X, T_1, T_2) be a space and S be a pairwise-δ-closed subset of X. If (A, B) is a δ-separation relative to X and $S = A \cup B$, then A is *ji*-δ-closed and B is *ij*-δ-closed.

Proof. Since S is pairwise-δ-closed $ji\text{-}cl_\delta A \subset ji\text{-}cl_\delta S = S$ and hence we have $ji\text{-}cl_\delta A = ji\text{-}cl_\delta A \cap S = (ji\text{-}cl_\delta A) \cap (A \cup B) = A$. This shows that A is *ji*-δ-closed. By using the same argument we can show that B is *ij*-δ-closed.

Theorem 3.9. Let (X, T_1, T_2) be a space and $\{F_m | m \in I\}$ be a family of sets δ-connected relative to X. If the pair (F_m, F_n) is not δ-separation relative to X for any $m, n \in I$, then $\cup\{F_m | m \in I\}$ is δ-connected relative to X.

Lemma 3.10. A function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij*-δ-*c*. if and only if $f: (X, T_{1\delta}, T_{2\delta}) \to (Y, \sigma_{1\delta}, \sigma_{2\delta})$ is pairwise continuous.

Definition 3.11. [7] A function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij-super-continuous* (*ij-sc*) if and only if the inverse of each *i*-open subset of Y is an *ij*-δ-open subset of X. The function f is said to be pairwise-super-continuous if it is both *ij-sc* and *ji-sc*.

Lemma 3.12. A function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-sc* if and only if $f: (X, T_{1\delta}, T_{2\delta}) \to (Y, \sigma_1, \sigma_2)$ is pairwise continuous.

Theorem 3.13. If $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is an *ij*-δ-continuous function and K is δ-connected relative to X, then $f(K)$ is δ-connected relative to Y.

Proof. Follows from Theorem 3.4 and Lemma 3.10.

Theorem 3.14. If $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-sc* and K is δ-connectd relative to X, then $f(K)$ is pairwise-connected.

Corollary 3.15. If $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is an *ij-sc* surjection and X is δ-connected, then Y is pairwise connected.

Theorem 3.16. Let $Y = \{0,1\}$, $\sigma_1 = \{Y, \varnothing, \{0\}\}$ and $\sigma_2 = \{Y, \varnothing, \{1\}\}$. Then the space (X, T_1, T_2) is δ-connected if and only if every *ij-sc* function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is constant.

Proof.

Necessity: Let (X, T_1, T_2) be δ-connected. Suppose that there exists an *ij-sc* function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ such that f is not constant. Then f is surjective and hence by Corollary 3.15, Y is pairwise-connected. However, by Theorem 2.11, Y is not pairwise-connected since (Y, σ_1, σ_2) is not θ-connected. This is a contradiction.

Sufficiency: Suppose that (X, T_1, T_2) is not δ-connected. There exists a δ-separation $[A|B]_\delta$ such that $X = A \cup B$. Define a function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ as follows: $f(A) = 0$ and $f(B) = 1$. Since $A \cap ij\text{-}cl_\delta B = \varnothing$ we have $ij\text{-}cl_\delta(f^{-1}(1)) = ij\text{-}cl_\delta B = (A \cup B) \cap ij\text{-}cl_\delta B = B = f^{-1}(1)$. Therefore, $f^{-1}(1)$ is 12-δ-closed where $\{1\}$ is 1-closed. Similarly, $f^{-1}(0)$ is 21-δ-closed where $\{0\}$ is 2-closed. This shows that $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is *ij-sc*. However f is not constant.

4. COMPARISONS

Lemma 4.1. For any subset A of a space (X, T_1, T_2) $i\text{-}cl\, A \subset ij\text{-}cl_\delta A \subset ij\text{-}cl_\theta A$.

Proof. Let $x \in i\text{-}cl\, A$, then for any *i*-open neighborhood U of x $A \cap U \neq \varnothing$. Since U is an *i*-open set we have $U \subset i\text{-}int(j\text{-}cl\, U)$ and so $A \cap i\text{-}int(j\text{-}cl\, U) \neq \varnothing$. This means $x \in ij\text{-}cl_\delta A$, hence $i\text{-}cl\, A \in ij\text{-}cl_\delta A$.

Now let $x \in ij\text{-}cl_\delta A$, then for every *i*-open set U of x we have $A \cap i\text{-}int(j\text{-}cl\, U) \neq \varnothing$ and therefore $A \cap (j\text{-}cl\, U) \neq \varnothing$. This means $x \in ij\text{-}cl_\theta A$. Thus $ij\text{-}cl_\delta A \subset ij\text{-}cl_\theta A$ and the lemma is proved.

Theorem 4.2. For a subset of a space (X, T_1, T_2) the following implications hold:

pairwise-connected $\Rightarrow$ δ-connected $\Rightarrow$ θ-connected.

Proof. This follows from Lemma 4.1.

Lemma 4.3. If the space (X, T_1, T_2) is ij-almost-regular (resp. ij-semi-regular), then $ij\text{-}cl_\delta A = ij\text{-}cl_\theta A$ (resp. $ij\text{-}cl_\delta A = i\text{-}cl\, A$), for any subset A of X.

Proof. By Lemma 4.1 we have $ij\text{-}cl_\delta A \subset ij\text{-}cl_\theta A$. Now, let $x \in ij\text{-}cl_\theta A$, then for every i-open neighborhood U of x we have $A \cap j\text{-}cl\, U \neq \varnothing$. Since (X, T_1, T_2) is ij-almost-regular $A \cap i\text{-}int(j\text{-}cl\, U) \neq \varnothing$ which means $x \in ij\text{-}cl_\delta A$. Therefore $ij\text{-}cl_\theta A = ij\text{-}cl_\delta A$. By the same way if the space (X, T_1, T_2) is ij-semi-regular we have $ij\text{-}cl_\delta A = i\text{-}cl\, A$.

Theorem 4.4. Let S be a subset of X. If the space (X, T_1, T_2) is ij-almost-regular (resp. ij-semi-regular) and S is θ-connected (resp. δ-connected) relative to X, then S is δ-connected relative to X (resp. pairwise connected).

Proof. Follows from Lemma 4.3.

Corollary 4.5. Let S be a subset of X. If the space (X, T_1, T_2) is ij-regular, then the following are equivalent:

(*a*) S is pairwise-connected.

(*b*) S is δ-connected relative to X.

(*c*) S is θ-connected relative to X.

Proof. Since ij-regular space is an ij-almost-regular and ij-semi-regular space, the proof follows from Theorem 4.2 and Theorem 4.4.

Definition 4.6. [3] A function $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *ij-almost-strongly-θ-continuous* ($ij\text{-}a.s.\theta.c$) if for each $x \in X$ and each i-open set V containing $f(x)$, there exists an i-open set U containing x such that $f(j\text{-}cl\, U) \subset i\text{-}int(j\text{-}cl\, V)$.

Theorem 4.7. If $f: (X, T_1, T_2) \to (Y, \sigma_1, \sigma_2)$ is $ij\text{-}a.s.\theta.c$ and K is θ-connected relative to X, then $f(K)$ is δ-connected relative to Y.

Proof. Suppose that $f(K)$ is not δ-connected. There exist nonempty subsets A^* and B^* of $f(K)$ such that $f(K) = A^* \cup B^*$ where $[A^* \cap ij\text{-}cl_\delta B^*] \cup [B^* \cap ji\text{-}cl_\delta A^*] = \varnothing$. Let $A = K \cap f^{-1}(A^*)$ and $B = K \cap f^{-1}(B^*)$, then $K = A \cup B$. Since K is θ-connected relative to X, without any loss of generality we may suppose that $x \in A \cap ij\text{-}cl_\theta B$. Then $f(x) \in A^* \subset Y - ij\text{-}cl_\delta B^*$. There exists an $ij\text{-}r.o$ set V containing $f(x)$ such that $V \cap B^* = \varnothing$. Since f is $ij\text{-}a.s.\theta.c$, there exists an i-open set U of x such that $f(j\text{-}cl\, U) \subset i\text{-}int(j\text{-}cl\, V) = V$. Since $x \in ij\text{-}cl_\theta B$, we have $\varnothing \neq j\text{-}cl\, U \cap B \subset j\text{-}cl\, U \cap f^{-1}(B^*)$. Thus we obtain $f(j\text{-}cl\, U) \cap B^* \neq \varnothing$. This is a contradiction.

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